Algorithmic Aspects of Embeddability

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joint work with

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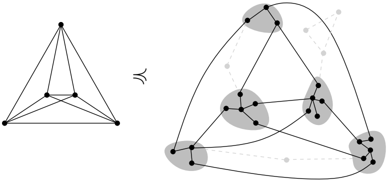
Poncelet Laboratory, Independent University of Moscow, March 30, 2016

Starting Point: Graphs & Planarity

- A graph (=1-dimensional complex) G is planar if it can be embedded into the plane $\mathbb{R}^2$ (equivalently, into the sphere S^2)
- Classical notion in topology, graph theory, discrete and computational geometry, theoretical computer science
- Combinatorics & Structure
  - Characterization of planar graphs by forbidden minors K_5, $K_{3,3}$ (Kuratowski 1930, K. Wagner 1937)



- Algorithms & Complexity
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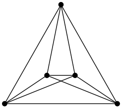

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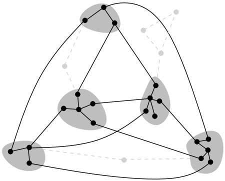

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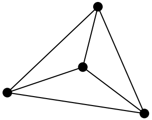

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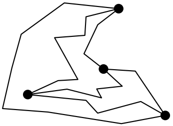
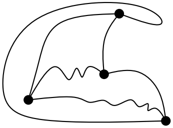


piecewise linear (PL)



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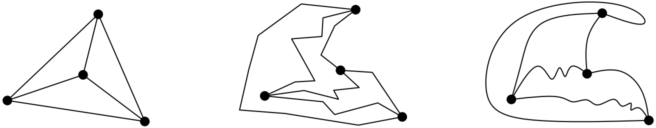

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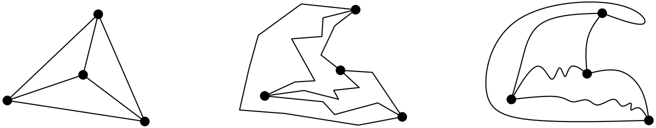


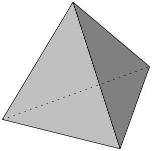
linear | piecewise linear (PL) | topological

- For graphs in the plane, TOP/PL/LINEAR embeddability are equivalent (only *one* notion of planarity).
  - TOP $\Rightarrow$ PL: easy compactness argument,
  - PL $\Rightarrow$ LINEAR: nontrivial [Steinitz,Fáry].

Different Types of Embeddings

Embeddings $X \hookrightarrow \mathbb{R}^d$ of a simplicial complex, $\dim X = k$

- Subtle differences in higher dimensions ($d \geq 3$)



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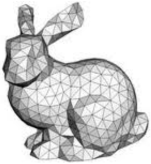
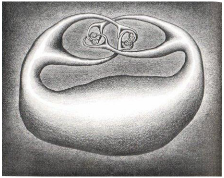


PL



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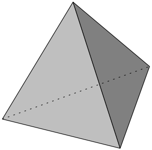

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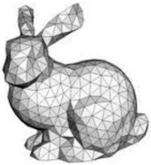

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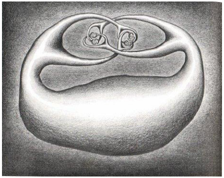

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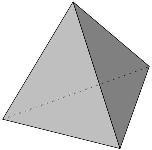

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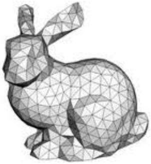

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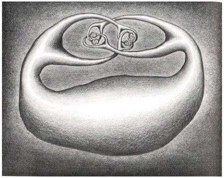

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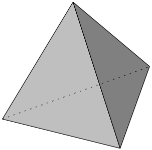

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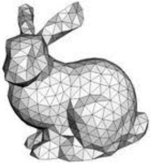

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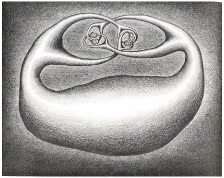

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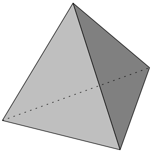

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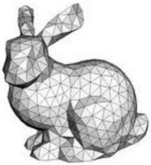

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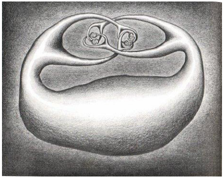

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- For algorithmic questions we consider PL embeddability

Algorithmic Embeddability Testing

$k \leq d$ fixed positive integers
$\text{EMBED}_{k \to d}$ is the following algorithmic problem:

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- $d \geq 2k+1$ trivial: embeds always (general position).
- For $d = 2k$, there exist k-dimensional complexes not embeddable into $\mathbb{R}^{2k}$:
  - complete k-complex $K^k_{2k+3} = \mathrm{skel}_k(\Delta^{2k+2})$ (all simplices of dimension $\leq k$ on $2k+3$ vertices)
  - complete multipartite k-complex $K^k_{3,\ldots,3}$
  - for $k \geq 2$, infinitely other minimally non-embeddable complexes (no straightforward analogue of Kuratowski)

Algorithmic Embeddability: Classical Results

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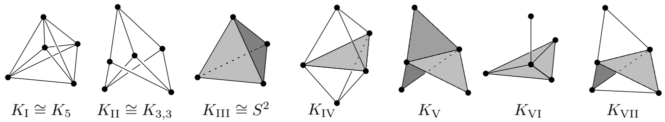
- $\mathrm{EMBED}_{1\to 2}$: $O(n)$-algorithm for graph planarity testing (Hopcroft, Tarjan 1974).
- $\mathrm{EMBED}_{2\to 2}$: characterization by forbidden subcomplexes (Halin, Jung 1964) yields $O(n)$ algorithm.



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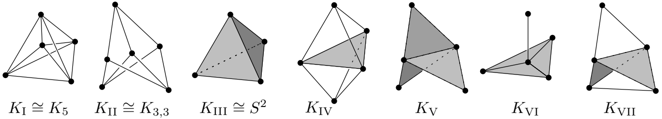

- van Kampen obstruction (van Kampen 1932; Shapiro, Wu), yields polynomial-time algorithm for $\mathrm{EMBED}_{k\to 2k}$, $k \geq 3$.

Current State of Knowledge: Complexity of EMBED$_{k\to d}$

k \ d	2	3	4	5	6	7	8	9	10	11	12	13	14
1	P												
2	P	D	NPh										
3		D	NPh	NPh	P								
4			NPh	und	NPh	NPh	P						
5				und	und	NPh	NPh	P	P				
6					und	und	NPh	NPh	NPh	P	P		
7						und	und	NPh	NPh	NPh	P	P	P

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Dividing line: metastable range $d \geq 3(k+1)/2$
(*small dimensions* $d = 2, 3$ somewhat exceptional)

The deleted product obstruction and Haefliger–Weber

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If K is a k-dimensional simplicial complex and $d \geq \frac{3(k+1)}{2}$ (metastable range) then K embeds in $\mathbb{R}^d$ iff there is an equivariant map $K^2_\Delta \to_{\mathbb{Z}_2} S^{d-1}$.

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Remark

For all (d, k) outside the metastable range, $d \geq 3$, the deleted product obstruction is known to be incomplete (Segal, Spież, Freedman, Krushkal, Teichner, A. Skopenkov).

New Results on Homotopy Classification and Extensions

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Obstruction Theory: Extending maps step by step

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- Plan: Knowing $[X^{(k-1)}, Y]$, compute $[X^{(k)}, Y]$.

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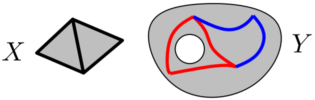
  - Extendable $f^{(k-1)}$ has to be homotopically trivial on the boundary of each k-simplex.



Obstruction Theory: Extending maps step by step

- X a simplicial complex, $X^{(k)}$ the k-skeleton (union of all simplices of dimension $\leq k$).
- Plan: Knowing $[X^{(k-1)}, Y]$, compute $[X^{(k)}, Y]$.
  - Suppose $f^{(k-1)}\colon X^{(k-1)} \to Y$ fixed; what are all $f^{(k)}\colon X^{(k)} \to Y$ extending $f^{(k-1)}$?
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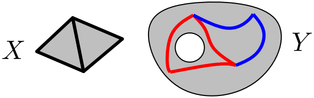
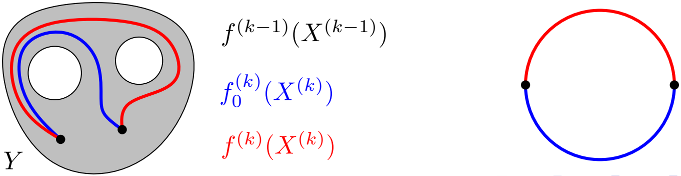


  - All possible $f^{(k)}$ have a "coset structure". From one extension $f_0^{(k)}$ we can get all by adding an element of $\pi_k(Y) = [S^k, Y]$ on each k-simplex of X.



Extending maps step by step, cont'd

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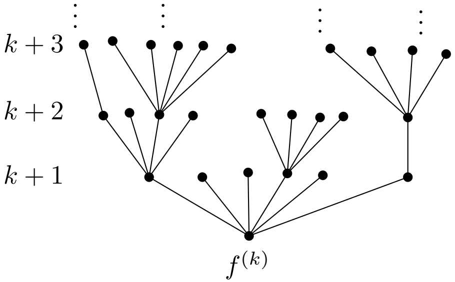

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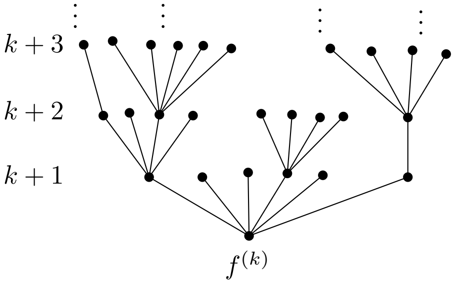

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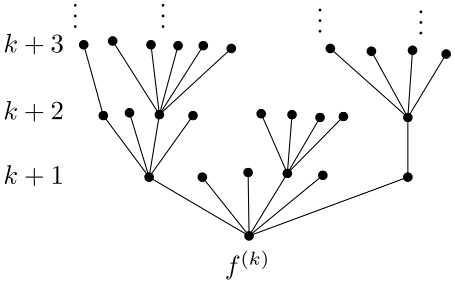

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- Primary obstruction allows us to jump two levels at a time: Given some $f^{(k)}$, it provides a finite description of all $f^{(k+1)}$ that extend $f^{(k)}$ and are extendable to some $f^{(k+2)}$.
  If Y is $(d-1)$-connected, this handles the case of $\dim X = d+1$ ($k = d-1$, $f^{(d-1)}\colon X^{(d-1)} \to Y$ is unique), but in general, the infinite branching problem doesn't go away.

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- Higher obstructions: if Y is sufficiently connected, then the set of all possible extensions has an additive structure that allows for a finite encoding; more conveniently formulated in the language of Postnikov systems

Postnikov Systems

Postnikov system for (simply connected) Y:

$$
\begin{tikzcd}
 & \vdots \\
 & P_2 \arrow[d, "p_2"] \\
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- $[X, Y] \cong [X, P_i]$ for $\dim X \le i$.
- If Y is r-connected then the stable stages P_i, $i \le 2r$ have a canonical H-space structure ("addition up to homotopy"), makes $[X, P_i]$ into a finitely generated abelian group.

Postnikov Systems, cont'd

- ith stage P_i obtained from previous stage as "twisted product" with an Eilenberg–Mac Lane space,

$$P_i = P_{i-1} \times_{k_{i-1}} K(\pi_i, i),$$

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- In the stable range, exact sequence of abelian groups

$$\begin{tikzcd}
{[SX, P_{i-1}]} \arrow[r] & {[X, K(\pi_i, i)]} \arrow[r] & {[X, P_i]} \arrow[d, "{[p_{i*}]}"] & \\
& & {[X, P_{i-1}]} \arrow[r, "{[k_{(i-1)*}]}"] & {[X, K(\pi_i, i+1)]}
\end{tikzcd}$$

where SX = suspension; inductively, compute $[X, P_i]$

- Challenges: Make everything algorithmic, handle homology computations for infinite simplicial sets (Eilenberg–Mac Lane spaces and Postnikov stages); use framework of objects with effective homology pioneered by Sergeraert, Rubio, and collaborators.

Sketch of Undecidability

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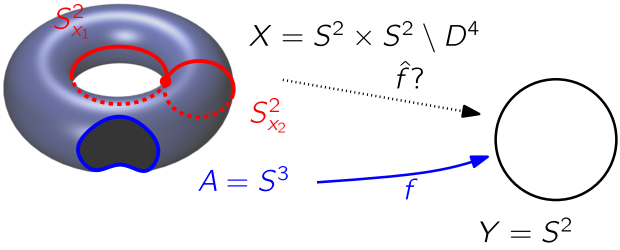
- How to encode *one quadratic equation* $x_1 x_2 = b$?
- $X = (S^2 \times S^2) \setminus D^4$, $A = \partial D^4 = S^3$, $Y = S^2$.
  $f\colon A \to Y$ given by $[f] = b \in \pi_3(Y) \cong \mathbb{Z}$.
  Any map $X \to Y$ determined by its restrictions to the "factors" $S^2_{x_i}$, these correspond to integers $x_i \in \pi_2(Y) \cong \mathbb{Z}$.
  f is extendable if there are choices x_1, x_2 such that $x_1 x_2 = b$ (*Whitehead products*)



Hardness of $\text{EMBED}_{2\to4}$: A Sketch

Theorem

It is NP-hard to decide whether a given 2-complex embeds into $\mathbb{R}^4$.

- Reduction from 3-SAT: for every 3-CNF formula φ, e.g.,

$$\varphi = (x_1 \vee \bar{x}_2 \vee x_4) \wedge (x_1 \vee \bar{x}_4 \vee x_5) \wedge \dots,$$

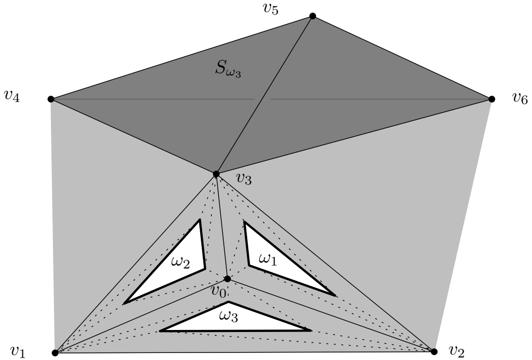
  construct a 2-dimensional simplicial complex K_φ such that

$$\varphi \text{ is satisfiable} \Leftrightarrow K_\varphi \hookrightarrow \mathbb{R}^4$$

- K_φ is built from clause gadgets and conflict gadgets
- Gadgets based on examples of Freedman, Krushkal and Teichner showing that the van Kampen obstruction is incomplete for embeddings into $\mathbb{R}^4$.

Clause Gadget

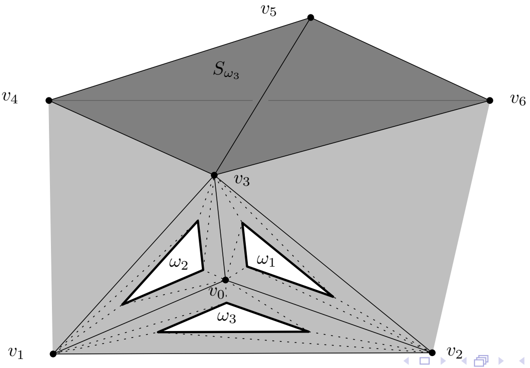
- start with K_7^2 (all triangles on 7 vertices)
- make small holes (openings) in the interiors of three triangles sharing a vertex
- for each opening, there is a complementary 2-sphere



Linking Lemma

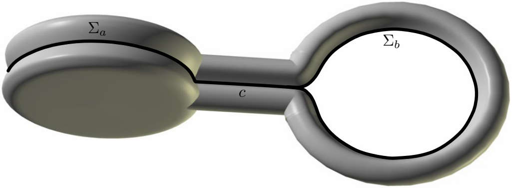
Lemma

1. *For every PL embedding $f\colon G \hookrightarrow \mathbb{R}^4$, there is an opening ω_i such that the images $f(\partial\omega_i)$ and $f(S_{\omega_i})$ have odd linking number.*
2. *For every i, there exists and embedding such that only $f(\partial\omega_i)$ and $f(S_{\omega_i})$ are linked.*



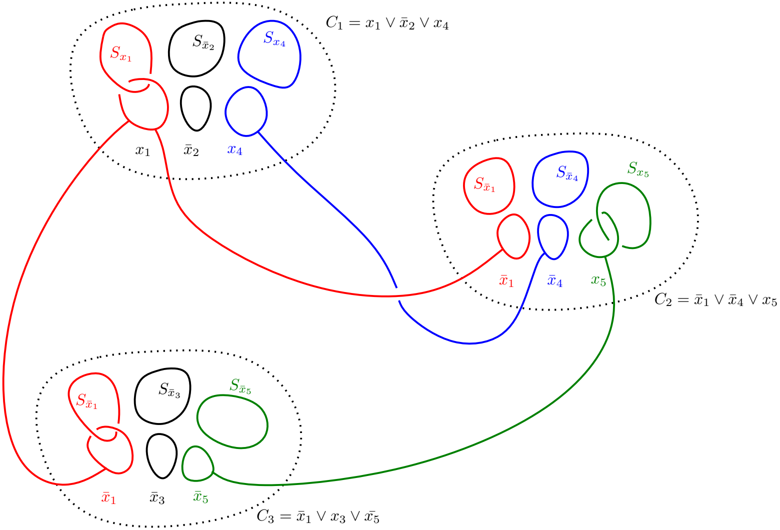
Conflict Gadget

- Squeezed torus, obtained by glueing an octagon to "two circles with a stick".



- Can be embedded into $\mathbb{R}^3$ if one of the circles is "free" (not linked with any obstacles); asymmetry in the embedding.
- Cannot be embedded into $\mathbb{R}^4$ if both circles are blocked (linked with 2-spheres).

Reduction Sketch



Algorithmic Embeddability in $\mathbb{R}^3$

- $\mathrm{EMBED}_{2\to3}$ and $\mathrm{EMBED}_{3\to3}$ can be reduced, possibly with exponential-time overhead, to the following question: Given a compact 3-manifold X with boundary, does it embed in S^3?
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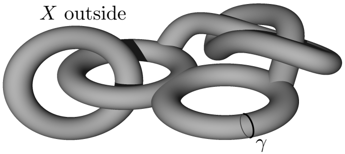
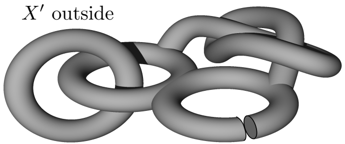
- Theorem (Fox): If X can be embedded in S^3, then there is an embedding such that the complement is a union of balls and handle bodies (solid tori).
- **Strategy**: "Guess" a meridian γ, glue a thickened disk to X along γ.





This preserves embeddability, simplifies ∂X. Recurse.

Algorithmic Embeddability in $\mathbb{R}^3$, cont'd

Key technical result, proved using normal surface theory:

Theorem (Short Meridians; Matoušek, Sedgwick, Tancer, W.)

Suppose that X is a 3-manifold with boundary[1] *that embeds in S^3. Then there exists (a possibly different) embedding of X for which there is a short meridian γ, i.e., an essential*[2] *normal curve $\gamma \subset \partial X$ bounding a disk in $S^3 \setminus X$ such that the length of γ, measured as the number of intersections of γ with the edges of the triangulation, is bounded by a computable function of the number of tetrahedra.*

---

[1] Caveat: We first need to do some preprocessing to ensure that X has certain helpful technical properties:

- X is *irreducible*, neither a ball nor an S^3,
- X has *incompressible boundary*,
- X is equipped with a 0-efficient triangulation.

[2] Meaning that γ does not bound a disk in ∂X.

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  If the embeddability test tells us $K \hookrightarrow \mathbb{R}^d$, can we compute an explicit PL embedding?
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