Axiomatizable classes of structures

Marina V. Semenova[1] Anna Zamojska-Dzienio[2]

[1]Institute of Mathematics of the Siberian Branch of Russian Academy of Sciences
[2]Faculty of Mathematics and Information Science, Warsaw University of Technology

Novosibirsk, October 2011

For a lattice L, let L^∂ denote the **dual** lattice of L. For an element $a \in L$, we put $\downarrow a = \{x \in L \mid x \leq a\}$.

For a lattice L, let L^∂ denote the **dual** lattice of L. For an element $a \in L$, we put $\downarrow a = \{x \in L \mid x \leq a\}$.

Definition

A pair $\langle X, C \rangle$, where X is a set and $C \colon \mathcal{P}(X) \to \mathcal{P}(X)$ is an operator on X, is a **closure space**, if the following conditions hold for all $A \subseteq B \subseteq X$:

1. $A \subseteq C(A)$;
2. $C^2(A) = C(A)$;
3. $C(A) \subseteq C(B)$.

For a lattice L, let L^∂ denote the **dual** lattice of L. For an element $a \in L$, we put $\downarrow a = \{x \in L \mid x \leq a\}$.

Definition

A pair $\langle X, C\rangle$, where X is a set and $C \colon \mathcal{P}(X) \to \mathcal{P}(X)$ is an operator on X, is a **closure space**, if the following conditions hold for all $A \subseteq B \subseteq X$:

1. $A \subseteq C(A)$;
2. $C^2(A) = C(A)$;
3. $C(A) \subseteq C(B)$.

A set $A \subseteq X$ is **closed**, if $C(A) = A$.

For a lattice L, let L^∂ denote the **dual** lattice of L. For an element $a \in L$, we put $\downarrow a = \{x \in L \mid x \leq a\}$.

Definition

A pair $\langle X, C\rangle$, where X is a set and $C\colon \mathcal{P}(X) \to \mathcal{P}(X)$ is an operator on X, is a **closure space**, if the following conditions hold for all $A \subseteq B \subseteq X$:

1. $A \subseteq C(A)$;
2. $C^2(A) = C(A)$;
3. $C(A) \subseteq C(B)$.

A set $A \subseteq X$ is **closed**, if $C(A) = A$.
The closure space $\langle X, C\rangle$ is **algebraic**, if
$C(A) = \bigcup\{C(F) \mid F \subseteq A \text{ is finite}\}$ for any $A \subseteq X$.

Let $\mathbb{L}(X, C)$ denote the set of all closed subsets of X. Ordered by inclusion, it forms a complete lattice, where

$$\bigwedge_{i \in I} A_i = \bigcap_{i \in I} A_i; \quad \bigvee_{i \in I} A_i = C(\bigcup_{i \in I} A_i)$$

for any set $\{A_i \in \mathbb{L}(X, C) \mid i \in I\}$.

Let $\mathbb{L}(X, C)$ denote the set of all closed subsets of X. Ordered by inclusion, it forms a complete lattice, where

$$\bigwedge_{i \in I} A_i = \bigcap_{i \in I} A_i; \quad \bigvee_{i \in I} A_i = C(\bigcup_{i \in I} A_i)$$

for any set $\{A_i \in \mathbb{L}(X, C) \mid i \in I\}$.

We call lattices of the form $\mathbb{L}(X, C)$ **closure lattices**.

Let $\mathbb{L}(X, C)$ denote the set of all closed subsets of X. Ordered by inclusion, it forms a complete lattice, where

$$\bigwedge_{i \in I} A_i = \bigcap_{i \in I} A_i; \quad \bigvee_{i \in I} A_i = C(\bigcup_{i \in I} A_i)$$

for any set $\{A_i \in \mathbb{L}(X, C) \mid i \in I\}$.

We call lattices of the form $\mathbb{L}(X, C)$ **closure lattices**.

Theorem

A lattice is complete if and only if it is isomorphic to a closure lattice.

Let $\mathbb{L}(X, C)$ denote the set of all closed subsets of X. Ordered by inclusion, it forms a complete lattice, where

$$\bigwedge_{i \in I} A_i = \bigcap_{i \in I} A_i; \quad \bigvee_{i \in I} A_i = C(\bigcup_{i \in I} A_i)$$

for any set $\{A_i \in \mathbb{L}(X, C) \mid i \in I\}$.

We call lattices of the form $\mathbb{L}(X, C)$ **closure lattices**.

Theorem

A lattice is complete if and only if it is isomorphic to a closure lattice.

Theorem

A lattice is algebraic if and only if it is isomorphic to the closure lattice of an algebraic closure space.

Definition

Let L be a complete lattice. A subset $A \subseteq L$ is a **complete meet subsemilattice** of L, if $\bigwedge X \in A$ for any $X \subseteq A$.

Definition

Let L be a complete lattice. A subset $A \subseteq L$ is a **complete meet subsemilattice** of L, if $\bigwedge X \in A$ for any $X \subseteq A$.

A complete meet subsemilattice $A \subseteq L$ is an **algebraic subset** of L, if $\bigvee X \in A$ for any non-empty up-directed subset X of A.

Definition

Let L be a complete lattice. A subset $A \subseteq L$ is a **complete meet subsemilattice** of L, if $\bigwedge X \in A$ for any $X \subseteq A$.

A complete meet subsemilattice $A \subseteq L$ is an **algebraic subset** of L, if $\bigvee X \in A$ for any non-empty up-directed subset X of A.

A binary relation R on a meet semilattice $\langle S, \wedge \rangle$ is **distributive**, if for any $a, b, c \in S$ relation $(c, a \wedge b) \in R$ implies that $c = a' \wedge b'$ for some $a', b' \in S$ such that $(a', a) \in R$ and $(b', b) \in R$.

For a meet semilattice $\langle S, \wedge, 1\rangle$ with unit and for any binary relation $R \subseteq S^2$, let $\mathrm{Sub}(S, R)$ denote the set of all R-**closed subsemilattices** of S;

For a meet semilattice $\langle S, \wedge, 1\rangle$ with unit and for any binary relation $R \subseteq S^2$, let $\mathrm{Sub}(S, R)$ denote the set of all R-**closed subsemilattices** of S; that is, $X \in \mathrm{Sub}(S, R)$ if and only if the following conditions hold:

- $\bigwedge F \in X$ for all finite $F \subseteq X$;
- $b \in X$ and $(a, b) \in R$ imply $a \in X$.

For a meet semilattice $\langle S, \wedge, 1\rangle$ with unit and for any binary relation $R \subseteq S^2$, let $\mathrm{Sub}(S, R)$ denote the set of all R-**closed subsemilattices** of S; that is, $X \in \mathrm{Sub}(S, R)$ if and only if the following conditions hold:

- $\bigwedge F \in X$ for all finite $F \subseteq X$;
- $b \in X$ and $(a, b) \in R$ imply $a \in X$.

For a complete lattice L, let $\mathrm{Sub}_c(L, R)$ denote the set of all **complete** R-**closed meet subsemilattices** of L, while $\mathrm{Sp}(L)$ denotes the set of all algebraic subsets of L.

For a meet semilattice $\langle S, \wedge, 1\rangle$ with unit and for any binary relation $R \subseteq S^2$, let $\mathrm{Sub}(S, R)$ denote the set of all R-**closed subsemilattices** of S; that is, $X \in \mathrm{Sub}(S, R)$ if and only if the following conditions hold:

- $\bigwedge F \in X$ for all finite $F \subseteq X$;
- $b \in X$ and $(a, b) \in R$ imply $a \in X$.

For a complete lattice L, let $\mathrm{Sub}_c(L, R)$ denote the set of all **complete** R-**closed meet subsemilattices** of L, while $\mathrm{Sp}(L)$ denotes the set of all algebraic subsets of L.

We write $\mathrm{Sub}(L)$ instead of $\mathrm{Sub}(L, =)$ and $\mathrm{Sub}_c(L)$ instead of $\mathrm{Sub}_c(L, =)$.

For a meet semilattice $\langle S, \wedge, 1 \rangle$ with unit and for any binary relation $R \subseteq S^2$, let $\mathrm{Sub}(S, R)$ denote the set of all R-**closed subsemilattices** of S; that is, $X \in \mathrm{Sub}(S, R)$ if and only if the following conditions hold:

- $\bigwedge F \in X$ for all finite $F \subseteq X$;
- $b \in X$ and $(a, b) \in R$ imply $a \in X$.

For a complete lattice L, let $\mathrm{Sub}_c(L, R)$ denote the set of all **complete** R-**closed meet subsemilattices** of L, while $\mathrm{Sp}(L)$ denotes the set of all algebraic subsets of L.

We write $\mathrm{Sub}(L)$ instead of $\mathrm{Sub}(L, =)$ and $\mathrm{Sub}_c(L)$ instead of $\mathrm{Sub}_c(L, =)$.

Ordered by inclusion, those sets form complete lattices.

Then lattice meet is just set-theoretic intersection. As for join, the following lemma holds.

Then lattice meet is just set-theoretic intersection. As for join, the following lemma holds.

Lemma

Let L be a meet semilattice and let $R \subseteq L^2$ be a distributive relation. Then the following holds.

1. $A \vee B = \{a \wedge b \mid a \in A,\ b \in B\}$ *for all* $A, B \in \mathrm{Sub}(L, R)$.
2. *If L is a complete lattice, then* $A \vee B = \{a \wedge b \mid a \in A,\ b \in B\}$ *for all* $A, B \in \mathrm{Sub}_c(L, R)$.
3. *If L is an upper continuous complete lattice, then* $A \vee B = \{a \wedge b \mid a \in A,\ b \in B\}$ *for all* $A, B \in \mathrm{Sp}(L)$.

For an arbitrary signature σ, let $\mathbf{K}(\sigma)$ denote the class of all structures of signature σ.

For an arbitrary signature σ, let $\mathbf{K}(\sigma)$ denote the class of all structures of signature σ.

Let also $\mathbf{T}(\sigma)$ denote the variety of σ-structures defined by the identity $\forall xy\ x = y$.

For an arbitrary signature σ, let $\mathbf{K}(\sigma)$ denote the class of all structures of signature σ.

Let also $\mathbf{T}(\sigma)$ denote the variety of σ-structures defined by the identity $\forall xy\ x = y$.

$$\mathbf{V}(\mathbf{K}) = \mathbf{HSP}(\mathbf{K}) = \mathbf{HP}_s\mathbf{S}(\mathbf{K}) = \mathbf{HP}_s(\mathbf{K}),$$

For an arbitrary signature σ, let $\mathbf{K}(\sigma)$ denote the class of all structures of signature σ.

Let also $\mathbf{T}(\sigma)$ denote the variety of σ-structures defined by the identity $\forall xy\ x = y$.

$$\mathbf{V}(\mathbf{K}) = \mathbf{HSP}(\mathbf{K}) = \mathbf{HP}_s\mathbf{S}(\mathbf{K}) = \mathbf{HP}_s(\mathbf{K}),$$

$$\mathbf{Q}(\mathbf{K}) = \mathbf{L}_s\mathbf{P}_s\mathbf{S}(\mathbf{K}) = \mathbf{L}_s\mathbf{P}_s(\mathbf{K}).$$

For an arbitrary signature σ, let $\mathbf{K}(\sigma)$ denote the class of all structures of signature σ.

Let also $\mathbf{T}(\sigma)$ denote the variety of σ-structures defined by the identity $\forall xy\ x = y$.

$$\mathbf{V}(\mathbf{K}) = \mathbf{HSP}(\mathbf{K}) = \mathbf{HP}_s\mathbf{S}(\mathbf{K}) = \mathbf{HP}_s(\mathbf{K}),$$

$$\mathbf{Q}(\mathbf{K}) = \mathbf{L}_s\mathbf{P}_s\mathbf{S}(\mathbf{K}) = \mathbf{L}_s\mathbf{P}_s(\mathbf{K}).$$

A class $\mathbf{K} \subseteq \mathbf{K}(\sigma)$ is a **prevariety**, if $\mathbf{K} = \mathbf{SP}(\mathbf{K}) = \mathbf{P}_s\mathbf{S}(\mathbf{K})$.

For an arbitrary signature σ, let $\mathbf{K}(\sigma)$ denote the class of all structures of signature σ.

Let also $\mathbf{T}(\sigma)$ denote the variety of σ-structures defined by the identity $\forall xy\ x = y$.

$$\mathbf{V}(\mathbf{K}) = \mathbf{HSP}(\mathbf{K}) = \mathbf{HP}_s\mathbf{S}(\mathbf{K}) = \mathbf{HP}_s(\mathbf{K}),$$

$$\mathbf{Q}(\mathbf{K}) = \mathbf{L}_s\mathbf{P}_s\mathbf{S}(\mathbf{K}) = \mathbf{L}_s\mathbf{P}_s(\mathbf{K}).$$

A class $\mathbf{K} \subseteq \mathbf{K}(\sigma)$ is a [**finitary**] **prevariety**, if
$\mathbf{K} = \mathbf{SP}(\mathbf{K}) = \mathbf{P}_s\mathbf{S}(\mathbf{K})$ [$\mathbf{K} = \mathbf{SP}^\omega(\mathbf{K}) = \mathbf{P}_s^\omega\mathbf{S}(\mathbf{K})$, respectively].

For an arbitrary signature σ, let $\mathbf{K}(\sigma)$ denote the class of all structures of signature σ.

Let also $\mathbf{T}(\sigma)$ denote the variety of σ-structures defined by the identity $\forall xy\ x = y$.

$$\mathbf{V}(\mathbf{K}) = \mathbf{HSP}(\mathbf{K}) = \mathbf{HP}_s\mathbf{S}(\mathbf{K}) = \mathbf{HP}_s(\mathbf{K}),$$

$$\mathbf{Q}(\mathbf{K}) = \mathbf{L}_s\mathbf{P}_s\mathbf{S}(\mathbf{K}) = \mathbf{L}_s\mathbf{P}_s(\mathbf{K}).$$

A class $\mathbf{K} \subseteq \mathbf{K}(\sigma)$ is a [**finitary**] **prevariety**, if $\mathbf{K} = \mathbf{SP}(\mathbf{K}) = \mathbf{P}_s\mathbf{S}(\mathbf{K})$ [$\mathbf{K} = \mathbf{SP}^\omega(\mathbf{K}) = \mathbf{P}_s^\omega\mathbf{S}(\mathbf{K})$, respectively].

According to B. Banaschewski and H. Herrlich, a class is a prevariety if and only if it can be defined by infinite implications.

Definition (Gorbunov)

Let $\mathbf{K}' \subseteq \mathbf{K} \subseteq \mathbf{K}(\sigma)$. Then $\mathbf{K}'$ is $\mathbf{K}$- **equational**, if $\mathbf{K}' = \mathbf{K} \cap \mathrm{Mod}(\Sigma)$ for some set Σ of identities of signature σ.

Definition (Gorbunov)

Let $\mathbf{K}' \subseteq \mathbf{K} \subseteq \mathbf{K}(\sigma)$. Then $\mathbf{K}'$ is $\mathbf{K}$-[quasi-]**equational**, if $\mathbf{K}' = \mathbf{K} \cap \mathrm{Mod}(\Sigma)$ for some set Σ of [quasi-] identities of signature σ.

Definition (Gorbunov)

Let $\mathbf{K}' \subseteq \mathbf{K} \subseteq \mathbf{K}(\sigma)$. Then $\mathbf{K}'$ is **K**-[quasi-]**equational**, if $\mathbf{K}' = \mathbf{K} \cap \mathrm{Mod}(\Sigma)$ for some set Σ of [quasi-] identities of signature σ.

Definition

Let $\mathbf{K}' \subseteq \mathbf{K} \subseteq \mathbf{K}(\sigma)$. Then $\mathbf{K}'$ is a **K-prevariety**, if $\mathbf{K}' = \mathbf{K} \cap \mathbf{A}$ for some prevariety $\mathbf{A} \subseteq \mathbf{K}(\sigma)$.

Definition (Gorbunov)

Let $\mathbf{K}' \subseteq \mathbf{K} \subseteq \mathbf{K}(\sigma)$. Then $\mathbf{K}'$ is $\mathbf{K}$-[quasi-]**equational**, if $\mathbf{K}' = \mathbf{K} \cap \mathrm{Mod}(\Sigma)$ for some set Σ of [quasi-] identities of signature σ.

Definition

Let $\mathbf{K}' \subseteq \mathbf{K} \subseteq \mathbf{K}(\sigma)$. Then $\mathbf{K}'$ is a [**finitary**] **$\mathbf{K}$-prevariety**, if $\mathbf{K}' = \mathbf{K} \cap \mathbf{A}$ for some [finitary] prevariety $\mathbf{A} \subseteq \mathbf{K}(\sigma)$.

Definition (Gorbunov)

Let $\mathbf{K}' \subseteq \mathbf{K} \subseteq \mathbf{K}(\sigma)$. Then $\mathbf{K}'$ is **K**-[quasi-]**equational**, if $\mathbf{K}' = \mathbf{K} \cap \mathrm{Mod}(\Sigma)$ for some set Σ of [quasi-] identities of signature σ.

Definition

Let $\mathbf{K}' \subseteq \mathbf{K} \subseteq \mathbf{K}(\sigma)$. Then $\mathbf{K}'$ is a [**finitary**] **K-prevariety**, if $\mathbf{K}' = \mathbf{K} \cap \mathbf{A}$ for some [finitary] prevariety $\mathbf{A} \subseteq \mathbf{K}(\sigma)$.

Equivalently, $\mathbf{K}'$ is a **K**-prevariety if and only if $\mathbf{K}' = \mathbf{K} \cap \mathbf{SP}(\mathbf{K}')$.

Definition (Gorbunov)

Let $\mathbf{K}' \subseteq \mathbf{K} \subseteq \mathbf{K}(\sigma)$. Then $\mathbf{K}'$ is $\mathbf{K}$-[quasi-]**equational**, if $\mathbf{K}' = \mathbf{K} \cap \mathrm{Mod}(\Sigma)$ for some set Σ of [quasi-] identities of signature σ.

Definition

Let $\mathbf{K}' \subseteq \mathbf{K} \subseteq \mathbf{K}(\sigma)$. Then $\mathbf{K}'$ is a [**finitary**] **K-prevariety**, if $\mathbf{K}' = \mathbf{K} \cap \mathbf{A}$ for some [finitary] prevariety $\mathbf{A} \subseteq \mathbf{K}(\sigma)$.

Equivalently, $\mathbf{K}'$ is a [finitary] $\mathbf{K}$-prevariety if and only if $\mathbf{K}' = \mathbf{K} \cap \mathbf{SP}(\mathbf{K}')$ [$\mathbf{K}' = \mathbf{K} \cap \mathbf{SP}^{\omega}(\mathbf{K}')$, respectively].

Let $\mathrm{Lv}(\mathbf{K})$ denote the class of all $\mathbf{K}$-equational subclasses of $\mathbf{K}$, Let $\mathrm{Lv}(\mathbf{K})$ denote the class of all $\mathbf{K}$-equational subclasses of $\mathbf{K}$, while $\mathrm{Lq}(\mathbf{K})$ denotes the class of all $\mathbf{K}$-quasi-equational subclasses of $\mathbf{K}$.

Let Lv(**K**) denote the class of all **K**-equational subclasses of **K**, while Lq(**K**) denotes the class of all **K**-quasi-equational subclasses of **K**.

Let also Lp(**K**) denote the class of all **K**-prevarieties.

Let $\mathrm{Lv}(\mathbf{K})$ denote the class of all $\mathbf{K}$-equational subclasses of $\mathbf{K}$, while $\mathrm{Lq}(\mathbf{K})$ denotes the class of all $\mathbf{K}$-quasi-equational subclasses of $\mathbf{K}$.

Let also $\mathrm{Lp}(\mathbf{K})$ [$\mathrm{Lp}^{\omega}(\mathbf{K})$, respectively] denote the class of all [finitary] $\mathbf{K}$-prevarieties.

Let Lv(**K**) denote the class of all **K**-equational subclasses of **K**, while Lq(**K**) denotes the class of all **K**-quasi-equational subclasses of **K**.

Let also Lp(**K**) [$\mathrm{Lp}^{\omega}(\mathbf{K})$, respectively] denote the class of all [finitary] **K**-prevarieties.

Ordered with respect to inclusion, all the three form complete lattices.

Let $\sigma = \{p_i \mid i \in I\}$ be a signature consisting of unary relation symbols only.

Let $\sigma = \{p_i \mid i \in I\}$ be a signature consisting of unary relation symbols only.

Furthermore, for any set $X \subseteq I$, let $\mathcal{A}_X$ denote a structure from $\mathbf{T}(\sigma)$ such that $\mathcal{A}_X \models \forall x \; p_i(x)$ iff $i \in X$.

Let $\sigma = \{p_i \mid i \in I\}$ be a signature consisting of unary relation symbols only.

Furthermore, for any set $X \subseteq I$, let $\mathcal{A}_X$ denote a structure from $\mathbf{T}(\sigma)$ such that $\mathcal{A}_X \models \forall x \; p_i(x)$ iff $i \in X$.

$\mathbf{T}(\sigma)$ consists of isomorphic copies of structures $\mathcal{A}_X$, $X \subseteq I$.

Lemma

For any signature $\sigma = \{p_i \mid i \in I\}$ containing unary relation symbols only, the following statements hold:

1. *For any sets $X, Y \subseteq I$, $\mathcal{A}_Y \in \mathbf{H}(\mathcal{A}_X)$ if and only if $X \subseteq Y$;*

Lemma

For any signature $\sigma = \{p_i \mid i \in I\}$ *containing unary relation symbols only, the following statements hold:*

1. *For any sets* $X, Y \subseteq I$, $\mathcal{A}_Y \in \mathbf{H}(\mathcal{A}_X)$ *if and only if* $X \subseteq Y$*;*
2. *For any set* $X \subseteq I$, $\mathbf{S}(\mathcal{A}_X) = \{\mathcal{A}_X\}$*;*

Lemma

For any signature $\sigma = \{p_i \mid i \in I\}$ *containing unary relation symbols only, the following statements hold:*

1. *For any sets* $X, Y \subseteq I$, $\mathcal{A}_Y \in \mathbf{H}(\mathcal{A}_X)$ *if and only if* $X \subseteq Y$;
2. *For any set* $X \subseteq I$, $\mathbf{S}(\mathcal{A}_X) = \{\mathcal{A}_X\}$;
3. *If* $X_j \subseteq I$ *for any* $j \in J$, *then* $\prod_{j \in J} \mathcal{A}_{X_j} \cong \mathcal{A}_X$, *where* $X = \bigcap_{j \in J} X_j$.

Let $\langle X, C \rangle$ be a closure space. We put

$$\sigma(X) = \{p_x \mid x \in X\}.$$

Let $\langle X, C\rangle$ be a closure space. We put

$$\sigma(X) = \{p_x \mid x \in X\}.$$

Let $\Sigma(X, C)$ consist of (in general infinite) implications of the form

$$\forall x \bigwedge_{a\in A} p_a(x) \rightarrow p_b(x), \quad A \subseteq X,\ b \in C(A).$$

Let $\langle X, C\rangle$ be a closure space. We put

$$\sigma(X) = \{p_x \mid x \in X\}.$$

Let $\Sigma(X, C)$ consist of (in general infinite) implications of the form

$$\forall x \bigwedge_{a\in A} p_a(x) \to p_b(x), \quad A \subseteq X,\ b \in C(A).$$

If the set X is finite, then the signature $\sigma(X)$ is finite, while $\Sigma(X, C)$ becomes a finite set of quasi-identities.

Let $\langle X, C \rangle$ be a closure space. We put

$$\sigma(X) = \{p_x \mid x \in X\}.$$

Let $\Sigma(X, C)$ consist of (in general infinite) implications of the form

$$\forall x \bigwedge_{a \in A} p_a(x) \to p_b(x), \quad A \subseteq X,\ b \in C(A).$$

If the set X is finite, then the signature $\sigma(X)$ is finite, while $\Sigma(X, C)$ becomes a finite set of quasi-identities.

The class $\mathrm{Mod}\big(\Sigma(X, C)\big)$ is a prevariety.

The class $\mathbf{K}(X, C) = \mathrm{Mod}(\Sigma(X, C)) \cap \mathbf{T}(\sigma(X))$ is also a prevariety.

The class $\mathbf{K}(X, C) = \mathrm{Mod}(\Sigma(X, C)) \cap \mathbf{T}(\sigma(X))$ is also a prevariety.

Lemma

For any closure space $\langle X, C \rangle$, the class $\mathbf{K}(X, C)$ consists of isomorphic copies of structures $\mathcal{A}_B$, where $B \in \mathbb{L}(X, C)$.

Suppose now that $\langle X, C \rangle$ is an algebraic closure space. Let $\Delta(X, C)$ consist of quasi-identities of the form

$$\forall x \bigwedge_{a \in A} p_a(x) \to p_b(x), \quad A \subseteq X \text{ is finite, } b \in C(A).$$

Suppose now that $\langle X, C\rangle$ is an algebraic closure space. Let $\Delta(X, C)$ consist of quasi-identities of the form

$$\forall x \bigwedge_{a\in A} p_a(x) \to p_b(x), \quad A \subseteq X \text{ is finite, } b \in C(A).$$

If the set X is finite, then the set $\Delta(X, C)$ is finite too.

Suppose now that $\langle X, C\rangle$ is an algebraic closure space. Let $\Delta(X, C)$ consist of quasi-identities of the form

$$\forall x \bigwedge_{a\in A} p_a(x) \to p_b(x), \quad A \subseteq X \text{ is finite, } b \in C(A).$$

If the set X is finite, then the set $\Delta(X, C)$ is finite too.

The class $\mathrm{Mod}(\Delta(X, C))$ is a quasivariety.

Suppose now that $\langle X, C \rangle$ is an algebraic closure space. Let $\Delta(X, C)$ consist of quasi-identities of the form

$$\forall x \bigwedge_{a \in A} p_a(x) \to p_b(x), \quad A \subseteq X \text{ is finite, } b \in C(A).$$

If the set X is finite, then the set $\Delta(X, C)$ is finite too.

The class $\mathrm{Mod}(\Delta(X, C))$ is a quasivariety.

Therefore, the class $\mathbf{A}(X, C) = \mathrm{Mod}(\Delta(X, C)) \cap \mathbf{T}(\sigma(X))$ is also a quasivariety.

Suppose now that $\langle X, C\rangle$ is an algebraic closure space. Let $\Delta(X, C)$ consist of quasi-identities of the form

$$\forall x \bigwedge_{a\in A} p_a(x) \to p_b(x), \quad A \subseteq X \text{ is finite, } b \in C(A).$$

If the set X is finite, then the set $\Delta(X, C)$ is finite too.

The class $\mathrm{Mod}(\Delta(X, C))$ is a quasivariety.

Therefore, the class $\mathbf{A}(X, C) = \mathrm{Mod}(\Delta(X, C)) \cap \mathbf{T}(\sigma(X))$ is also a quasivariety.

Lemma

For any algebraic closure space $\langle X, C\rangle$, the class $\mathbf{A}(X, C)$ consists of isomorphic copies of structures $\mathcal{A}_B$, where $B \in \mathbb{L}(X, C)$.

Proposition (Gorbunov)

For any complete lattice L, there are a signature σ consisting only of unary relation symbols and a prevariety $\mathbf{K} \subseteq \mathbf{T}(\sigma)$ such that $L^{\partial} \cong \mathrm{Lv}(\mathbf{K})$ and $\mathrm{Sub}_c(L) \cong \mathrm{Lp}(\mathbf{K}) = \mathrm{Lq}(\mathbf{K})$.

Proposition (Gorbunov)

For any complete lattice L, there are a signature σ consisting only of unary relation symbols and a prevariety $\mathbf{K} \subseteq \mathbf{T}(\sigma)$ such that $L^\partial \cong \mathrm{Lv}(\mathbf{K})$ and $\mathrm{Sub}_c(L) \cong \mathrm{Lp}(\mathbf{K}) = \mathrm{Lq}(\mathbf{K})$.

Sketch of proof: Since the lattice L is complete, there is a closure space $\langle X, C \rangle$ such that $L^\partial \cong \mathbb{L}(X, C)$.

Proposition (Gorbunov)

For any complete lattice L, there are a signature σ consisting only of unary relation symbols and a prevariety $\mathbf{K} \subseteq \mathbf{T}(\sigma)$ such that $L^\partial \cong \mathrm{Lv}(\mathbf{K})$ and $\mathrm{Sub}_c(L) \cong \mathrm{Lp}(\mathbf{K}) = \mathrm{Lq}(\mathbf{K})$.

Sketch of proof: Since the lattice L is complete, there is a closure space $\langle X, C \rangle$ such that $L^\partial \cong \mathbb{L}(X, C)$.

Let $\sigma = \sigma(X)$ and let $\mathbf{K} = \mathbf{K}(X, C)$.

Proposition (Gorbunov)

For any complete lattice L, there are a signature σ consisting only of unary relation symbols and a prevariety $\mathbf{K} \subseteq \mathbf{T}(\sigma)$ such that $L^\partial \cong \mathrm{Lv}(\mathbf{K})$ and $\mathrm{Sub}_c(L) \cong \mathrm{Lp}(\mathbf{K}) = \mathrm{Lq}(\mathbf{K})$.

Sketch of proof: Since the lattice L is complete, there is a closure space $\langle X, C \rangle$ such that $L^\partial \cong \mathbb{L}(X, C)$.

Let $\sigma = \sigma(X)$ and let $\mathbf{K} = \mathbf{K}(X, C)$. Then $\mathbf{K}$ is a prevariety.

Proposition (Gorbunov)

For any complete lattice L, there are a signature σ consisting only of unary relation symbols and a prevariety $\mathbf{K} \subseteq \mathbf{T}(\sigma)$ such that $L^\partial \cong \mathrm{Lv}(\mathbf{K})$ and $\mathrm{Sub}_c(L) \cong \mathrm{Lp}(\mathbf{K}) = \mathrm{Lq}(\mathbf{K})$.

Sketch of proof: Since the lattice L is complete, there is a closure space $\langle X, C \rangle$ such that $L^\partial \cong \mathbb{L}(X, C)$.

Let $\sigma = \sigma(X)$ and let $\mathbf{K} = \mathbf{K}(X, C)$. Then $\mathbf{K}$ is a prevariety.

Let $\psi \colon L \to \mathbb{L}(X, C)$ be an isomorphism. The class $\mathbf{K}$ consists of isomorphic copies of structures $\mathcal{A}_{\psi(a)}$, where $a \in L$.

Define a map $\varphi\colon \mathbb{L}(X, C) \to \mathrm{Lv}(\mathbf{K})$ by the rule

$$\varphi\colon B \mapsto \{\mathcal{A}_F \in \mathbf{T}(\sigma) \mid F \in \mathbb{L}(X, C) \text{ and } B \subseteq F\}, \quad B \in \mathbb{L}(X, C).$$

Define a map $\varphi\colon \mathbb{L}(X, C) \to \mathrm{Lv}(\mathbf{K})$ by the rule

$$\varphi\colon B \mapsto \{\mathcal{A}_F \in \mathbf{T}(\sigma) \mid F \in \mathbb{L}(X, C) \text{ and } B \subseteq F\}, \quad B \in \mathbb{L}(X, C).$$

It is well-defined and establishes a dual isomorphism, whence $L^\partial \cong \mathrm{Lv}(\mathbf{K})$.

Define a map $\varphi\colon \mathbb{L}(X, C) \to \mathrm{Lv}(\mathbf{K})$ by the rule

$$\varphi\colon B \mapsto \{\mathcal{A}_F \in \mathbf{T}(\sigma) \mid F \in \mathbb{L}(X, C) \text{ and } B \subseteq F\}, \quad B \in \mathbb{L}(X, C).$$

It is well-defined and establishes a dual isomorphism, whence $L^\partial \cong \mathrm{Lv}(\mathbf{K})$.

Define a map $\varphi'\colon \mathrm{Sub}_c(L) \to \mathrm{Lp}(\mathbf{K})$ by the rule

$$\varphi'\colon B \mapsto \{\mathcal{A}_{\psi(b)} \in \mathbf{T}(\sigma) \mid b \in B\}, \quad B \in \mathrm{Sub}_c(L).$$

Define a map $\varphi\colon \mathbb{L}(X, C) \to \mathrm{Lv}(\mathbf{K})$ by the rule

$$\varphi\colon B \mapsto \{\mathcal{A}_F \in \mathbf{T}(\sigma) \mid F \in \mathbb{L}(X, C) \text{ and } B \subseteq F\}, \quad B \in \mathbb{L}(X, C).$$

It is well-defined and establishes a dual isomorphism, whence $L^\partial \cong \mathrm{Lv}(\mathbf{K})$.

Define a map $\varphi'\colon \mathrm{Sub}_c(L) \to \mathrm{Lp}(\mathbf{K})$ by the rule

$$\varphi'\colon B \mapsto \{\mathcal{A}_{\psi(b)} \in \mathbf{T}(\sigma) \mid b \in B\}, \quad B \in \mathrm{Sub}_c(L).$$

It is well-defined and is a lattice isomorphism, whence $\mathrm{Sub}_c(L) \cong \mathrm{Lp}(\mathbf{K})$.

Define a map $\varphi\colon \mathbb{L}(X, C) \to \mathrm{Lv}(\mathbf{K})$ by the rule

$$\varphi\colon B \mapsto \{\mathcal{A}_F \in \mathbf{T}(\sigma) \mid F \in \mathbb{L}(X, C) \text{ and } B \subseteq F\}, \quad B \in \mathbb{L}(X, C).$$

It is well-defined and establishes a dual isomorphism, whence $L^\partial \cong \mathrm{Lv}(\mathbf{K})$.

Define a map $\varphi'\colon \mathrm{Sub}_c(L) \to \mathrm{Lp}(\mathbf{K})$ by the rule

$$\varphi'\colon B \mapsto \{\mathcal{A}_{\psi(b)} \in \mathbf{T}(\sigma) \mid b \in B\}, \quad B \in \mathrm{Sub}_c(L).$$

It is well-defined and is a lattice isomorphism, whence $\mathrm{Sub}_c(L) \cong \mathrm{Lp}(\mathbf{K})$.

$\varphi'(B)$ is closed under the operator $\mathbf{L}_s \cap \mathbf{K}$ for any $B \in \mathrm{Sub}_c(L)$, whence $\varphi'(B) \in \mathrm{Lq}(\mathbf{K})$. Therefore, $\mathrm{Lp}(\mathbf{K}) = \mathrm{Lq}(\mathbf{K})$.

Theorem (Gorbunov)

Let $\mathbf{K}$ *be a prevariety and let* $\mathbf{K}' \subseteq \mathbf{K}$ *be l-projectively complete in* $\mathbf{K}$*. Then for any non-empty subclass* $\mathbf{A} \subseteq \mathbf{K}$*,*

$$\mathbf{Q}(\mathbf{A}) \cap \mathbf{K}' = (\mathbf{L}_s \cap \mathbf{K}')(\mathbf{P}_s \cap \mathbf{K}')(\mathbf{S} \cap \mathbf{K}')(\mathbf{A}).$$

In particular, a non-empty subclass $\mathbf{A} \subseteq \mathbf{K}'$ *is* $\mathbf{K}'$*-quasi-equational if and only if* $\mathbf{A}$ *is closed under operators* $\mathbf{L}_s \cap \mathbf{K}'$*,* $\mathbf{S} \cap \mathbf{K}'$*, and* $\mathbf{P}_s \cap \mathbf{K}'$*.*

Theorem (Gorbunov)

Let $\mathbf{K}$ be a prevariety and let $\mathbf{K}' \subseteq \mathbf{K}$ be l-projectively complete in $\mathbf{K}$. Then for any non-empty subclass $\mathbf{A} \subseteq \mathbf{K}$,

$$\mathbf{Q}(\mathbf{A}) \cap \mathbf{K}' = (\mathbf{L}_s \cap \mathbf{K}')(\mathbf{P}_s \cap \mathbf{K}')(\mathbf{S} \cap \mathbf{K}')(\mathbf{A}).$$

In particular, a non-empty subclass $\mathbf{A} \subseteq \mathbf{K}'$ is $\mathbf{K}'$-quasi-equational if and only if $\mathbf{A}$ is closed under operators $\mathbf{L}_s \cap \mathbf{K}'$, $\mathbf{S} \cap \mathbf{K}'$, and $\mathbf{P}_s \cap \mathbf{K}'$.

Corollary

For any complete upper continuous lattice L, there is a signature σ consisting only of unary relation symbols and a prevariety $\mathbf{K} \subseteq \mathbf{T}(\sigma)$ such that $\mathrm{Sp}(L)$ embeds into $\mathrm{Lq}(\mathbf{K})$.

Proposition

For any meet semilattice $\langle S, \wedge, 1\rangle$ with unit, there is a signature σ consisting only of unary relation symbols and a finitary prevariety $\mathbf{K} \subseteq \mathbf{T}(\sigma)$ such that $\mathrm{Sub}(S) \cong \mathrm{Lp}^{\omega}(\mathbf{K})$.

Proposition

For any meet semilattice $\langle S, \wedge, 1\rangle$ with unit, there is a signature σ consisting only of unary relation symbols and a finitary prevariety $\mathbf{K} \subseteq \mathbf{T}(\sigma)$ such that $\mathrm{Sub}(S) \cong \mathrm{Lp}^{\omega}(\mathbf{K})$.

Sketch of proof: Let $\sigma = \{p_a \mid a \in S\}$ consist of unary relation symbols and let the class $\mathbf{K}$ consist of isomorphic copies of structures $\mathcal{A}_{\downarrow b}$, where $b \in S$.

Proposition

For any meet semilattice $\langle S, \wedge, 1\rangle$ with unit, there is a signature σ consisting only of unary relation symbols and a finitary prevariety $\mathbf{K} \subseteq \mathbf{T}(\sigma)$ such that $\mathrm{Sub}(S) \cong \mathrm{Lp}^{\omega}(\mathbf{K})$.

Sketch of proof: Let $\sigma = \{p_a \mid a \in S\}$ consist of unary relation symbols and let the class $\mathbf{K}$ consist of isomorphic copies of structures $\mathcal{A}_{\downarrow b}$, where $b \in S$.
The class $\mathbf{K}$ is a finitary prevariety: for any $n < \omega$ and for any b_0, $\ldots$, $b_{n-1} \in S$, one has $\mathcal{A}_{\downarrow b_0} \times \ldots \times \mathcal{A}_{\downarrow b_{n-1}} \cong \mathcal{A}_{\downarrow b}$, where $b = b_0 \wedge \ldots \wedge b_{n-1}$, while $\mathbf{S}(\mathcal{A}_{\downarrow a}) = \{\mathcal{A}_{\downarrow a}\}$ for any $a \in S$.

Proposition

For any meet semilattice $\langle S, \wedge, 1\rangle$ with unit, there is a signature σ consisting only of unary relation symbols and a finitary prevariety $\mathbf{K} \subseteq \mathbf{T}(\sigma)$ such that $\mathrm{Sub}(S) \cong \mathrm{Lp}^{\omega}(\mathbf{K})$.

Sketch of proof: Let $\sigma = \{p_a \mid a \in S\}$ consist of unary relation symbols and let the class $\mathbf{K}$ consist of isomorphic copies of structures $\mathcal{A}_{\downarrow b}$, where $b \in S$.
The class $\mathbf{K}$ is a finitary prevariety: for any $n < \omega$ and for any b_0, $\ldots$, $b_{n-1} \in S$, one has $\mathcal{A}_{\downarrow b_0} \times \ldots \times \mathcal{A}_{\downarrow b_{n-1}} \cong \mathcal{A}_{\downarrow b}$, where $b = b_0 \wedge \ldots \wedge b_{n-1}$, while $\mathbf{S}(\mathcal{A}_{\downarrow a}) = \{\mathcal{A}_{\downarrow a}\}$ for any $a \in S$.

Define a map $\varphi\colon \mathrm{Sub}(S) \to \mathrm{Lp}^{\omega}(\mathbf{K})$ by the rule

$$\varphi\colon B \mapsto \{\mathcal{A}_{\downarrow b} \in \mathbf{T}(\sigma) \mid b \in B\}, \quad B \in \mathrm{Sub}(S).$$

Proposition

For any meet semilattice $\langle S, \wedge, 1\rangle$ with unit, there is a signature σ consisting only of unary relation symbols and a finitary prevariety $\mathbf{K} \subseteq \mathbf{T}(\sigma)$ such that $\mathrm{Sub}(S) \cong \mathrm{Lp}^{\omega}(\mathbf{K})$.

Sketch of proof: Let $\sigma = \{p_a \mid a \in S\}$ consist of unary relation symbols and let the class $\mathbf{K}$ consist of isomorphic copies of structures $\mathcal{A}_{\downarrow b}$, where $b \in S$.

The class $\mathbf{K}$ is a finitary prevariety: for any $n < \omega$ and for any $b_0, \ldots, b_{n-1} \in S$, one has $\mathcal{A}_{\downarrow b_0} \times \ldots \times \mathcal{A}_{\downarrow b_{n-1}} \cong \mathcal{A}_{\downarrow b}$, where $b = b_0 \wedge \ldots \wedge b_{n-1}$, while $\mathbf{S}(\mathcal{A}_{\downarrow a}) = \{\mathcal{A}_{\downarrow a}\}$ for any $a \in S$.

Define a map $\varphi\colon \mathrm{Sub}(S) \to \mathrm{Lp}^{\omega}(\mathbf{K})$ by the rule

$$\varphi\colon B \mapsto \{\mathcal{A}_{\downarrow b} \in \mathbf{T}(\sigma) \mid b \in B\}, \quad B \in \mathrm{Sub}(S).$$

It is a lattice isomorphism.

Proposition (Gorbunov, Tumanov)

For any complete dually algebraic lattice L, there are a signature σ consisting only of unary relation symbols and a quasivariety $\mathbf{K} \subseteq \mathbf{T}(\sigma)$ such that $L \cong \mathrm{Lv}(\mathbf{K})$.

Proposition (Gorbunov, Tumanov)

For any complete dually algebraic lattice L, there are a signature σ consisting only of unary relation symbols and a quasivariety $\mathbf{K} \subseteq \mathbf{T}(\sigma)$ such that $L \cong \mathrm{Lv}(\mathbf{K})$.

Sketch of proof: Since the *dual* lattice L^{∂} is algebraic, there is an algebraic closure space $\langle X, C \rangle$ such that $L^{\partial} \cong \mathbb{L}(X, C)$.

Proposition (Gorbunov, Tumanov)

For any complete dually algebraic lattice L, there are a signature σ consisting only of unary relation symbols and a quasivariety $\mathbf{K} \subseteq \mathbf{T}(\sigma)$ such that $L \cong \mathrm{Lv}(\mathbf{K})$.

Sketch of proof: Since the *dual* lattice L^∂ is algebraic, there is an algebraic closure space $\langle X, C \rangle$ such that $L^\partial \cong \mathbb{L}(X, C)$.

Let $\sigma = \sigma(X)$ and let $\mathbf{K} = \mathbf{A}(X, C)$. Then $\mathbf{K}$ is a quasivariety.

Proposition (Gorbunov, Tumanov)

For any complete dually algebraic lattice L, there are a signature σ consisting only of unary relation symbols and a quasivariety $\mathbf{K} \subseteq \mathbf{T}(\sigma)$ such that $L \cong \mathrm{Lv}(\mathbf{K})$.

Sketch of proof: Since the *dual* lattice L^∂ is algebraic, there is an algebraic closure space $\langle X, C \rangle$ such that $L^\partial \cong \mathbb{L}(X, C)$.

Let $\sigma = \sigma(X)$ and let $\mathbf{K} = \mathbf{A}(X, C)$. Then $\mathbf{K}$ is a quasivariety.

We define a map $\varphi \colon \mathbb{L}(X, C) \to \mathrm{Lv}(\mathbf{K})$ by the rule

$$\varphi \colon B \mapsto \{\mathcal{A}_F \in \mathbf{T}(\sigma) \mid F \in \mathbb{L}(X, C) \text{ and } B \subseteq F\}, \quad B \in \mathbb{L}(X, C).$$

Then φ establishes a dual isomorphism between $\mathbb{L}(X, C)$ and $\mathrm{Lv}(\mathbf{K})$.

It was shown by V. A. Gorbunov that for any quasivariety $\mathbf{K}$, the lattice $\mathrm{Lv}(\mathbf{K})$ is complete and dually algebraic.

It was shown by V. A. Gorbunov that for any quasivariety $\mathbf{K}$, the lattice $\mathrm{Lv}(\mathbf{K})$ is complete and dually algebraic.

Corollary

The class of complete dually algebraic lattices coincides with the class of lattices of relative equational classes of quasivarieties.

Definition (Pal'chunov)

Let $\mathbf{K}$ be a class of structures of signature σ and let Δ be a set of first-order sentences of the same signature. A class $\mathbf{K}'$ is **axiomatizable in K relatively to** Δ, if $\mathbf{K}' = \mathbf{K} \cap \mathrm{Mod}(\Sigma)$ for some set $\Sigma \subseteq \Delta$.

Definition (Pal'chunov)

Let $\mathbf{K}$ be a class of structures of signature σ and let Δ be a set of first-order sentences of the same signature. A class $\mathbf{K}'$ is **axiomatizable in K relatively to** Δ, if $\mathbf{K}' = \mathbf{K} \cap \mathrm{Mod}(\Sigma)$ for some set $\Sigma \subseteq \Delta$.

A class $\mathbf{K} \subseteq \mathbf{K}(\sigma)$ is axiomatizable if and only if it is axiomatizable in $\mathbf{K}(\sigma)$ relatively to the set of all first-order sentences.

Definition (Pal'chunov)

Let $\mathbf{K}$ be a class of structures of signature σ and let Δ be a set of first-order sentences of the same signature. A class $\mathbf{K}'$ is **axiomatizable in K relatively to** Δ, if $\mathbf{K}' = \mathbf{K} \cap \mathrm{Mod}(\Sigma)$ for some set $\Sigma \subseteq \Delta$.

A class $\mathbf{K} \subseteq \mathbf{K}(\sigma)$ is axiomatizable if and only if it is axiomatizable in $\mathbf{K}(\sigma)$ relatively to the set of all first-order sentences.

Furthermore, for any set Δ of sentences and any class $\mathbf{K} \subseteq \mathbf{K}(\sigma)$, the set of all axiomatizable in $\mathbf{K}$ classes relatively to Δ forms a complete lattice $\mathbb{A}(\mathbf{K}, \Delta)$.

Corollary

For any complete lattice L, there are a signature σ, a prevariety $\mathbf{K} \subseteq \mathbf{K}(\sigma)$, and a set Δ of first-order sentences of the same signature such that $L \cong \mathbb{A}(\mathbf{K}, \Delta)$.

Corollary

For any complete lattice L, there are a signature σ*, a prevariety* $\mathbf{K} \subseteq \mathbf{K}(\sigma)$*, and a set* Δ *of first-order sentences of the same signature such that* $L \cong \mathbb{A}(\mathbf{K}, \Delta)$*.*

Proof.

Take σ and $\mathbf{K}$ as in the proof of Proposition and take the set of all identities of signature σ as Δ. □

Corollary

For any complete lattice L, there are a signature σ, a prevariety $\mathbf{K} \subseteq \mathbf{K}(\sigma)$, and a set Δ of first-order sentences of the same signature such that $L \cong \mathbb{A}(\mathbf{K}, \Delta)$.

Proof.

Take σ and $\mathbf{K}$ as in the proof of Proposition and take the set of all identities of signature σ as Δ. □

Corollary was proved by D. E. Pal'chunov only for at most countable complete lattices L. This has lead him to ask whether any complete lattice is isomorphic to a lattice of relatively axiomatizable classes, cf. Problem 1 in [*D. E. Pal'chunov*, Lattices of relatively axiomatizable classes, Lecture Notes in Artificial Intelligence, **4390** (2007), 221–239.]

Corollary

The class of complete dually algebraic lattices coincides with the class of lattices of the form $\mathbb{A}(\mathbf{K}, \Delta)$, *where* $\mathbf{K}$ *is a quasivariety and* Δ *is a set of first-order sentences.*

Corollary

The class of complete dually algebraic lattices coincides with the class of lattices of the form $\mathbb{A}(\mathbf{K}, \Delta)$, *where* $\mathbf{K}$ *is a quasivariety and* Δ *is a set of first-order sentences.*

Corollary

For any finite lattice L, *there are a finite signature* σ *and a set* Δ *of first-order sentences of* σ *such that* $L \cong \mathbb{A}(\mathbf{K}(\sigma), \Delta)$.

Corollary

The class of complete dually algebraic lattices coincides with the class of lattices of the form $\mathbb{A}(\mathbf{K}, \Delta)$*, where* $\mathbf{K}$ *is a quasivariety and* Δ *is a set of first-order sentences.*

Corollary

For any finite lattice L*, there are a finite signature* σ *and a set* Δ *of first-order sentences of* σ *such that* $L \cong \mathbb{A}(\mathbf{K}(\sigma), \Delta)$*.*

The latter Corollary was proved by D. E. Pal'chunov.

Proposition (Gorbunov, Tumanov)

For any complete algebraic lattice L, there is a signature σ consisting only of unary relation symbols and a quasivariety $\mathbf{K} \subseteq \mathbf{T}(\sigma)$ such that $\mathrm{Sp}(L) \cong \mathrm{Lq}(\mathbf{K})$ and $\mathrm{Sub}_c(L) \cong \mathrm{Lp}(\mathbf{K})$.

Proposition (Gorbunov, Tumanov)

For any complete algebraic lattice L, there is a signature σ consisting only of unary relation symbols and a quasivariety $\mathbf{K} \subseteq \mathbf{T}(\sigma)$ such that $\mathrm{Sp}(L) \cong \mathrm{Lq}(\mathbf{K})$ and $\mathrm{Sub}_c(L) \cong \mathrm{Lp}(\mathbf{K})$.

Sketch of proof: There is an algebraic closure space $\langle X, C \rangle$ such that $L \cong \mathbb{L}(X, C)$; let $\psi \colon L \to \mathbb{L}(X, C)$ be an isomorphism. Let $\sigma = \sigma(X)$ and let $\mathbf{K} = \mathbf{A}(X, C)$. Then $\mathbf{K}$ is a quasivariety. The class $\mathbf{K}$ consists of isomorphic copies of structures $\mathcal{A}_{\psi(a)}$, where $a \in L$.

Proposition (Gorbunov, Tumanov)

For any complete algebraic lattice L, there is a signature σ consisting only of unary relation symbols and a quasivariety $\mathbf{K} \subseteq \mathbf{T}(\sigma)$ such that $\mathrm{Sp}(L) \cong \mathrm{Lq}(\mathbf{K})$ and $\mathrm{Sub}_c(L) \cong \mathrm{Lp}(\mathbf{K})$.

Sketch of proof: There is an algebraic closure space $\langle X, C\rangle$ such that $L \cong \mathbb{L}(X, C)$; let $\psi\colon L \to \mathbb{L}(X, C)$ be an isomorphism. Let $\sigma = \sigma(X)$ and let $\mathbf{K} = \mathbf{A}(X, C)$. Then $\mathbf{K}$ is a quasivariety. The class $\mathbf{K}$ consists of isomorphic copies of structures $\mathcal{A}_{\psi(a)}$, where $a \in L$.

We define a map $\varphi\colon \mathrm{Sub}_c(L) \to \mathrm{Lp}(\mathbf{K})$ by the rule

$$\varphi\colon B \mapsto \{\mathcal{A}_{\psi(b)} \in \mathbf{T}(\sigma) \mid b \in B\}, \quad B \in \mathrm{Sp}(L).$$

Then φ is well-defined and it is a lattice isomorphism, whence $\mathrm{Sub}_c(L) \cong \mathrm{Lp}(\mathbf{K})$. Moreover, the restriction of φ on $\mathrm{Sp}(L)$ defines an isomorphism from $\mathrm{Sp}(L)$ onto $\mathrm{Lq}(\mathbf{K})$.

It is known that quasivariety lattices are completely join-semidistributive and dually algebraic (V. A. Gorbunov). In contrast, lattices of the form $\mathrm{Lq}(\mathbf{K})$ and $\mathrm{Lp}(\mathbf{K})$, where $\mathbf{K}$ is a prevariety, are neither join-semidistributive nor even lower continuous.

It is known that quasivariety lattices are completely join-semidistributive and dually algebraic (V. A. Gorbunov). In contrast, lattices of the form $\mathrm{Lq}(\mathbf{K})$ and $\mathrm{Lp}(\mathbf{K})$, where $\mathbf{K}$ is a prevariety, are neither join-semidistributive nor even lower continuous.

Corollary

There are prevarieties $\mathbf{K}$ *such that neither* $\mathrm{Lq}(\mathbf{K})$ *nor* $\mathrm{Lp}(\mathbf{K})$ *embed into a quasivariety lattice.*