FLEXIBLE VARIETIES

Mikhail Zaidenberg
(Institut Fourier, Grenoble, France)
After
I. Arzhantsev, F. Bogomolov, H. Flenner, S. Kaliman, I. Karzhemanov, T. Kishimoto, F. Kutzschebauch, K. Kuyumzhiyan, A. Perepechko, Yu. Prokhorov, and the speaker

Nezavisimyj Universitet
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X - an algebraic variety $/\mathbb{k}$, $\dim X \geq 2$, $\mathbb{k} = \bar{\mathbb{k}}$, $\mathrm{char}(\mathbb{k}) = 0$

Definitions

• $x \in X_{\mathrm{reg}}$ is *FLEXIBLE* if $T_x X$ is spanned by the tangent vectors to the orbits $H.x$ of one-parameter unipotent subgroups $H \subseteq \mathrm{Aut}(X)$

• X is *FLEXIBLE* if every smooth point $x \in X_{\mathrm{reg}}$ is

• $\mathrm{SAut}(X)$ - *THE SPECIAL AUTOMORPHISM GROUP* - the subgroup of $\mathrm{Aut}(X)$ generated by all one-parameter unipotent subgroups

First examples

• $\mathrm{SAut}(\mathbb{A}^1) = \mathrm{Transl}(\mathbb{A}^1) \subseteq \mathrm{Aff}(\mathbb{A}^1)$ - an algebraic group
- it acts transitively, not 2-transitively on $\mathbb{A}^1$

• $\mathrm{SAut}(\mathbb{A}^n)$, $n \geq 2$, - a non-algebraic group
- it acts ∞-*TRANSITIVELY* on $\mathbb{A}^n$
i.e. m-transitively $\forall m \geq 1$

• $\mathrm{SAut}(\mathbb{A}^2)$ contains the shears

$$(x, y) \mapsto (x, y + P(x)), \quad P \in \Bbbk[x]$$

THEOREM

$\forall X$ affine, $\dim X \geq 2$*, the following are equivalent :*

(i) *$SAut(X)$ is transitive on X_{reg}*

(ii) *$SAut(X)$ is ∞-transitive on X_{reg}*

(iii) *X is flexible*

Remark

- An algebraic group G cannot act ∞-transitively on X
- G cannot act 3-transitively on affine X (Borel - Knop)

2. Examples of flexibility
2.1. Homogeneous varieties

Notation : $G^\vee$ - the group of characters of G

Theorem

G/H is flexible if $G^\vee = \{1\}$, e.g. $\forall\, G$ semisimple

Corollary

G/H affine, $\dim G/H \geq 2 \Rightarrow \mathrm{SAut}(G/H)$ is ∞-transitive

Theorem

G/P - flag variety, $G/P \hookrightarrow \mathbb{P}^n$ - equivariant embedding
$X = \mathrm{AffCone}(G/P) \Rightarrow X$ is flexible

Theorem

X - affine, toric, non-degenerate
$\Rightarrow X$ is flexible

Theorem

G - semisimple, X - smooth, affine
$G : X$ with an open orbit
$\Rightarrow X$ is flexible

Indication : $\exists \tilde{G} \supseteq G$ s.t. $\tilde{G}^\vee = \{1\}$ and
$\tilde{G} : X$ is transitive (Luna's Étale Slice Theorem)

Definition

Y - affine variety, $f \in \mathcal{O}(Y)$
SUSPENSION :
$X = \mathrm{susp}_f(Y) \subseteq Y \times \mathbb{A}^2$ is given by $uv - f(y) = 0$

Theorem

Y - affine, flexible
$\Rightarrow X = \mathrm{susp}_f(Y)$ *is flexible* $\forall f \in \mathcal{O}(Y)$

Theorem

X - smooth, affine, flexible
$\Rightarrow$ *the tensor bundles* $(TX)^{\otimes a} \otimes (T^*X)^{\otimes b}$ *are flexible*

Theorem

$X_r = \{A \in \mathrm{Mat}(m,n) \,|\, \mathrm{rk}(A) \leq r\}$*is flexible* $\forall r$

Indication :
The action

$$\mathrm{SL}_m \times \mathrm{SL}_n : \mathrm{Mat}(m,n), \quad (A_1, A_2).B = A_1 B A_2$$

is transitive on $X_r \backslash X_{r-1}$
where X_{r-1} =sing (X_r) if $r < \min\{m,n\}$

Theorem

X - affine, flexible, $\dim X \geq 2$
$Z \subseteq X_{\mathrm{reg}}$ - finite
• $\Rightarrow \exists$ *a curve $C \subseteq X_{\mathrm{reg}}$, $C \simeq \mathbb{A}^1$, s.t. $Z \subseteq C$*
and C has prescribed jets at the points of Z
• *If $Y \subseteq X$ is closed with* $\mathrm{codim}_X Y \geq 2$ *s.t* $Z \cap Y = \emptyset$
$\Rightarrow \exists C$ *as before s.t.* $C \cap Y = \emptyset$

Indication :
$\forall \mathbb{G}_a$-orbit $O \simeq \mathbb{A}^1$ in X_{reg} $\exists Z' \subseteq O$ with $\mathrm{card}\, Z' = \mathrm{card}\, Z$
$\exists g \in \mathrm{SAut}(X)$ s.t. $g(Z') = Z$ and so $C = g(O) \supseteq Z$
To prescribe jets one needs an interpolation theorem

4. Interpolation by automorphisms

Theorem

X - affine, $\dim X \geq 2$
ω - an algebraic volume form on X_{reg}
$\mathrm{SAut}(X)$ *has an open orbit $O \subseteq X$*
$\Rightarrow \forall Z \subseteq O$ *- finite,*
$\forall\, (j_p^m \,|\, p \in Z)$ *- prescribed jets of automorphisms preserving ω and mapping Z into O*
$\exists g \in \mathrm{SAut}(X)$ *with given jets j_p^m, $p \in Z$*

Definition

A - an affine algebra $/\mathbb{k}$, $X = \mathrm{Spec}\, A$ - affine
$\partial \in \mathrm{Der}\, A$ - a *locally nilpotent derivation* (LND)
(i.e. $\forall a \in A\ \exists n : \partial^n(a) = 0$)
$H = \exp(\mathbb{k}\partial) \subseteq \mathrm{SAut}(A)$ - one parameter unipotent subgroup with infinitesimal generator ∂

$f \in A^H = \ker \partial \Rightarrow f\partial \in \mathrm{LND}(A)$ - a *REPLICA* of ∂
$H_f = \exp(\mathbb{k}f\partial)$ - a *REPLICA* of H

Example

$$X = \mathbb{A}^3 = \operatorname{Spec} \Bbbk[X, Y, Z]$$

$$\partial = X\frac{\partial}{\partial Y} + Y\frac{\partial}{\partial Z}$$

$$f = Y^2 - 2XZ \in \ker \partial$$

The famous *Nagata automorphism* is the replica

$$H_f(1) = \exp(f\partial) \in \mathrm{SAut}(\mathbb{A}^3)$$

It is *wild* (*Nagata Conjecture* - I. Shestakov and U. Umirbaev, 2004)

6. Algebraically generated groups

Definition

$G \subseteq \mathrm{Aut}(X)$ is *ALGEBRAICALLY GENERATED* if

$$G = \langle H_i \,|\, i \in I \rangle$$

where $H_i \subseteq \mathrm{Aut}(X)$ are connected, algebraic $\forall i$

If $H_i \simeq \mathbb{G}_a$ $\forall i$ then G is $\mathbb{G}_a$- *GENERATED*

Examples

- $\mathrm{SAut}(X)$ is $\mathbb{G}_a$-generated
- G - connected affine algebraic group
  G is $\mathbb{G}_a$-generated $\Leftrightarrow$ $G^{\vee} = 1$ (V. Popov, 2010)
- G - semi-simple $\Rightarrow$ G is $\mathbb{G}_a$-generated

Proposition

$G \subseteq \mathrm{Aut}(X)$ - an algebraically generated group $\Rightarrow$
- *the orbits of G are locally closed*
- *the function $x \mapsto \dim(G.x)$ is lower semicontinuous on X w.r.t. Zariski topology.*

Notation

For $\mathcal{N} \subseteq \mathrm{LND}(A)$,

$G = \langle \mathcal{N} \rangle$ means $G = \langle \exp(\mathbb{k}\partial) \,|\, \partial \in \mathcal{N} \rangle$

Proposition

$G = \langle \mathcal{N} \rangle \subseteq \mathrm{Aut}(X)$ - $\mathbb{G}_a$-generated
$\mathcal{N}$ closed under conjugation in G
$\Rightarrow \exists \partial_1, \ldots, \partial_s \in \mathcal{N}$ s.t.

$$T_x(G.x) = \mathrm{span}(\partial(x) \,|\, \partial \in \mathcal{N}) \quad \forall x \in X$$

Definition

$x \in X_{\mathrm{reg}}$ is *G-FLEXIBLE* if $T_x X = \mathrm{span}(\partial(x) \,|\, \partial \in \mathcal{N})$

Corollary

(a) *$x \in X$ is G-flexible $\Leftrightarrow$ the orbit $G.x$ is open*

(b) *An open G-orbit is unique and consists of all G-flexible points*

For $G = \mathrm{SAut}(X)$ this gives $(i) \Leftrightarrow (iii)$ of the Main Theorem

Corollary

$SAut(X)$ is transitive on $X_{\mathrm{reg}} \Leftrightarrow X$ is flexible

7. Rosenlicht Separation and Kleiman Transversality Theorems

'Rosenlicht Theorem'

$G \subseteq \mathrm{SAut}(X)$ - $\mathbb{G}_a$-*generated*
$\Rightarrow \exists f_1, \dots, f_m \in \mathcal{O}(X)^G$ *separating general* G*-orbits*

'Kleiman TransversalityTheorem'

$G \subseteq \mathrm{SAut}(X)$ - $\mathbb{G}_a$-*generated*
$\exists O = G.x$ - *an open orbit*
$\Rightarrow \forall Y, Z \subseteq O$ *locally closed*
$\exists U_1, \dots, U_s \subseteq G$ - *one-parameter unipotent subgroups*
s.t. for a general

$$(h_1, \dots, h_s) \in U_1 \times \dots \times U_s$$

$(h_1 \cdot \dots \cdot h_s).Z_{\mathrm{reg}}$ *meets* Y_{reg} *transversally.*

Saturation by replicas

Definition

$G = \langle \mathcal{N} \rangle \subseteq \mathrm{SAut}(X)$
We say that $\mathcal{N}$ is *SATURATED* if

1. $\mathcal{N}$ is closed under conjugation in G
2. $\mathcal{N}$ is closed under taking replicas
   i.e. $\forall \partial \in \mathcal{N}$, $\forall f \in \ker \partial$, $f\partial \in \mathcal{N}$

Notation

$Y \subseteq X$ closed
$G_{\mathcal{N},Y} = \langle f\partial \,|\, \partial \in \mathcal{N},\ f \in \ker \partial,\ f|Y = 0 \rangle \subseteq \mathrm{Stab}_G(Y)$

Theorem

Let $G = \langle \mathcal{N} \rangle \subseteq \mathrm{SAut}(X)$ *where* $\mathcal{N}$ *is saturated s.t.* $\exists$ *an open orbit* $O = G.x_0$
$\Rightarrow \forall x \in O$ *the tangent presentation*

$$G_{\mathcal{N},x} \to \mathrm{SL}(T_x O)\,, \quad g \longmapsto dg(x)$$

is surjective

Definitions

THE MAKAR-LIMANOV INVARIANT of X is

$$\mathrm{ML}(X) = \mathcal{O}(X)^{\mathrm{SAut}(X)}$$

THE FIELD MAKAR-LIMANOV INVARIANT of X is

$$\mathrm{FML}(X) = \mathrm{Frac}\,(\mathcal{O}(X))^{\mathrm{SAut}(X)}$$

9. Unirationality of flexible varieties

Theorem (Liendo Conjecture)

$\exists x \in X$ flexible $\Leftrightarrow$ $\mathrm{SAut}(X)$ has an open orbit $\Leftrightarrow$ $\mathrm{FML}(X) = \Bbbk$
$\Rightarrow$ X is unirational

Remark

Flexibility implies neither rationality (A. Liendo) nor stable rationality (V. Popov) :
For $n \geq 4$ $\exists$ $F \subseteq \mathrm{SL}(n, \mathbb{C})$ - a finite subgroup
s. t. $X = \mathrm{SL}(n, \mathbb{C})/F$ is a smooth affine variety, flexible, unirational, not stably rational

10. Bogomolov's Conjecture

Conjecture

X is unirational $\Leftrightarrow$ X has a stably flexible model
(i.e. $\exists n \geq 0$ s.t. $X \times \mathbb{A}^n$ has a flexible model)

Examples (F. Bogomolov, I. Karzhemanov, K. Kuyumzhiyan, 2012)

1. $\mathbb{P}^n/G$ is stably flexible $\forall$ finite subgroup $G \subseteq \mathrm{PGL}_n$
2. X flexible $\Rightarrow$ X/G is stably flexible $\forall G \subseteq \mathrm{SAut}(X)$ finite
3. $\forall$ smooth cubic $X_3 \subseteq \mathbb{P}^{n+1}$, $n \geq 2$, is stably flexible
4. $\forall$ singular quartic with a double line $X_4 \subseteq \mathbb{P}^{n+1}$, $n \geq 3$ is stably flexible
5. $\forall$ intersection of 3 quadrics in $\mathbb{P}^{n+1}$, $n \geq 4$, is stably flexible.

11. Oka-Grauert-Gromov Principle

Definition

$h : X \to B$ - a smooth, surjective holomorphic map of smooth complex varieties
The *OGG PRINCIPLE* holds for h if

1. $\forall$ continuous section of h is homotopic to a holomorphic one
2. $\forall$ two holomorphic sections of h that are homotopic via continuous sections are homotopic via holomorphic ones

Theorem

X, B - smooth affine varieties $/\mathbb{C}$
$G \subseteq \mathrm{SAut}(X)$ - a $\mathbb{G}_a$-generated subgroup
s.t. $B = X/G$ - a geometric factor with smooth orbit map
$h : X \to B$
$\Rightarrow$ the OGG Principle holds for h

Indication :

The fibers of h are flexible $\Rightarrow \exists$ a *Gromov dominating spray* for h

12. Gizatullin surfaces

Definition

X - a *Gizatullin surface* if X is a normal affine surface that admits a completion by a chain of smooth rational curves

Gizatullin Theorem

Let X be a normal affine surface $\not\cong \mathbb{A}^1 \times (\mathbb{A}^1 \setminus \{0\})$
X is Gizatullin $\Leftrightarrow \mathrm{SAut}(X)$ *has an open orbit O in X*

In fact $X \setminus O$ is finite. Does $O = X_{\mathrm{reg}}$?

Gizatullin Conjecture : *Any Gizatullin surface is flexible*

- True for surfaces $xy - p(z) = 0$ in $\mathbb{A}^3$ (L. Makar-Limanov, 1970, or flexibility of suspensions)
- True for the *Danilov-Gizatullin surfaces* $\mathbb{F}_n \setminus S$, where S is an ample section in a Hirzebruch surface $\mathbb{F}_n \to \mathbb{P}^1$ (F. Donzelli, 1912)
- False in positive characteristic (V. Danilov-M. Gizatullin, 1975)
- False in zero characteristic (S. Kovalenko, 2012)