Modes and modals.
Varieties generated by modes of submodes.

Agata Pilitowska   Anna Zamojska-Dzienio

Faculty of Mathematics and Information Science, Warsaw University of Technology

Novosibirsk, October 2011

For a given set A denote by $\mathcal{P}_{>0}A$ the family of all non-empty subsets of A.

For a given set A denote by $\mathcal{P}_{>0}A$ the family of all non-empty subsets of A.

For any n-ary operation $\omega : A^n \longrightarrow A$ we define **the complex** (**or power**) **operation** $\omega : \mathcal{P}_{>0}A^n \longrightarrow \mathcal{P}_{>0}A$ in the following way:

For a given set A denote by $\mathcal{P}_{>0}A$ the family of all non-empty subsets of A.

For any n-ary operation $\omega : A^n \rightarrow A$ we define **the complex** (**or power**) **operation** $\omega : \mathcal{P}_{>0}A^n \rightarrow \mathcal{P}_{>0}A$ in the following way:

$$\omega(A_1, \ldots, A_n) := \{\omega(a_1, \ldots, a_n) \mid a_i \in A_i\},$$

For a given set A denote by $\mathcal{P}_{>0}A$ the family of all non-empty subsets of A.

For any n-ary operation $\omega : A^n \to A$ we define **the complex** (**or power**) **operation** $\omega : \mathcal{P}_{>0}A^n \to \mathcal{P}_{>0}A$ in the following way:

$$\omega(A_1, \dots, A_n) := \{\omega(a_1, \dots, a_n) \mid a_i \in A_i\},$$

where $\emptyset \neq A_1, \dots, A_n \subseteq A$.

For a given set A denote by $\mathcal{P}_{>0}A$ the family of all non-empty subsets of A.

For any n-ary operation $\omega : A^n \to A$ we define **the complex** (**or power**) **operation** $\omega : \mathcal{P}_{>0}A^n \to \mathcal{P}_{>0}A$ in the following way:

$$\omega(A_1, \dots, A_n) := \{\omega(a_1, \dots, a_n) \mid a_i \in A_i\},$$

where $\emptyset \neq A_1, \dots, A_n \subseteq A$.

The power (**complex** or **global**) **algebra** of an algebra (A, Ω) is the algebra $(\mathcal{P}_{>0}A, \Omega)$.

The complex operation is a natural generalization of the multiplication of cosets of a subgroup of a group introduced by Frobenius.

The complex operation is a natural generalization of the multiplication of cosets of a subgroup of a group introduced by Frobenius.

Besides group theory, power operations appeared also in other algebraic theories. For example, the set of ideals of a distributive lattice $(L, \vee, \wedge)$ again forms a lattice, where meets and joins are precisely the power operations of $\vee$ and $\wedge$. In formal language theory the product of two languages is the power operation of concatenation of words.

Closely related to power algebras of sets are complex algebras of subalgebras.

Closely related to power algebras of sets are complex algebras of subalgebras.

Let AS be the set of all (non-empty) subalgebras of (A, Ω).

Closely related to power algebras of sets are complex algebras of subalgebras.

Let AS be the set of all (non-empty) subalgebras of (A, Ω). In general, the family AS has not to be closed under complex operations. However if it does, (AS, Ω) is a subalgebra of the algebra $(\mathcal{P}_{>0}A, \Omega)$ and is called **the algebra of subalgebras** of (A, Ω).

Closely related to power algebras of sets are complex algebras of subalgebras.

Let AS be the set of all (non-empty) subalgebras of (A, Ω). In general, the family AS has not to be closed under complex operations. However if it does, (AS, Ω) is a subalgebra of the algebra $(\mathcal{P}_{>0}A, \Omega)$ and is called **the algebra of subalgebras** of (A, Ω).

For example, if an algebra (A, Ω) is **entropic**, i.e. any two of its operations commute, then its algebra of subalgebras is always defined.

Some properties of an algebra (A, Ω) may remain invariant under power construction but obviously not all of them.

In particular, not all identities true in (A, Ω) will be satisfied in $(\mathcal{P}_{>0}A, \Omega)$ or in (AS, Ω).

For example, the power algebra of a group is not again a group [Grätzer and Lakser, 1988].

For an arbitrary variety $\mathcal{V}$, let $\mathcal{V}\Sigma$ denote the variety generated by power algebras of algebras in $\mathcal{V}$,

For an arbitrary variety $\mathcal{V}$, let $\mathcal{V}\Sigma$ denote the variety generated by power algebras of algebras in $\mathcal{V}$, i.e.,

$$\mathcal{V}\Sigma := \mathrm{HSP}(\{(\mathcal{P}_{>0}A, \Omega) \mid (A, \Omega) \in \mathcal{V}\}).$$

For an arbitrary variety $\mathcal{V}$, let $\mathcal{V}\Sigma$ denote the variety generated by power algebras of algebras in $\mathcal{V}$, i.e.,

$$\mathcal{V}\Sigma := \mathrm{HSP}(\{(\mathcal{P}_{>0}A, \Omega) \mid (A, \Omega) \in \mathcal{V}\}).$$

If additionally for every algebra in $\mathcal{V}$, its algebra of subalgebras is defined, let $\mathcal{V}\mathcal{S}$ denote the variety generated by algebras of subalgebras of algebras in $\mathcal{V}$,

For an arbitrary variety $\mathcal{V}$, let $\mathcal{V}\Sigma$ denote the variety generated by power algebras of algebras in $\mathcal{V}$, i.e.,

$$\mathcal{V}\Sigma := \mathrm{HSP}(\{(\mathcal{P}_{>0}A, \Omega) \mid (A, \Omega) \in \mathcal{V}\}).$$

If additionally for every algebra in $\mathcal{V}$, its algebra of subalgebras is defined, let $\mathcal{V}\mathcal{S}$ denote the variety generated by algebras of subalgebras of algebras in $\mathcal{V}$, i.e.,

$$\mathcal{V}\mathcal{S} := \mathrm{HSP}(\{(AS, \Omega) \mid (A, \Omega) \in \mathcal{V}\}).$$

For an arbitrary variety $\mathcal{V}$, let $\mathcal{V}\Sigma$ denote the variety generated by power algebras of algebras in $\mathcal{V}$, i.e.,

$$\mathcal{V}\Sigma := \mathrm{HSP}(\{(\mathcal{P}_{>0}A, \Omega) \mid (A, \Omega) \in \mathcal{V}\}).$$

If additionally for every algebra in $\mathcal{V}$, its algebra of subalgebras is defined, let $\mathcal{V}\mathcal{S}$ denote the variety generated by algebras of subalgebras of algebras in $\mathcal{V}$, i.e.,

$$\mathcal{V}\mathcal{S} := \mathrm{HSP}(\{(AS, \Omega) \mid (A, \Omega) \in \mathcal{V}\}).$$

This happens for example in the case $\mathcal{V}$ is **entropic**, i.e. consists of entropic algebras.

For an arbitrary variety $\mathcal{V}$, let $\mathcal{V}\Sigma$ denote the variety generated by power algebras of algebras in $\mathcal{V}$, i.e.,

$$\mathcal{V}\Sigma := \mathrm{HSP}(\{(\mathcal{P}_{>0}A, \Omega) \mid (A, \Omega) \in \mathcal{V}\}).$$

If additionally for every algebra in $\mathcal{V}$, its algebra of subalgebras is defined, let $\mathcal{V}\mathcal{S}$ denote the variety generated by algebras of subalgebras of algebras in $\mathcal{V}$, i.e.,

$$\mathcal{V}\mathcal{S} := \mathrm{HSP}(\{(AS, \Omega) \mid (A, \Omega) \in \mathcal{V}\}).$$

This happens for example in the case $\mathcal{V}$ is **entropic**, i.e. consists of entropic algebras.

It is clear that, $\mathcal{V}\mathcal{S} \subseteq \mathcal{V}\Sigma$.

For an arbitrary variety $\mathcal{V}$, let $\mathcal{V}\Sigma$ denote the variety generated by power algebras of algebras in $\mathcal{V}$, i.e.,

$$\mathcal{V}\Sigma := \mathrm{HSP}(\{(\mathcal{P}_{>0}A, \Omega) \mid (A, \Omega) \in \mathcal{V}\}).$$

If additionally for every algebra in $\mathcal{V}$, its algebra of subalgebras is defined, let $\mathcal{V}\mathcal{S}$ denote the variety generated by algebras of subalgebras of algebras in $\mathcal{V}$, i.e.,

$$\mathcal{V}\mathcal{S} := \mathrm{HSP}(\{(AS, \Omega) \mid (A, \Omega) \in \mathcal{V}\}).$$

This happens for example in the case $\mathcal{V}$ is **entropic**, i.e. consists of entropic algebras.

It is clear that, $\mathcal{V}\mathcal{S} \subseteq \mathcal{V}\Sigma$.

Moreover, $\mathcal{V} \subseteq \mathcal{V}\Sigma$, because every algebra (A, Ω) can be embedded into $(\mathcal{P}_{>0}A, \Omega)$ by $x \mapsto \{x\}$.

G. Grätzer and H. Lakser determined the identities satisfied by the variety $\mathcal{V}\Sigma$ in relation to identities true in $\mathcal{V}$.

G. Grätzer and H. Lakser determined the identities satisfied by the variety $\mathcal{V}\Sigma$ in relation to identities true in $\mathcal{V}$.

We call a term t of the language of a variety $\mathcal{V}$ **linear**, if every variable occurs in t at most once. An identity $t \approx u$ is called **linear**, if both terms t and u are linear.

G. Grätzer and H. Lakser determined the identities satisfied by the variety $\mathcal{V}\Sigma$ in relation to identities true in $\mathcal{V}$.

We call a term t of the language of a variety $\mathcal{V}$ **linear**, if every variable occurs in t at most once. An identity $t \approx u$ is called **linear**, if both terms t and u are linear.

Theorem (Grätzer and Lakser, 1988)

Let $\mathcal{V}$ be a variety of algebras. The variety $\mathcal{V}\Sigma$ satisfies precisely those identities resulting through identification of variables from the linear identities true in $\mathcal{V}$.

G. Grätzer and H. Lakser determined the identities satisfied by the variety $\mathcal{V}\Sigma$ in relation to identities true in $\mathcal{V}$.

We call a term t of the language of a variety $\mathcal{V}$ **linear**, if every variable occurs in t at most once. An identity $t \approx u$ is called **linear**, if both terms t and u are linear.

Theorem (Grätzer and Lakser, 1988)

Let $\mathcal{V}$ be a variety of algebras. The variety $\mathcal{V}\Sigma$ satisfies precisely those identities resulting through identification of variables from the linear identities true in $\mathcal{V}$.

Corollary (Grätzer and Lakser, 1988)

Let $\mathcal{V}$ be a variety of algebras. Then $\mathcal{V}\Sigma = \mathcal{V}$ if and only if $\mathcal{V}$ is defined by a set of linear identities.

A general similar characterization for varieties $\mathcal{V}\mathcal{S}$ is still not known. Though $\mathcal{V}\mathcal{S}$ satisfies the linear identities true in $\mathcal{V}$, it is usually very difficult to determine which non-linear identities true in $\mathcal{V}$ are also satisfied in $\mathcal{V}\mathcal{S}$.

A general similar characterization for varieties $\mathcal{VS}$ is still not known. Though $\mathcal{VS}$ satisfies the linear identities true in $\mathcal{V}$, it is usually very difficult to determine which non-linear identities true in $\mathcal{V}$ are also satisfied in $\mathcal{VS}$.

An algebra (A, Ω) is **idempotent** if each singleton is a subalgebra,

A general similar characterization for varieties $\mathcal{VS}$ is still not known. Though $\mathcal{VS}$ satisfies the linear identities true in $\mathcal{V}$, it is usually very difficult to determine which non-linear identities true in $\mathcal{V}$ are also satisfied in $\mathcal{VS}$.

An algebra (A, Ω) is **idempotent** if each singleton is a subalgebra, i.e. the following identities are satisfied in (A, Ω) for every n-ary $\omega \in \Omega$

$$\omega(x, \dots, x) \approx x.$$

A general similar characterization for varieties $\mathcal{VS}$ is still not known. Though $\mathcal{VS}$ satisfies the linear identities true in $\mathcal{V}$, it is usually very difficult to determine which non-linear identities true in $\mathcal{V}$ are also satisfied in $\mathcal{VS}$.

An algebra (A, Ω) is **idempotent** if each singleton is a subalgebra, i.e. the following identities are satisfied in (A, Ω) for every n-ary $\omega \in \Omega$

$$\omega(x, \dots, x) \approx x.$$

A variety $\mathcal{V}$ of algebras is called **idempotent** if every algebra in $\mathcal{V}$ is idempotent.

Note that the property of entropicity may also be expressed by means of identities:

Note that the property of entropicity may also be expressed by means of identities:

$$\omega(\varphi(x_{11},\dots,x_{n1}),\dots,\varphi(x_{1m},\dots,x_{nm})) \approx \\ \varphi(\omega(x_{11},\dots,x_{1m}),\dots,\omega(x_{n1},\dots,x_{nm})).$$

for every m-ary $\omega \in \Omega$ and n-ary $\varphi \in \Omega$.

Note that the property of entropicity may also be expressed by means of identities:

$$\omega(\varphi(x_{11},\ldots,x_{n1}),\ldots,\varphi(x_{1m},\ldots,x_{nm})) \approx \varphi(\omega(x_{11},\ldots,x_{1m}),\ldots,\omega(x_{n1},\ldots,x_{nm})).$$

for every m-ary $\omega \in \Omega$ and n-ary $\varphi \in \Omega$.

It is known that for an idempotent and entropic variety $\mathcal{V}$, the variety $\mathcal{VS}$ is also idempotent and entropic [Romanowska and Smith, 1981].

Note that the property of entropicity may also be expressed by means of identities:

$$\omega(\varphi(x_{11}, \dots, x_{n1}), \dots, \varphi(x_{1m}, \dots, x_{nm})) \approx \\ \varphi(\omega(x_{11}, \dots, x_{1m}), \dots, \omega(x_{n1}, \dots, x_{nm})).$$

for every m-ary $\omega \in \Omega$ and n-ary $\varphi \in \Omega$.

It is known that for an idempotent and entropic variety $\mathcal{V}$, the variety $\mathcal{V}\mathcal{S}$ is also idempotent and entropic [Romanowska and Smith, 1981].

Since entropic identities are linear it follows that in this case the variety $\mathcal{V}\Sigma$ is entropic too, but very rarely is again idempotent.

By Grätzer-Lakser Theorem an idempotent law is satisfied in the variety $\mathcal{V}\Sigma$ if and only if it is a consequence of linear identities true in $\mathcal{V}$.

By Grätzer-Lakser Theorem an idempotent law is satisfied in the variety $\mathcal{V}\Sigma$ if and only if it is a consequence of linear identities true in $\mathcal{V}$.

On the other hand, if $\mathcal{V}$ is idempotent, then $\mathcal{V} \subseteq \mathcal{V}\mathcal{S} \subseteq \mathcal{I}\mathcal{V} \subseteq \mathcal{V}\Sigma$, where $\mathcal{I}\mathcal{V}$ is the idempotent subvariety of $\mathcal{V}\Sigma$.

By Grätzer-Lakser Theorem an idempotent law is satisfied in the variety $\mathcal{V}\Sigma$ if and only if it is a consequence of linear identities true in $\mathcal{V}$.

On the other hand, if $\mathcal{V}$ is idempotent, then $\mathcal{V} \subseteq \mathcal{V}\mathcal{S} \subseteq \mathcal{I}\mathcal{V} \subseteq \mathcal{V}\Sigma$, where $\mathcal{I}\mathcal{V}$ is the idempotent subvariety of $\mathcal{V}\Sigma$.

But the inclusion $\mathcal{V} \subseteq \mathcal{V}\mathcal{S}$ does not hold in general. For example, for the variety $\mathcal{A}$ of Abelian groups $(A, \cdot, ^{-1})$ defined as inverse semigroups, $\mathcal{A}\mathcal{S}$ is idempotent and entropic [Pilitowska, 1998], whence $\mathcal{A} \not\subseteq \mathcal{A}\mathcal{S}$. This example also shows that the variety $\mathcal{V}\mathcal{S}$ can be idempotent, while $\mathcal{V}$ is not.

By Grätzer-Lakser Theorem an idempotent law is satisfied in the variety $\mathcal{V}\Sigma$ if and only if it is a consequence of linear identities true in $\mathcal{V}$.

On the other hand, if $\mathcal{V}$ is idempotent, then $\mathcal{V} \subseteq \mathcal{V}\mathcal{S} \subseteq \mathcal{I}\mathcal{V} \subseteq \mathcal{V}\Sigma$, where $\mathcal{I}\mathcal{V}$ is the idempotent subvariety of $\mathcal{V}\Sigma$.

But the inclusion $\mathcal{V} \subseteq \mathcal{V}\mathcal{S}$ does not hold in general. For example, for the variety $\mathcal{A}$ of Abelian groups $(A, \cdot, ^{-1})$ defined as inverse semigroups, $\mathcal{A}\mathcal{S}$ is idempotent and entropic [Pilitowska, 1998], whence $\mathcal{A} \nsubseteq \mathcal{A}\mathcal{S}$. This example also shows that the variety $\mathcal{V}\mathcal{S}$ can be idempotent, while $\mathcal{V}$ is not.

If a variety $\mathcal{V}$ is defined by a set of linear identities, then $\mathcal{V}\mathcal{S} \subseteq \mathcal{V}$.

By Grätzer-Lakser Theorem an idempotent law is satisfied in the variety $\mathcal{V}\Sigma$ if and only if it is a consequence of linear identities true in $\mathcal{V}$.

On the other hand, if $\mathcal{V}$ is idempotent, then $\mathcal{V} \subseteq \mathcal{V}\mathcal{S} \subseteq \mathcal{I}\mathcal{V} \subseteq \mathcal{V}\Sigma$, where $\mathcal{I}\mathcal{V}$ is the idempotent subvariety of $\mathcal{V}\Sigma$.

But the inclusion $\mathcal{V} \subseteq \mathcal{V}\mathcal{S}$ does not hold in general. For example, for the variety $\mathcal{A}$ of Abelian groups $(A, \cdot, ^{-1})$ defined as inverse semigroups, $\mathcal{A}\mathcal{S}$ is idempotent and entropic [Pilitowska, 1998], whence $\mathcal{A} \not\subseteq \mathcal{A}\mathcal{S}$. This example also shows that the variety $\mathcal{V}\mathcal{S}$ can be idempotent, while $\mathcal{V}$ is not.

If a variety $\mathcal{V}$ is defined by a set of linear identities, then $\mathcal{V}\mathcal{S} \subseteq \mathcal{V}$. Hence, if an idempotent variety $\mathcal{V}$ is defined by a set of linear identities, then $\mathcal{V}\mathcal{S} = \mathcal{V}$.

Conjecture (Pilitowska, 1996)

An idempotent variety $\mathcal{V}$, in which every algebra has the algebra of subalgebras, coincides with $\mathcal{VS}$ if and only if $\mathcal{V}$ has a basis consisting of idempotent and linear identities.

Conjecture (Pilitowska, 1996)

An idempotent variety $\mathcal{V}$, in which every algebra has the algebra of subalgebras, coincides with $\mathcal{VS}$ if and only if $\mathcal{V}$ has a basis consisting of idempotent and linear identities.

This statement for non-idempotent varieties is false. Let $\mathcal{V}$ be the non-idempotent variety of entropic groupoids satisfying $(xx)y = xy$ and $y(xx) = yx$. It was shown by Adaricheva, Pilitowska and Stanovský (2008) that $\mathcal{V} = \mathcal{VS}$ and $\mathcal{V}$ satisfies the two non-linear identities which cannot be deduced from any set of linear identities true in $\mathcal{V}$.

Hence very natural classes for investigating algebras of subalgebras are varieties of **modes** - idempotent and entropic algebras. Modes and algebras of subalgebras of modes were introduced and investigated in detail by A. Romanowska and J.D.H. Smith, e.g. monographs *Modal theory* (1985) and *Modes* (2002).

Hence very natural classes for investigating algebras of subalgebras are varieties of **modes** - idempotent and entropic algebras. Modes and algebras of subalgebras of modes were introduced and investigated in detail by A. Romanowska and J.D.H. Smith, e.g. monographs *Modal theory* (1985) and *Modes* (2002).

We proved the following theorem.

Theorem (Main Theorem)

Let $\mathcal{M}$ be a variety of modes and let the variety $\mathcal{M}\Sigma$ be locally finite. The variety $\mathcal{M}\mathcal{S}$ satisfies precisely the consequences of the idempotent and linear identities true in $\mathcal{M}$.

Let (A, Ω) be an algebra. The set $\mathcal{P}_{>0}A$ also carries a join semilattice structure under the set-theoretical union $\cup$. By adding the operation $\cup$ to the set of fundamental operations of the power algebra of (A, Ω) we obtain **the extended power algebra** $(\mathcal{P}_{>0}A, \Omega, \cup)$.

Let (A, Ω) be an algebra. The set $\mathcal{P}_{>0}A$ also carries a join semilattice structure under the set-theoretical union $\cup$. By adding the operation $\cup$ to the set of fundamental operations of the power algebra of (A, Ω) we obtain **the extended power algebra** $(\mathcal{P}_{>0}A, \Omega, \cup)$.

B. Jónsson and A. Tarski proved that complex operations **distribute** over the union $\cup$,

Let (A, Ω) be an algebra. The set $\mathcal{P}_{>0}A$ also carries a join semilattice structure under the set-theoretical union $\cup$. By adding the operation $\cup$ to the set of fundamental operations of the power algebra of (A, Ω) we obtain **the extended power algebra** $(\mathcal{P}_{>0}A, \Omega, \cup)$.

B. Jónsson and A. Tarski proved that complex operations **distribute** over the union $\cup$, i.e. for each n-ary operation $\omega \in \Omega$ and non-empty subsets $A_1, \ldots, A_i, \ldots, A_n, B_i$ of A

$$\omega(A_1, \ldots, A_i \cup B_i, \ldots, A_n) = \omega(A_1, \ldots, A_i, \ldots, A_n) \cup \omega(A_1, \ldots, B_i, \ldots, A_n),$$

for any $1 \leq i \leq n$.

Power algebras have also the following two elementary properties for any non-empty subsets $A_i \subseteq B_i$ and A_{ij} of A for $1 \leq i \leq n$, $1 \leq j \leq r$:

Power algebras have also the following two elementary properties for any non-empty subsets $A_i \subseteq B_i$ and A_{ij} of A for $1 \leq i \leq n$, $1 \leq j \leq r$:

$$\omega(A_1, \ldots, A_n) \subseteq \omega(B_1, \ldots, B_n),$$

Power algebras have also the following two elementary properties for any non-empty subsets $A_i \subseteq B_i$ and A_{ij} of A for $1 \leq i \leq n$, $1 \leq j \leq r$:

$$\omega(A_1, \ldots, A_n) \subseteq \omega(B_1, \ldots, B_n),$$
$$\omega(A_{11}, \ldots, A_{n1}) \cup \ldots \cup \omega(A_{1r}, \ldots, A_{nr}) \subseteq$$
$$\omega(A_{11} \cup \ldots \cup A_{1r}, \ldots, A_{n1} \cup \ldots \cup A_{nr}).$$

Power algebras have also the following two elementary properties for any non-empty subsets $A_i \subseteq B_i$ and A_{ij} of A for $1 \leq i \leq n$, $1 \leq j \leq r$:

$$\omega(A_1, \ldots, A_n) \subseteq \omega(B_1, \ldots, B_n),$$
$$\omega(A_{11}, \ldots, A_{n1}) \cup \ldots \cup \omega(A_{1r}, \ldots, A_{nr}) \subseteq$$
$$\omega(A_{11} \cup \ldots \cup A_{1r}, \ldots, A_{n1} \cup \ldots \cup A_{nr}).$$

It is easy to see that both properties hold also for all derived operations t and we obtain the inclusion

$$t(A_1, \ldots, A_i, \ldots, A_n) \cup t(A_1, \ldots, B_i, \ldots, A_n) \subseteq t(A_1, \ldots, A_i \cup B_i, \ldots, A_n)$$

that generalizes the distributive law.

Let $(\mathcal{P}_{>0}M, \Omega, \cup)$ be the extended power algebra of a mode (M, Ω).

Let $(\mathcal{P}_{>0}M, \Omega, \cup)$ be the extended power algebra of a mode (M, Ω).

Denote by $\mathcal{I}$ a (quasi)variety of all idempotent τ-algebras of type $\tau : \Omega \dot{\cup} \{\cup\} \to \mathbb{N}$.

Let $(\mathcal{P}_{>0}M, \Omega, \cup)$ be the extended power algebra of a mode (M, Ω).

Denote by $\mathcal{I}$ a (quasi)variety of all idempotent τ-algebras of type $\tau : \Omega \uplus \{\cup\} \to \mathbb{N}$.

Then $Con_{\mathcal{I}}(\mathcal{P}_{>0}M)$ is the set of all congruence relations γ on $(\mathcal{P}_{>0}M, \Omega, \cup)$, such that the quotient $(\mathcal{P}_{>0}M^{\gamma}, \Omega)$ is idempotent.

Let $(\mathcal{P}_{>0}M, \Omega, \cup)$ be the extended power algebra of a mode (M, Ω).

Denote by $\mathcal{I}$ a (quasi)variety of all idempotent τ-algebras of type $\tau : \Omega \,\dot\cup\, \{\cup\} \to \mathbb{N}$.

Then $Con_{\mathcal{I}}(\mathcal{P}_{>0}M)$ is the set of all congruence relations γ on $(\mathcal{P}_{>0}M, \Omega, \cup)$, such that the quotient $(\mathcal{P}_{>0}M^{\gamma}, \Omega)$ is idempotent.

$Con_{\mathcal{I}}(\mathcal{P}_{>0}M)$ is an algebraic subset of the lattice of all congruences of $(\mathcal{P}_{>0}M, \Omega, \cup)$. The least element in $(Con_{\mathcal{I}}(\mathcal{P}_{>0}M), \subseteq)$ is called **the** $\mathcal{I}$**-replica congruence** of $(\mathcal{P}_{>0}M, \Omega, \cup)$.

For $1 \leq i \leq k$ and $k \geq 2$, let t_i be m_i-ary terms. By **the composition term** $t_1 \circ t_2 \circ \ldots \circ t_k$ of the terms $t_1, t_2, \ldots, t_k$ is meant an $m := m_1 \cdot \ldots \cdot m_k$ -ary term defined by the rule:

For $1 \leq i \leq k$ and $k \geq 2$, let t_i be m_i-ary terms. By **the composition term** $t_1 \circ t_2 \circ \ldots \circ t_k$ of the terms $t_1, t_2, \ldots, t_k$ is meant an $m := m_1 \cdot \ldots \cdot m_k$ -ary term defined by the rule:

$$t_1 \circ t_2(\overline{x}_1, \ldots, \overline{x}_{m_1}) := t_1(t_2(\overline{x}_1), \ldots, t_2(\overline{x}_{m_1})),$$
$$t_1 \circ \ldots \circ t_k(\overline{x}_1, \ldots, \overline{x}_r) := t_1 \circ \ldots \circ t_{k-1}(t_k(\overline{x}_1), \ldots, t_k(\overline{x}_r)),$$

where $r = m_1 \cdot \ldots \cdot m_{k-1}$ and $\overline{x}_i = (x_{i1}, \ldots, x_{im_k})$, for $i = 1, \ldots, r$.

For $1 \leq i \leq k$ and $k \geq 2$, let t_i be m_i-ary terms. By **the composition term** $t_1 \circ t_2 \circ \ldots \circ t_k$ of the terms $t_1, t_2, \ldots, t_k$ is meant an $m := m_1 \cdot \ldots \cdot m_k$ -ary term defined by the rule:

$$t_1 \circ t_2(\overline{x}_1, \ldots, \overline{x}_{m_1}) := t_1(t_2(\overline{x}_1), \ldots, t_2(\overline{x}_{m_1})),$$
$$t_1 \circ \ldots \circ t_k(\overline{x}_1, \ldots, \overline{x}_r) := t_1 \circ \ldots \circ t_{k-1}(t_k(\overline{x}_1), \ldots, t_k(\overline{x}_r)),$$

where $r = m_1 \cdot \ldots \cdot m_{k-1}$ and $\overline{x}_i = (x_{i1}, \ldots, x_{im_k})$, for $i = 1, \ldots, r$.

Note that for a mode (M, Ω) and a non-empty subset X of M

$$t_1 \circ \ldots \circ t_k(X, \ldots, X) = t_{\sigma(1)} \circ \ldots \circ t_{\sigma(k)}(X, \ldots, X),$$

for any permutation σ of the set $\{1, \ldots, k\}$.

For $1 \leq i \leq k$ and $k \geq 2$, let t_i be m_i-ary terms. By **the composition term** $t_1 \circ t_2 \circ \ldots \circ t_k$ of the terms $t_1, t_2, \ldots, t_k$ is meant an $m := m_1 \cdot \ldots \cdot m_k$ -ary term defined by the rule:

$$t_1 \circ t_2(\overline{x}_1, \ldots, \overline{x}_{m_1}) := t_1(t_2(\overline{x}_1), \ldots, t_2(\overline{x}_{m_1})),$$
$$t_1 \circ \ldots \circ t_k(\overline{x}_1, \ldots, \overline{x}_r) := t_1 \circ \ldots \circ t_{k-1}(t_k(\overline{x}_1), \ldots, t_k(\overline{x}_r)),$$

where $r = m_1 \cdot \ldots \cdot m_{k-1}$ and $\overline{x}_i = (x_{i1}, \ldots, x_{im_k})$, for $i = 1, \ldots, r$.

Note that for a mode (M, Ω) and a non-empty subset X of M

$$t_1 \circ \ldots \circ t_k(X, \ldots, X) = t_{\sigma(1)} \circ \ldots \circ t_{\sigma(k)}(X, \ldots, X),$$

for any permutation σ of the set $\{1, \ldots, k\}$.

For any derived operation t, we have $X \subseteq t(X, \ldots, X)$.

Remark

Let (M, Ω) be a mode. For $1 \leq i \leq k$, let t_i be m_i-ary terms and $\emptyset \neq X \subseteq M$. For the composition term $t = t_1 \circ t_2 \circ \ldots \circ t_k$ we have

$$t_i(\underbrace{X, X, \ldots, X}_{m_i}) \subseteq t(\underbrace{X, X, \ldots, X}_{m_1 \cdot \ldots \cdot m_k}),$$

for each $1 \leq i \leq k$.

Remark

Let (M, Ω) be a mode. For $1 \leq i \leq k$, let t_i be m_i-ary terms and $\emptyset \neq X \subseteq M$. For the composition term $t = t_1 \circ t_2 \circ \ldots \circ t_k$ we have

$$t_i(\underbrace{X, X, \ldots, X}_{m_i}) \subseteq t(\underbrace{X, X, \ldots, X}_{m_1 \cdot \ldots \cdot m_k}),$$

for each $1 \leq i \leq k$.

Now we define a binary relation ρ on the set $\mathcal{P}_{>0}M$ in the following way:

$$X \;\rho\; Y \quad \Leftrightarrow \quad \begin{array}{l} \text{there exist a } k\text{-ary term } t \text{ and an } m\text{-ary term } s \\ \text{both of type } \Omega \text{ such that} \\ X \subseteq t(Y, Y, \ldots, Y) \quad \text{and} \quad Y \subseteq s(X, X, \ldots, X). \end{array}$$

Theorem

Let (M, Ω) be a mode. The relation ρ is the $\mathcal{I}$-replica congruence of $(\mathcal{P}_{>0}M, \Omega, \cup)$.

Theorem

Let (M, Ω) be a mode. The relation ρ is the $\mathcal{I}$-replica congruence of $(\mathcal{P}_{>0}M, \Omega, \cup)$.

Note that if B is a subalgebra of $(\mathcal{P}_{>0}M, \Omega, \cup)$ then the restriction $\rho_B := \rho \cap B^2$ is a congruence on $(B, \Omega, \cup)$. Moreover, for every $X \in B$ and Ω-term t, $t(X, \ldots, X) \in B$. Hence, ρ_B is the $\mathcal{I}$-replica congruence of $(B, \Omega, \cup)$.

Theorem

Let (M, Ω) be a mode. The relation ρ is the $\mathcal{I}$-replica congruence of $(\mathcal{P}_{>0}M, \Omega, \cup)$.

Note that if B is a subalgebra of $(\mathcal{P}_{>0}M, \Omega, \cup)$ then the restriction $\rho_B := \rho \cap B^2$ is a congruence on $(B, \Omega, \cup)$. Moreover, for every $X \in B$ and Ω-term t, $t(X, \ldots, X) \in B$. Hence, ρ_B is the $\mathcal{I}$-replica congruence of $(B, \Omega, \cup)$.

Let $\mathcal{P}^{<\omega}_{>0}M$ be the set of all finite non-empty subsets of a mode (M, Ω). Then $\mathcal{P}^{<\omega}_{>0}M$ is a subalgebra of the extended power algebra $(\mathcal{P}_{>0}M, \Omega, \cup)$.

Let (M, Ω) be a mode and let $\emptyset \neq X \subseteq M$ and $\Delta \subseteq \Omega$. For any $n \in \mathbb{N}$ let us define sets $X^{[n]_\Delta}$ in the following way:

$$X^{[0]_\Delta} := X,$$

$$X^{[n+1]_\Delta} := \bigcup_{\delta \in \Delta} \delta(X^{[n]_\Delta}, \dots, X^{[n]_\Delta}) = (X^{[n]_\Delta})^{[1]_\Delta}.$$

If $\Delta = \Omega$ we will use the abbreviated notation $X^{[n]}$ instead of $X^{[n]_\Omega}$.

Let (M, Ω) be a mode and let $\emptyset \neq X \subseteq M$ and $\Delta \subseteq \Omega$. For any $n \in \mathbb{N}$ let us define sets $X^{[n]_\Delta}$ in the following way:

$$X^{[0]_\Delta} := X,$$

$$X^{[n+1]_\Delta} := \bigcup_{\delta \in \Delta} \delta(X^{[n]_\Delta}, \dots, X^{[n]_\Delta}) = (X^{[n]_\Delta})^{[1]_\Delta}.$$

If $\Delta = \Omega$ we will use the abbreviated notation $X^{[n]}$ instead of $X^{[n]_\Omega}$.

It is well known that

$$\langle X \rangle = \bigcup_{n \in \mathbb{N}} X^{[n]},$$

where $\langle X \rangle$ denotes the subalgebra of (M, Ω) generated by X.

As proved by A. Romanowska and J.D.H. Smith (1981), for each n-ary complex operation $\omega \in \Omega$ and non-empty subsets $X_1, \ldots, X_n$ of M

$$\langle \omega(X_1, \ldots, X_n) \rangle = \omega(\langle X_1 \rangle, \ldots, \langle X_n \rangle).$$

As proved by A. Romanowska and J.D.H. Smith (1981), for each n-ary complex operation $\omega \in \Omega$ and non-empty subsets $X_1, \ldots, X_n$ of M

$$\langle \omega(X_1, \ldots, X_n) \rangle = \omega(\langle X_1 \rangle, \ldots, \langle X_n \rangle).$$

In particular, if the subsets $X_1, \ldots, X_n$ are finite, then the subalgebra $\omega(\langle X_1 \rangle, \ldots, \langle X_n \rangle)$ is finitely generated.

As proved by A. Romanowska and J.D.H. Smith (1981), for each n-ary complex operation $\omega \in \Omega$ and non-empty subsets $X_1, \ldots, X_n$ of M

$$\langle \omega(X_1, \ldots, X_n) \rangle = \omega(\langle X_1 \rangle, \ldots, \langle X_n \rangle).$$

In particular, if the subsets $X_1, \ldots, X_n$ are finite, then the subalgebra $\omega(\langle X_1 \rangle, \ldots, \langle X_n \rangle)$ is finitely generated.

Now we define the second binary relation on the set $\mathcal{P}_{>0}M$:

$$X \ \alpha \ Y \quad \Leftrightarrow \quad \langle X \rangle = \langle Y \rangle.$$

Theorem

For a mode (M, Ω)*, the relation* α *belongs to the set* $Con_{\mathcal{J}}(\mathcal{P}_{>0}M)$.

Theorem

For a mode (M, Ω), the relation α belongs to the set $Con_{\mathcal{J}}(\mathcal{P}_{>0}M)$.

Lemma

Let (M, Ω) be a mode, Δ be a finite subset of Ω and $\gamma \in Con_{\mathcal{J}}(\mathcal{P}_{>0}M)$. Then $X \, \gamma \, X^{[n]_\Delta}$, for any $n \in \mathbb{N}$.

Theorem

For a mode (M, Ω), the relation α belongs to the set $Con_{\mathfrak{J}}(\mathcal{P}_{>0}M)$.

Lemma

Let (M, Ω) be a mode, Δ be a finite subset of Ω and $\gamma \in Con_{\mathfrak{J}}(\mathcal{P}_{>0}M)$. Then $X \; \gamma \; X^{[n]_\Delta}$, for any $n \in \mathbb{N}$.

Theorem

Let (M, Ω) be a mode. The congruences α and ρ restricted to the subalgebra $\mathcal{P}_{>0}^{<\omega}M$ of $(\mathcal{P}_{>0}M, \Omega, \cup)$ coincide:

$$\alpha_{\mathcal{P}_{>0}^{<\omega}M} = \rho_{\mathcal{P}_{>0}^{<\omega}M}.$$

As was shown by A. Romanowska and J.D.H. Smith (1981), if (M, Ω) is a mode, then the algebra (MS, Ω) of all non-empty subalgebras of (M, Ω) is a mode satisfying each linear identity true in (M, Ω). So, if $\mathcal{M}$ is a variety of modes, then $\mathcal{MS} = \mathrm{HSP}(\{(MS, \Omega) \mid (M, \Omega) \in \mathcal{M}\})$ is also a variety of modes satisfying each linear identity true in $\mathcal{M}$. But the mode (MS, Ω) may also satisfy some non-linear identities.

Example

Consider the groupoid $G = (\{a, b, c, d\}, \cdot)$ with the following multiplication table:

$\cdot$	a	b	c	d
a	a	a	b	b
b	b	b	a	a
c	d	d	c	c
d	c	c	d	d

Example

Consider the groupoid $G = (\{a, b, c, d\}, \cdot)$ with the following multiplication table:

$$
\begin{array}{c|cccc}
\cdot & a & b & c & d \\
\hline
a & a & a & b & b \\
b & b & b & a & a \\
c & d & d & c & c \\
d & c & c & d & d
\end{array}
$$

The groupoid satisfies the identity: $x = (xy)y$ and has 7 subalgebras: $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$, $\{a, b\}$, $\{c, d\}$, G.

Example

Consider the groupoid $G = (\{a, b, c, d\}, \cdot)$ with the following multiplication table:

$\cdot$	a	b	c	d
a	a	a	b	b
b	b	b	a	a
c	d	d	c	c
d	c	c	d	d

The groupoid satisfies the identity: $x = (xy)y$ and has 7 subalgebras: $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$, $\{a, b\}$, $\{c, d\}$, G. Note that $(\{a\}G)G = \{a, b\} \neq \{a\}$, so $(GS, \cdot)$ does not satisfy the identity $x = (xy)y$.

Example

Consider the groupoid $G = (\{a, b, c, d\}, \cdot)$ with the following multiplication table:

$\cdot$	a	b	c	d
a	a	a	b	b
b	b	b	a	a
c	d	d	c	c
d	c	c	d	d

The groupoid satisfies the identity: $x = (xy)y$ and has 7 subalgebras: $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$, $\{a, b\}$, $\{c, d\}$, G. Note that $(\{a\}G)G = \{a, b\} \neq \{a\}$, so $(GS, \cdot)$ does not satisfy the identity $x = (xy)y$. However, straightforward calculations show that it satisfies the non-linear identity: $xy = ((xy)y)y$, true also in $(G, \cdot)$.

Definition

A *modal* is an algebra $(M, \Omega, +)$ such that (M, Ω) is a mode, $(M, +)$ is a (join) semilattice with semilattice order $\leq$, i.e. $x \leq y \;\Leftrightarrow\; x + y = y$, and the operations $\omega \in \Omega$ distribute over $+$ i.e.

$$\begin{aligned} & \omega(x_1, \dots, x_i + y_i, \dots, x_n) = \\ = \;\; & \omega(x_1, \dots, x_i, \dots, x_n) + \omega(x_1, \dots, y_i, \dots, x_n). \end{aligned}$$

Definition

A *modal* is an algebra $(M, \Omega, +)$ such that (M, Ω) is a mode, $(M, +)$ is a (join) semilattice with semilattice order $\leq$, i.e. $x \leq y \;\Leftrightarrow\; x + y = y$, and the operations $\omega \in \Omega$ distribute over $+$ i.e.

$$\begin{aligned} & \omega(x_1, \ldots, x_i + y_i, \ldots, x_n) = \\ = \;& \omega(x_1, \ldots, x_i, \ldots, x_n) + \omega(x_1, \ldots, y_i, \ldots, x_n). \end{aligned}$$

For a given algebra (A, Ω), the set AS of all non-empty subalgebras of (A, Ω) forms a (join) semilattice $(AS, +)$, where $+$ is obtained by setting

$$A_1 + A_2 := \langle A_1 \cup A_2 \rangle,$$

for any $A_1, A_2 \in AS$.

A. Romanowska and J.D.H. Smith proved that in the case of modes, these two structures, mode and semilattice, are related by distributive laws. In this way, for all modes (M, Ω) one obtains the algebras $(MS, \Omega, +)$ that provide basic examples of modals.

A. Romanowska and J.D.H. Smith proved that in the case of modes, these two structures, mode and semilattice, are related by distributive laws. In this way, for all modes (M, Ω) one obtains the algebras $(MS, \Omega, +)$ that provide basic examples of modals.

Other examples of modals are given by quotient algebras $(\mathcal{P}_{>0}M^{\gamma}, \Omega, \cup)$, where $\gamma \in Con_{\mathcal{I}}(\mathcal{P}_{>0}M)$.

A. Romanowska and J.D.H. Smith proved that in the case of modes, these two structures, mode and semilattice, are related by distributive laws. In this way, for all modes (M, Ω) one obtains the algebras $(MS, \Omega, +)$ that provide basic examples of modals.

Other examples of modals are given by quotient algebras $(\mathcal{P}_{>0}M^{\gamma}, \Omega, \cup)$, where $\gamma \in Con_{\mathcal{I}}(\mathcal{P}_{>0}M)$.

Let MP be the set of all finitely generated subalgebras of a mode (M, Ω). The algebra $(MP, \Omega, +)$ is a subalgebra of the modal $(MS, \Omega, +)$ and for any variety $\mathcal{M}$ of modes, the variety $\mathcal{MP} := \mathrm{HSP}(\{(MP, \Omega) \mid (M, \Omega) \in \mathcal{M}\})$ is a subvariety of $\mathcal{MS}$.

Theorem

Let (M, Ω) be a mode. The quotient algebra $(\mathcal{P}_{>0}M^{\alpha}, \Omega, \cup)$ is isomorphic to the modal $(MS, \Omega, +)$ of all non-empty subalgebras of (M, Ω).

Theorem

Let (M, Ω) be a mode. The quotient algebra $(\mathcal{P}_{>0}M^{\alpha}, \Omega, \cup)$ is isomorphic to the modal $(MS, \Omega, +)$ of all non-empty subalgebras of (M, Ω).

Corollary

Let (M, Ω) be a mode. The quotient algebra $(\mathcal{P}_{>0}^{<\omega}M^{\alpha}, \Omega, \cup)$ is isomorphic to the modal $(MP, \Omega, +)$ of all finitely generated subalgebras of (M, Ω).

Let $\mathcal{V}$ be a variety of algebras. By Grätzer-Lakser Theorem, the variety $\mathcal{V}\Sigma = \mathrm{HSP}(\{(\mathcal{P}_{>0}A, \Omega) \mid (A, \Omega) \in \mathcal{V}\})$ satisfies precisely the consequences of the linear identities holding in $\mathcal{V}$. The same proof applies to its subvariety

$$\mathcal{V}\Sigma_{<\omega} := \mathrm{HSP}(\{(\mathcal{P}_{>0}^{<\omega}A, \Omega) \mid (A, \Omega) \in \mathcal{V}\})$$

of power algebras of finite subsets.

Let $\mathcal{V}$ be a variety of algebras. By Grätzer-Lakser Theorem, the variety $\mathcal{V}\Sigma = \mathrm{HSP}(\{(\mathcal{P}_{>0}A, \Omega) \mid (A, \Omega) \in \mathcal{V}\})$ satisfies precisely the consequences of the linear identities holding in $\mathcal{V}$. The same proof applies to its subvariety

$$\mathcal{V}\Sigma_{<\omega} := \mathrm{HSP}(\{(\mathcal{P}^{<\omega}_{>0}A, \Omega) \mid (A, \Omega) \in \mathcal{V}\})$$

of power algebras of finite subsets.

Corollary

Let $\mathcal{V}$ be a variety of algebras. The varieties $\mathcal{V}\Sigma$ and $\mathcal{V}\Sigma_{<\omega}$ coincide.

Let $\mathcal{M}$ be a variety of Ω-modes and consider the variety

$$\rho\mathcal{M}\Sigma^{\cup}_{<\omega} := \mathrm{HSP}(\{(\mathcal{P}^{<\omega}_{>0} M^{\rho}, \Omega, \cup) \mid (M, \Omega) \in \mathcal{M}\}).$$

Let $\mathcal{M}$ be a variety of Ω-modes and consider the variety

$$\rho\mathcal{M}\Sigma^{\cup}_{<\omega} := \mathrm{HSP}(\{(\mathcal{P}^{<\omega}_{>0} M^{\rho}, \Omega, \cup) \mid (M, \Omega) \in \mathcal{M}\}).$$

Lemma

Let (M, Ω) be a mode and B be a subalgebra of $(\mathcal{P}^{<\omega}_{>0} M, \Omega, \cup)$. The $\mathcal{I}$-replica of $(B, \Omega, \cup)$ belongs to the variety $\rho\mathcal{M}\Sigma^{\cup}_{<\omega}$.

Let J be any set and let for each $j \in J$, $(M_j, \Omega) \in \mathcal{M}$. One obtains that the relation $\rho_\sqcap$ defined on the set $\prod_{j\in J} \mathcal{P}^{<\omega}_{>0} M_j$ in the following way:

$$X \; \rho_\sqcap \; Y \quad \Leftrightarrow \quad \begin{array}{l} \text{there exist a } k\text{-ary } \Omega\text{-term } t \text{ and an } m\text{-ary } \Omega\text{-term } s \\ \text{such that for each } j \in J \\ X(j) \subseteq t(Y(j), \ldots, Y(j)) \quad \text{and} \\ Y(j) \subseteq s(X(j), \ldots, X(j)) \end{array}$$

is the $\mathcal{J}$-replica of $(\prod_{j\in J} \mathcal{P}^{<\omega}_{>0} M_j, \Omega, \cup)$.

Lemma

Let $\mathcal{M}$ be a variety of Ω-modes, J be a finite set and let for each $j \in J$, $(M_j, \Omega) \in \mathcal{M}$. The mapping

$$h : (\prod_{j \in J} \mathcal{P}^{<\omega}_{>0} M_j)^{\rho\sqcap} \to \prod_{j \in J} \mathcal{P}^{<\omega}_{>0} M_j^{\rho}; \quad X^{\rho\sqcap} \mapsto \prod_{j \in J} X(j)^{\rho}$$

is an embedding of $((\prod_{j \in J} \mathcal{P}^{<\omega}_{>0} M_j)^{\rho\sqcap}, \Omega, \cup)$ *into* $(\prod_{j \in J} \mathcal{P}^{<\omega}_{>0} M_j^{\rho}, \Omega, \cup)$.

Lemma

Let $\mathcal{M}$ be a variety of Ω-modes, J be a finite set and let for each $j \in J$, $(M_j, \Omega) \in \mathcal{M}$. The mapping

$$h : (\prod_{j\in J} \mathcal{P}^{<\omega}_{>0} M_j)^{\rho\sqcap} \to \prod_{j\in J} \mathcal{P}^{<\omega}_{>0} M_j^{\rho}; \quad X^{\rho\sqcap} \mapsto \prod_{j\in J} X(j)^{\rho}$$

is an embedding of $((\prod_{j\in J} \mathcal{P}^{<\omega}_{>0} M_j)^{\rho\sqcap}, \Omega, \cup)$ into $(\prod_{j\in J} \mathcal{P}^{<\omega}_{>0} M_j^{\rho}, \Omega, \cup)$.

If a set J is not finite then Lemma is not longer true.

Corollary

Let $\mathcal{M}$ be a variety of Ω-modes, J be a finite set and let for each $j \in J$, $(M_j, \Omega) \in \mathcal{M}$. The $\mathcal{I}$-replica of $(\prod_{j\in J} \mathcal{P}_{>0}^{<\omega} M_j, \Omega, \cup)$ belongs to the variety $\rho\mathcal{M}\Sigma_{<\omega}^{\cup}$.

Corollary

Let $\mathcal{M}$ be a variety of Ω-modes, J be a finite set and let for each $j \in J$, $(M_j, \Omega) \in \mathcal{M}$. The $\mathcal{J}$-replica of $(\prod_{j\in J} \mathcal{P}^{<\omega}_{>0} M_j, \Omega, \cup)$ belongs to the variety $\rho\mathcal{M}\Sigma^{\cup}_{<\omega}$.

Let $\mathcal{M}\Sigma^{\cup}_{<\omega}$ denote the variety generated by extended power algebras of finite subsets of algebras from $\mathcal{M}$, i.e.,

$$\mathcal{M}\Sigma^{\cup}_{<\omega} := \mathrm{HSP}(\{(P^{<\omega}_{>0} M, \Omega, \cup) \mid (M, \Omega) \in \mathcal{M}\})$$

Corollary

Let $\mathcal{M}$ be a variety of Ω-modes, J be a finite set and let for each $j \in J$, $(M_j, \Omega) \in \mathcal{M}$. The $\mathcal{I}$-replica of $(\prod_{j \in J} \mathcal{P}^{<\omega}_{>0} M_j, \Omega, \cup)$ belongs to the variety $\rho\mathcal{M}\Sigma^{\cup}_{<\omega}$.

Let $\mathcal{M}\Sigma^{\cup}_{<\omega}$ denote the variety generated by extended power algebras of finite subsets of algebras from $\mathcal{M}$, i.e.,

$$\mathcal{M}\Sigma^{\cup}_{<\omega} := \mathrm{HSP}(\{(P^{<\omega}_{>0} M, \Omega, \cup) \mid (M, \Omega) \in \mathcal{M}\})$$

and let $\mathcal{I}\mathcal{M}\Sigma^{\cup}_{<\omega}$ denote the idempotent subvariety of $\mathcal{M}\Sigma^{\cup}_{<\omega}$.

It is known, that

$$\mathcal{IM}\Sigma^{\cup}_{<\omega} = \mathrm{HSP}(\{F_{\mathcal{IM}\Sigma^{\cup}_{<\omega}}(n) \mid n \in \mathbb{N}\}),$$

where $F_{\mathcal{IM}\Sigma^{\cup}_{<\omega}}(n)$ denotes the free $\mathcal{IM}\Sigma^{\cup}_{<\omega}$-algebra on n generators and each free algebra $F_{\mathcal{IM}\Sigma^{\cup}_{<\omega}}(n)$ is the idempotent replica of the free $\mathcal{M}\Sigma^{\cup}_{<\omega}$-algebra $F_{\mathcal{M}\Sigma^{\cup}_{<\omega}}(n)$.

It is known, that

$$\mathcal{IM}\Sigma^{\cup}_{<\omega} = \mathrm{HSP}(\{F_{\mathcal{IM}\Sigma^{\cup}_{<\omega}}(n) \mid n \in \mathbb{N}\}),$$

where $F_{\mathcal{IM}\Sigma^{\cup}_{<\omega}}(n)$ denotes the free $\mathcal{IM}\Sigma^{\cup}_{<\omega}$-algebra on n generators and each free algebra $F_{\mathcal{IM}\Sigma^{\cup}_{<\omega}}(n)$ is the idempotent replica of the free $\mathcal{M}\Sigma^{\cup}_{<\omega}$-algebra $F_{\mathcal{M}\Sigma^{\cup}_{<\omega}}(n)$.

If the variety $\mathcal{M}\Sigma^{\cup}_{<\omega}$ is locally finite then, for each $n \in \mathbb{N}$,

$$F_{\mathcal{M}\Sigma^{\cup}_{<\omega}}(n) \in \mathrm{HSP}_{fin}(\{(P^{<\omega}_{>0}M, \Omega, \cup) \mid (M, \Omega) \in \mathcal{M}\}).$$

Hence, the idempotent replica of $F_{\mathcal{M}\Sigma^{\cup}_{<\omega}}(n)$ belongs to the variety $\rho\mathcal{M}\Sigma^{\cup}_{<\omega}$.

Theorem

Let $\mathcal{M}$ be a variety of Ω-modes such that $\mathcal{M}\Sigma^{\cup}_{<\omega}$ is locally finite. Then

$$\mathcal{IM}\Sigma^{\cup}_{<\omega} = \rho\mathcal{M}\Sigma^{\cup}_{<\omega}.$$

Theorem

Let $\mathcal{M}$ be a variety of Ω-modes such that $\mathcal{M}\Sigma^{\cup}_{<\omega}$ is locally finite. Then

$$\mathcal{I}\mathcal{M}\Sigma^{\cup}_{<\omega} = \rho\mathcal{M}\Sigma^{\cup}_{<\omega}.$$

The following three varieties of modals:

$$\rho\mathcal{M}\Sigma^{\cup}_{<\omega},$$

$$\mathrm{HSP}(\{(MP,\Omega,+) \mid (M,\Omega) \in \mathcal{M}\}), \text{ and}$$

$$\mathrm{HSP}(\{(\mathcal{P}^{<\omega}_{>0}M^{\alpha},\Omega,\cup) \mid (M,\Omega) \in \mathcal{M}\})$$

coincide.

In particular they satisfy the same identities involving the operations of Ω. Hence for varieties

$$\rho\mathcal{M}\Sigma_{<\omega} := \mathrm{HSP}(\{(\mathcal{P}^{<\omega}_{>0} M^{\rho}, \Omega) \mid (M, \Omega) \in \mathcal{M}\}),$$

$$\alpha\mathcal{M}\Sigma_{<\omega} := \mathrm{HSP}(\{(\mathcal{P}^{<\omega}_{>0} M^{\alpha}, \Omega) \mid (M, \Omega) \in \mathcal{M}\})$$

we obtain

$$\mathcal{M}\Sigma = \mathcal{M}\Sigma_{<\omega} \supseteq \rho\mathcal{M}\Sigma_{<\omega} = \alpha\mathcal{M}\Sigma_{<\omega} = \mathcal{M}\mathcal{P}.$$

In particular they satisfy the same identities involving the operations of Ω. Hence for varieties

$$\rho\mathcal{M}\Sigma_{<\omega} := \mathrm{HSP}(\{(\mathcal{P}^{<\omega}_{>0} M^{\rho}, \Omega) \mid (M, \Omega) \in \mathcal{M}\}),$$

$$\alpha\mathcal{M}\Sigma_{<\omega} := \mathrm{HSP}(\{(\mathcal{P}^{<\omega}_{>0} M^{\alpha}, \Omega) \mid (M, \Omega) \in \mathcal{M}\})$$

we obtain

$$\mathcal{M}\Sigma = \mathcal{M}\Sigma_{<\omega} \supseteq \rho\mathcal{M}\Sigma_{<\omega} = \alpha\mathcal{M}\Sigma_{<\omega} = \mathcal{M}\mathcal{P}.$$

Theorem

Let $\mathcal{M}$ be a variety of modes such that the variety $\mathcal{M}\Sigma^{\cup}_{<\omega}$ is locally finite. Then

$$\mathcal{M}\mathcal{S} = \mathcal{M}\mathcal{P}.$$

Let $(A, \Omega, \cup) \in \mathcal{M}\Sigma^{\cup}_{<\omega}$ be an algebra generated by a set $X \subseteq A$. An element $a \in A$ is said to be in **disjunctive form** if it is a sum of a finite number of elements from $\langle X \rangle$, where $\langle X \rangle$ denotes the subalgebra of (A, Ω) generated by X.

Let $(A, \Omega, \cup) \in \mathcal{M}\Sigma^{\cup}_{<\omega}$ be an algebra generated by a set $X \subseteq A$. An element $a \in A$ is said to be in **disjunctive form** if it is a sum of a finite number of elements from $\langle X \rangle$, where $\langle X \rangle$ denotes the subalgebra of (A, Ω) generated by X.

Lemma (Disjunctive Form Lemma)

Let $\mathcal{M}$ be a variety of Ω-modes and $(A, \Omega, \cup) \in \mathcal{M}\Sigma^{\cup}_{<\omega}$ be an algebra generated by a set $X \subseteq A$. For each $a \in A$, there exist $a_1, \ldots, a_p \in \langle X \rangle$ such that $a = a_1 \cup \ldots \cup a_p$.

Theorem

Let $\mathcal{M}$ be a variety of Ω-modes. The variety $\mathcal{M}\Sigma_{<\omega}$ is locally finite if and only if the variety $\mathcal{M}\Sigma^{\cup}_{<\omega}$ is locally finite.

Theorem

Let $\mathcal{M}$ be a variety of Ω-modes. The variety $\mathcal{M}\Sigma_{<\omega}$ is locally finite if and only if the variety $\mathcal{M}\Sigma^{\cup}_{<\omega}$ is locally finite.

Theorem (Main Theorem)

Let $\mathcal{M}$ be a variety of modes such that the variety $\mathcal{M}\Sigma$ is locally finite. The variety $\mathcal{M}\mathcal{S}$ satisfies precisely the consequences of the idempotent and linear identities true in $\mathcal{M}$.

Theorem

Let $\mathcal{M}$ be a variety of Ω-modes. The variety $\mathcal{M}\Sigma_{<\omega}$ is locally finite if and only if the variety $\mathcal{M}\Sigma^{\cup}_{<\omega}$ is locally finite.

Theorem (Main Theorem)

Let $\mathcal{M}$ be a variety of modes such that the variety $\mathcal{M}\Sigma$ is locally finite. The variety $\mathcal{M}\mathcal{S}$ satisfies precisely the consequences of the idempotent and linear identities true in $\mathcal{M}$.

Corollary

Let $\mathcal{M}$ be a variety of modes such that the variety $\mathcal{M}\Sigma$ is locally finite. Then $\mathcal{M} = \mathcal{M}\mathcal{S}$ if and only if $\mathcal{M}$ is defined only by idempotent and linear identities.

Question

Is the result true in more general case, i.e. for an arbitrary variety of modes?

Question

Is the result true in more general case, i.e. for an arbitrary variety of modes?

The assumption of the local finiteness of $\mathcal{M}\Sigma$ is not essential.