Scrolls and hyperbolicity

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Content

1. Preliminaries : Kobayashi metric

Every complex space X possesses a unique *Kobayashi pseudometric* k_X satisfying the following axioms.

(i) In the unit disc $X = \Delta$, k_X coincides with the Poincaré metric ;

(ii) every holomorphic map $\varphi : \Delta \to X$ is a contraction :

$$\varphi^*(k_X) \le k_\Delta \,;$$

(iii) k_X is maximal among the pseudometrics on X satisfying (i) and (ii).

Remark : Every holomorphic map $\varphi : X \to Y$ is a contraction :

$$\varphi^*(k_Y) \le k_X \,.$$

Kobayashi hyperbolicity

Definition

X is called *Kobayashi hyperbolic* (or simply *hyperbolic*) if k_X is a metric i.e.

$$k_X(p, q) = 0 \Longleftrightarrow p = q.$$

Examples

$$k_{\mathbb{C}^n} \equiv 0, \quad k_{\mathbb{P}^n} \equiv 0, \quad k_{\mathbb{T}^n} \equiv 0\,,$$

where $\mathbb{T}^n = \mathbb{C}^n/\Lambda$ is a complex torus.

Schottky-Landau Theorem : $\mathbb{C} \setminus \{0, 1\}$ *is hyperbolic.*

Brody-Kiernan-Kobayashi-Kwack Theorem

If X is compact then the following conditions are equivalent.

- X is Kobayashi hyperbolic;
- *Little Picard Theorem holds for X i.e.*

$$\forall f : \mathbb{C} \to X, \quad f = \mathrm{cst};$$

- *Big Picard Theorem holds for X i.e.*

$$\forall f : \Delta \setminus \{0\} \to X \ \ \exists \bar{f} : \Delta \to X,$$

$$\bar{f}|(\Delta \setminus \{0\}) = f;$$

- *Montel Theorem holds for X i.e.* *the topological espace $\mathrm{HOL}(\Delta, X)$ is compact.*
- *For any complex space Y, the space $\mathrm{HOL}(Y, X)$ is compact.*

Brody Theorems

Definition

Let M be a hermitian compact complex variety. A *Brody curve* in M is an entire curve $\varphi : \mathbb{C} \to M$ satisfying

$$||\varphi'(z)|| \leq 1 = ||\varphi'(0)|| \quad \forall z \in \mathbb{C}\,.$$

Brody Theorem

M is hyperbolic if and only if it admits no Brody curve.

Brody Stability Theorem

Let X be a compact subspace of a complex space Z. If X is hyperbolic then any compact subspace $X' \subseteq Z$ sufficiently close to X is hyperbolic as well.

Classical Conjectures

Kobayashi Conjecture ′70

A very general hypersurface in $\mathbb{P}^n$ of sufficiently high degree (of degree $d \geq 2n-1$) is Kobayashi hyperbolic.

Green-Griffiths-Lang Conjecture ′80

All entire curves in a projective variety of general type are contained in a (common) proper subvariety.

Definition

A projective variety X is said to be of *general type* if for $m \gg 1$ the pluricanonical linear system $|mK_X|$ defines a birational embedding $\varphi_{|mK_X|} : X \dashrightarrow \mathbb{P}^n$.

Bloch Conjecture

Theorem

Let X be a projective variety. If X is irregular i.e., $q_1(X) = h^{1,0}(X) > \dim(X)$, *then any entire curve in X is contained in a proper subvariety, which* à priori *depends on the curve.*

Theorem (Bogomolov ′78, McQuillen ′98)

Let S be a projective surface of general type with $c_1^2(S) > c_2(S)$. *Then the set of all rational and elliptic curves in S is finite. Furthermore, any entire curve in S is contained in one of these curves.*

Projective hypersurfaces

Theorem (McQuillen, Demailly-El Goul's ′98, Paun ′08)

A very generic surface in $\mathbb{P}^3$ of degree $d \geq 18$ is Kobayashi hyperbolic.

A remarkable progress in higher dimensions is due to J.-P. Demailly, Y.-T. Siu, S. Diverio, J. Merker, E. Rousseau, S. Trapani e.a.

Theorem (Diverio-Merker-Rousseau, Diverio-Trapani ′10)

Let X be a very general hypersurface in $\mathbb{P}^{n+1}$ of degree $d \geq 2^{n^5}$. Then there exists a subvariety Y in X of codimension at least 2 which contains the image of any entire curve $\mathbb{C} \to X$.

This confirms the Green-Griffiths Conjecture in the setting of the Kobayashi Conjecture.

Corollary (Diverio-Trapani '10)
A very generic hypersurface in $\mathbb{P}^4$ of degree $d \geq 593$ is Kobayashi hyperbolic.

Theorem (Demailly '10)

Any entire curve in a variety of general type satisfies (a large number of) algebraic differential equations.

Examples of hyperbolic surfaces in $\mathbb{P}^3$

Such examples were constructed by

Brody-Green '77, Nadel '89, Masuda-Noguchi '96, Khoai '96 ;

El Goul '96	$\forall d \geq 14$,
Siu-Yeung '96, Demailly-El Goul '97	$\forall d \geq 11$;
Duval '99, Shirosaki-Fujimoto '00	$\forall d = 2k \geq 8$;
Shiffman-Z' '03, '05, Z' '07	$\forall d \geq 8$;
Duval '05	$d = 6$;
Ciliberto-Z' '11	$\forall d \geq 6$.

Examples

Example (Duval '99, Shirosaki-Fujimoto '00)
The surface in $\mathbb{P}^3$ with equation

$$Q(X_0, X_1, X_2)^2 - P(X_2, X_3) = 0\,,$$

where Q, P are generic homogeneous forms of degree 4 resp. 8, is hyperbolic.

Deformation method

Let $\pi : V \to \Delta$ be a family of compact varieties over the unit disc. Assume that V is smooth and the generic fiber $V_c = \pi^{-1}(c)$ $(c \in \Delta)$ is non-hyperbolic and so contains a Brody curve. By Brody's Stability Theorem, every special fiber contains a limiting Brody curve and so is not hyperbolic either.

The classical Hurwitz Theorem imposes restrictions on the position of a limiting Brody curve w.r.t. reducible singularities of the special fiber V_0.

Proposition

Let $\mathrm{br}(V_0)$ *be the set of double points of* V_0 *and* $\overline{\mathrm{br}(V_0)}$ *its Zariski closure. Consider a sequence of Brody curves* $f_n : \mathbb{C} \to V_{c_n}$ *which converges to a limiting Brody curve* $f : \mathbb{C} \to V_0$*. By Hurwitz Theorem, there is an alternative :*

$$\text{either} \quad f(\mathbb{C}) \cap \mathrm{br}(V_0) = \emptyset \quad \text{or} \quad f(\mathbb{C}) \subseteq \overline{\mathrm{br}(V_0)}.$$

Therefore

if $\overline{\mathrm{br}(V_0)}$ *and* $V_0 \setminus \mathrm{br}(V_0)$ *are both hyperbolic then every fiber* V_c $(c \neq 0)$ *sufficiently close to* V_0 *is hyperbolic too.*

This can be applied to the pencil

$$\{X_t\}_{t \in \mathbb{P}^1} = \langle X_0, X_\infty \rangle$$

generated by hypersurfaces X_0 and X_∞ in $\mathbb{P}^n$ of the same degree.

Attention : we don't have a good control over the base points $X_0 \cap X_\infty$ of the pencil.

A criterion of hyperbolicity

Proposition

Suppose that $\overline{\mathrm{br}(V_0)}$ *is hyperbolic, and there is a* $\mathbb{P}^1$*-fibration* $\pi : V_0^{\mathrm{norm}} \to E$ *of the normalisation of* V_0 *to a hyperbolic variety* E *such that every fiber meets the inverse image of* $\mathrm{br}(V_0)$ *in at least 3 distinct points. Then* $V_0 \setminus \mathrm{br}(V_0)$ *is hyperbolic, and so every fiber* V_c $(c \neq 0)$ *sufficiently close to* V_0 *is also hyperbolic.*

Examples

Example (Shiffman-Z. ′03) There is an abelian surface immersed in $\mathbb{P}^3$ with a non-normal singular image X of degree 8. *Generic small deformations of X are hyperbolic.*

Example (Shiffman-Z. ′05) Let X_0 be the union of two cones in general position in $\mathbb{P}^3$ over smooth plane quartics $C', C'' \subseteq \mathbb{P}^2$. Then *generic small deformations of X_0 are hyperbolic.*

Example (Z. ′07) Let $C \subseteq \mathbb{P}^2$ be a hyperbolic curve of degree $d \geq 4$, and let $X_0 \subseteq \mathbb{P}^3$ be a cone over C. Then *generic small deformations of the double cone $2X_0$ are hyperbolic surfaces of degree $2d \geq 8$.*

Example (Duval ′04) *There exists a hyperbolic sextic $X = X_\varepsilon \subseteq \mathbb{P}^3$.* Its construction involves a five step successive deformation i.e., X_ε varies in a family depending on five parameters.

Scrolls

Let E be a projective variety, and $V \to E$ be a vector bundle of rank 2 over E. The projectivization $S = \mathbb{P}_E(V) \to E$ is a $\mathbb{P}^1$-bundle over E. Consider a birational morphism $\varphi : S \to \mathbb{P}^n$ such that the image of every fiber is a projective line in $\mathbb{P}^n$. Then $\Sigma = \varphi(S)$ is called a *scroll*.
Let $G(1, n)$ be the Grassmannian of lines in $\mathbb{P}^n$. There is a natural morphism

$$\rho : E \to G(1, n) \quad \text{such that} \quad \deg \Sigma = \deg \rho(E).$$

If Σ is smooth then a generic hyperplane section H of Σ is smooth and isomorphic to E.

The setting

Let $S \subseteq \mathbb{P}^{n+k}$ be a smooth scroll of dimension $n-1$, and let Σ be a generic projection of S to $\mathbb{P}^n$. Then Σ is a hypersurface of $\mathbb{P}^n$ with normalization S, and Σ has only ordinary singularities.

For instance, if $n = 3$ then Σ is a ruled surface in $\mathbb{P}^3$, i.e. Σ is covered by lines, and E is a smooth curve of genus g. We say in this case that Σ is a *scroll of genus* g. Such a scroll Σ has at worst singularities along an irreducible curve Δ_Σ, and a generic point of Δ_Σ is a double point of Σ. Besides, Σ can have some number t of triple points, which are at the same time triple points of Δ_Σ, and some number p of pinch points with local equation $x^2 - y^2 z = 0$.

Nonspecial scrolls in $\mathbb{P}^3$

Theorem (Arrondo, Pereira, Sols '89, Calabri, Ciliberto, Flamini, Miranda '06)

There exists a scroll $\Sigma \subseteq \mathbb{P}^3$ *of genus* g *and degree* d *with ordinary singularities if*

- $g \geq 2$ *and* $d \geq 2g+2$, *or*
- $g = 1$ *and* $d \geq 5$, *or*
- $g = 0$ *and* $d \geq 4$.

Scrolls that we use

We use the following scrolls with ordinary singularities and an irreducible double curve :

- an elliptic quintic scroll ;
- a sextic scroll of genus 2 ;
- a septic scroll of genus 2.

Proposition

Let $\Sigma \subseteq \mathbb{P}^n$ be a hypersurface of degree d, which is a scroll with ordinary singularities. Suppose that :

(i) *the base E of Σ and the double locus Δ_Σ are both hyperbolic;*

(ii) *for a generic hypersurface $X \subseteq \mathbb{P}^n$ of degree d, every ruling F of Σ meets $\mathrm{br}(\Sigma)$ in at least 3 points off X.*

Consider the pencil $(X_t)_{t\in\mathbb{P}^1}$ generated by $X = X_\infty$ and $\Sigma = X_0$. Then the members X_t sufficiently close to Σ are hyperbolic.

Theorem (Ciliberto-Shiffman-Z. '05, '10)

For any $d \geq 6$ there exists a hyperbolic surface $S \subseteq \mathbb{P}^3$ of degree d, and for any $d \geq 12$ there exists a hyperbolic 3-fold $T \subseteq \mathbb{P}^4$ of degree d.

Definition

Let $X \subseteq \mathbb{P}^n$ be a projective variety. We say that X is *algebraically hyperbolic* if any morphism $A \to X$ from an abelian variety A is constant. We say that X is *algebraically hyperbolic in Demailly sense*, or *Demailly algebraically hyperbolic*, if for a positive real $\varepsilon = \varepsilon(X)$ and for any algebraic curve $C \subseteq X$,

$$2\mathrm{genus}(C) - 2 \geq \varepsilon \deg(C).$$

In particular, if the genus of C is bounded above then also the degree is.

Algebraic hyperbolicity versus hyperbolicity

If X is Demailly algebraically hyperbolic then it is also algebraically hyperbolic.

Both properties are open in the countable Zariski topology.

Kobayashi hyperbolicity implies Demailly algebraic hyperbolicity.

Hence, if there exists a hyperbolic hypersurface of degree d in $\mathbb{P}^n$ then a very generic hypersurface of degree d is Demailly algebraically hyperbolic.

Demailly algebraically hyperbolic hypersurfaces

Theorem (Xu '94-'96, Voisin '96-'99, Clemens-Ran '05)

A very generic surface of degree $d \geq 5$ in $\mathbb{P}^3$ and a very generic 3-fold of degree $d \geq 6$ in $\mathbb{P}^4$ are algebraically hyperbolic.

Are they also Demailly algebraically hyperbolic ?
No example of a hyperbolic quintique surface in $\mathbb{P}^3$ or a hyperbolic sextic 3-fold in $\mathbb{P}^4$ is known.

Corollary

A very generic hypersurface of degree ≥ 6 *in* $\mathbb{P}^3$ *or of degree* ≥ 12 *in* $\mathbb{P}^4$ *is Demailly algebraically hyperbolic.*

The proof exploites unions of cones of degree ≥ 4 in $\mathbb{P}^3$ (of degree ≥ 6 in $\mathbb{P}^4$, respectively) and also sextic and septic scrolls of genus 2 in $\mathbb{P}^3$.

Pasienca's estimates

For generic hypersurfaces of sufficiently high degree, the inequality of Demailly algebraic hyperbolicity holds even in a stronger form.

Theorem (Pacienza '04)

Let $X \subseteq \mathbb{P}^n$ be a very generic hypersurface of degree d. Then for any algebraic curve C in X we have the inequality

$$2g(C) - 2 \geq \deg(C)$$

provided one of the following conditions is satisfied :

- $n = 3$ *and* $d \geq 6$,
- $n = 4$ *and* $d \geq 7$,
- $n = 5$ *and* $d \geq 9$,
- $n \geq 6$ *and* $d \geq 2n - 2$.

The technique of proof is borrowed in the work of Claire Voisin.