Eliminating Multiple Intersections and Counterexamples to the Topological Tverberg Conjecture

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joint work with

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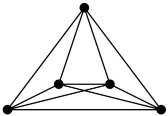
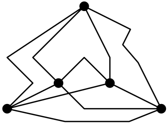
Setting: Maps from Simplicial Complexes to $\mathbb{R}^d$

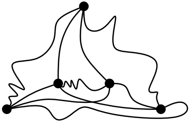
- K a finite simplicial complex
- $f\colon K \to \mathbb{R}^d$ a linear / piecewise-linear (PL) / continuous map



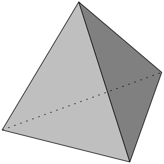
linear

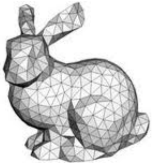


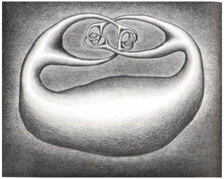
piecewise-linear (PL)



continuous







[Picture from Hocking & Young, Topology, pp. 176-177]

Question

Under which conditions does there exist a (PL) map $f\colon K \to \mathbb{R}^d$ *without self-intersections of high multiplicity?*

r-fold Intersection Points

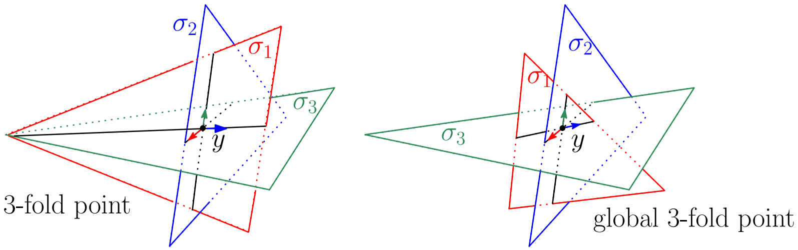
$f\colon K \to \mathbb{R}^d$, $r \geq 2$

- $y \in \mathbb{R}^d$ is an r-fold point of f if it has r distinct preimages

$$y = f(x_1) = \cdots = f(x_r), \quad x_i \in K, \quad x_i \neq x_j, i \neq j$$

- $y \in \mathbb{R}^d$ is a global r-fold point[1] of f if it has preimages in r *pairwise disjoint simplices* of K,

$$y \in f(\sigma_1) \cap \cdots \cap f(\sigma_r), \quad \sigma_i \cap \sigma_j = \emptyset, i \neq j$$



[1] With respect to a fixed triangulation.

(r-)Embeddings & Almost-(r-)Embeddings

- embedding $f\colon K \hookrightarrow \mathbb{R}^d$ = map without 2-fold points
- almost-embedding $f\colon K \to \mathbb{R}^d$ = map without *global* 2-fold points
- r-embedding $f\colon K \hookrightarrow \mathbb{R}^d$ = map without r-fold points
- almost-r-embedding $f\colon K \to \mathbb{R}^d$ = map without *global* r-fold points

Question

Necessary and sufficient conditions for (almost-)r-embeddability?

- Classical case $r = 2$:
  - Vanishing of the van Kampen obstruction gives a *complete* (necessary and sufficient) criterion for embeddability if $\dim K = m,\ d = 2m,\ m \neq 2$
  - Generalization: Haefliger–Weber Theorem: deleted product criterion *complete* in the metastable range $d \geq 3(m+1)/2$.
- Higher multiplicities $r \geq 3$?

History: Tverberg's Theorem

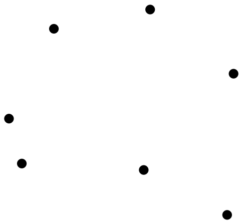
Theorem (Tverberg 1966)

Let $r \geq 2, d \geq 1$. *Set* $N := (d+1)(r-1)$.
Every $S \subseteq \mathbb{R}^d$ *with* $|S| \geq N+1$ *has an r-Tverberg partition, i.e.,*

$$S = A_1 \sqcup \ldots \sqcup A_r$$

with

$$\mathrm{conv}(A_1) \cap \ldots \cap \mathrm{conv}(A_r) \neq \emptyset.$$



$d = 2, r = 3, N + 1 = 7$

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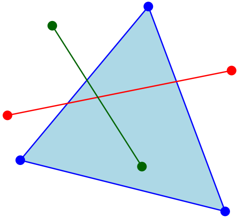

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Motivation: Topological Tverberg Conjecture

Theorem (Tverberg, equivalent form)

Let $r \geq 2, d \geq 1$, $N = (d+1)(r-1)$, $\sigma^N =$ *N-dimensional simplex*
Then every linear map $f : \sigma^N \to \mathbb{R}^d$ *has a global r-fold point.*

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- Continuous maps? [Bajmoczy–Bárány and Tverberg, 1979]

Conjecture (Topological Tverberg Conjecture)

Let $r \geq 2$, $d \geq 1$, and $N = (d+1)(r-1)$.
Then there is no almost-r-embedding $\sigma^N \to \mathbb{R}^d$, i.e., every continuous map $f : \sigma^N \to \mathbb{R}^d$ has a global r-fold point.

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True for

- $r = 2$ [Bajmoczy–Bárány 1979]
- r prime [Bárány–Shlosman–Szűcs 1981]
- $r = p^n$ prime power [Özaydin 1987][Volovikov 1996]

Long-standing open problem:

- What if r not a prime power?

Other topological Tverberg-type problems

Many variants of (topological) Tverberg-type problems/results, e.g., generalized Van Kampen–Flores-type theorem [Sarkaria; Volovikov]

Proposition (Gromov; Blagojević–Frick–Ziegler)

Let $r \geq 2$, $d \geq 1$, $m = \lceil \frac{r-1}{r} d \rceil$, $M := (d+2)(r-1)$. If there is an almost-r-embedding $g \colon : \mathrm{skel}_m(\sigma^M) \to \mathbb{R}^d$ then there exists an almost r-embedding $f \colon \sigma^M \to \mathbb{R}^{d+1}$.

Corollary (Van Kampen; Flores; Sarkaria; Volovikov)

If r is a prime power then there is no almost r-embedding $g \colon \mathrm{skel}_m(\sigma^M) \to \mathbb{R}^d$

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Proof of the proposition.

Given g, extend arbitrarily to $\hat{g} \colon \sigma^M \to \mathbb{R}^d$. Define $f \colon \sigma^M \to \mathbb{R}^D$ by $f(x) = (\hat{g}(x), \mathrm{dist}(x, K))$. If $y \in f(\sigma_1) \cap \cdots \cap f(\sigma_r)$ is a global r-fold point of f, then one σ_i has dimension $\leq m$ (pigeonholing), hence all σ_i do, hence y is a global r-fold point of g. □

Deleted Product Criterion

Lemma (Necessity of the Deleted Product Criterion)

If there exists a map $f : K \to \mathbb{R}^d$ without global r-fold points then there exists an equivariant map

$$\begin{array}{rcl} \widetilde{f}\colon K^r_\Delta & \to_{\mathfrak{S}_r} & (\mathbb{R}^d)^r \setminus \delta_r(\mathbb{R}^d) \simeq_{\mathfrak{S}_r} S^{d(r-1)-1} \\ (x_1, \dots, x_r) & \mapsto & (f(x_1), \dots, f(x_r)) \end{array}$$

where

- deleted product
  $$K^r_\Delta := \bigcup \{\sigma_1 \times \cdots \times \sigma_r \mid \sigma_i \cap \sigma_j = \emptyset, 1 \leq i < j \leq r\} \subset K^r$$
- thin diagonal $\delta_r(\mathbb{R}^d) = \{(y, \dots, y)\colon y \in \mathbb{R}^d\}$
- symmetric group $\mathfrak{S}_r$ acts by permuting components[2]

[2] The action is free on K^r_Δ for all r, not free on $S^{d(r-1)-1}$.

The Generalized Van Kampen Obstruction

Lemma

Suppose $\dim K_\Delta^r = n := d(r-1)$. *Then there exists an equivariant map* $F\colon K_\Delta^r \to_{\mathfrak{S}_r} (\mathbb{R}^d)^r \setminus \delta_r(\mathbb{R}^d) \simeq S^{n-1}$ *if and only if* $\mathfrak{o}(K_\Delta^r) = 0$.

- r-fold Van Kampen obstruction $\mathfrak{o}(K_\Delta^r) \in H^n_{\mathfrak{S}_r}(K_\Delta^r; \mathcal{Z})$

  ($\mathcal{Z}$ = integers with $\mathfrak{S}_r$-action given by $\pi \cdot a = (\operatorname{sgn}\pi)^d a$
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- Given $G\colon K^r_\Delta \to_{\mathfrak{S}_r} (\mathbb{R}^d)^r$ in general position, $\mathfrak{o}(K^r_\Delta) = [\varphi_G]$,

  $$\varphi_G(\sigma_1 \times \cdots \times \sigma_r) := G(\sigma_1 \times \cdots \times \sigma_r) \bullet \delta_r(\mathbb{R}^d) \in \mathbb{Z}$$

  algebraic intersection number with thin diagonal w.r.t. specified orientations, defines $\varphi_G \in Z^n_{\mathfrak{S}_r}(K^r_\Delta; \mathcal{Z})$

The Generalized Van Kampen Obstruction, cont'd

Caveat:

- $\mathfrak{o}(K^r_\Delta) = 0$ implies the existence of an equivariant map $F\colon K^r_\Delta \to_{\mathfrak{S}_r} (\mathbb{R}^d)^r \setminus \delta_r(\mathbb{R}^d)$

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- Implies the topological Tverberg conjecture for prime powers
- How about non-prime-powers?
- Can one show sufficiency of the deleted product obstruction, under suitable conditions?

Sufficiency of the Deleted Product Criterion for $r = 2$

Recall: almost-embedding = map without global 2-fold points

Theorem (Van Kampen–Shapiro–Wu)

Let K be a simplicial complex, $m := \dim K \geq 3$.

(VK1) *There exists an almost-embedding $f\colon K \to \mathbb{R}^{2m}$ iff there exists an equivariant map $K^2_\Delta \to_{\mathfrak{S}_2} S^{2m-1}$.*

(VK2) *If there an almost-embedding $f\colon K \to \mathbb{R}^{2m}$ then there exists an embedding $g\colon K \hookrightarrow \mathbb{R}^{2m}$; moreover, g can be taken to be piecewise-linear.*

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- Remains true for $m = 1$, (Hanani–Tutte Theorem), but with different proof method
- Fails for $m = 2$ [Freedman–Krushkal–Teichner]

Our Result: Sufficiency of the Deleted Product Criterion

Theorem (Mabillard–W.)

Let $k \geq 3$, $\dim K = m = (r-1)k$, $d = rk$. *Then the following are equivalent:*

(i) *There exists an almost* r*-embedding* $f : K \to \mathbb{R}^d$ *(no global* r*-fold points)*

(ii) *There exists an equivariant map* $F\colon K^r_\Delta \to_{\mathfrak{S}_r} S^{d(r-1)-1}$.

(iii) $\mathfrak{o}(K^r_\Delta) = 0$.

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(i) *There exists an almost r-embedding $f : K \to \mathbb{R}^d$ (no global r-fold points)*

(ii) *There exists an equivariant map $F \colon K^r_\Delta \to_{\mathfrak{S}_r} S^{d(r-1)-1}$.*

(iii) $\mathfrak{o}(K^r_\Delta) = 0$.

Theorem (Avvakumov–Mabillard–Skopenkov–W.)

The statements are equivalent also for $k \geq 2$ (codimension 2), provided $r \geq 3$.

Corollary

There is an algorithm to decide if a given K as above admits an almost r-embedding to $\mathbb{R}^d$; the running time is polynomial in the size (number of simplices) of K if r and m are fixed.

Motivation: Özaydin's Theorem

Theorem (**Özaydin**)

Let $d \geq 1$ and $r \geq 2$ not a prime power. Suppose $\mathfrak{S}_r$ acts freely on a cell complex X of dimension $d(r-1)$. There exists an equivariant map $F\colon X \to_{\mathfrak{S}_r} S^{d(r-1)-1}$.

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Example

$X = K^r_\Delta$, if $\dim K \leq \frac{r-1}{r} d$, or $K = \sigma^{(d+1)(r-1)}$.

Guiding Question

Özaydin + Sufficiency of Deleted Product Criterion
= Counterexamples to the topological Tverberg conjecture?

Özaydin & the Codimension 3 Barrier

Corollary

If r is not a prime power then $K^r_\Delta \to_{\mathfrak{S}_r} S^{d(r-1)-1}$, *whenever* $\dim K^r_\Delta \leq d(r-1)$, *e.g., if* $\dim K \leq \frac{r-1}{r}d$ *or if* $K = \sigma^N$, $N = (d+1)(r-1)$.

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Difficulty: **Codimension barrier difficulty**! Sufficiency of the deleted product criterion applies only in codimension at least 2!

Counterexamples 1: Frick's solution

Theorem (Frick)

Suppose $r \geq 6$ is not a prime power. Then there exists an almost r-embedding $f : \sigma^{(3r+2)(r-1)} \to \mathbb{R}^{3r+1}$ without r-Tverberg point.

Counterexamples 1: Frick's solution

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- Minimal counterexample: almost-6-embedding $\sigma^{100} \to \mathbb{R}^{19}$.

Proposition (Gromov; Blagojević–Frick–Ziegler)

Let $r \geq 2$, $d \geq 1$, $m = \lceil \frac{r-1}{r} d \rceil$, $M := (d+2)(r-1)$. If there is an almost-r-embedding $g : : \mathrm{skel}_m(\sigma^M) \to \mathbb{R}^d$ then there exists an almost r-embedding $f : \sigma^M \to \mathbb{R}^{d+1}$.

Proof of Frick's theorem.

Codimension of $\mathrm{skel}_m(\sigma^M)$ equals $d - m = 3$, so g exists by Özaydin & sufficiency of the DPC in codimension 3. □

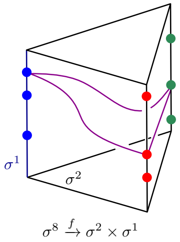
- Sufficiency of DPC in codimension 2 imples improved counterexample, almost 6-embedding $\sigma^{70} \to \mathbb{R}^{13}$

Counterexamples 2: Prismatic Maps

Theorem (Avvakumov–Mabillard–Skopenkov–W.)

Suppose $r \geq 6$ is not a prime power and let $N := (2r+1)(r-1)$
Then there exists a map $f\colon \sigma^N \to \mathbb{R}^{2r}$ without r-Tverberg point.

- Use restricted family of prismatic maps $f\colon \sigma^N \to \sigma^{2(r-1)} \times \sigma^2$.



- Structure of the maps forces all r-Tverberg points to lie on a "colorful" subcomplex C of dimension $2(r-1)$; apply Özaydin plus a relative version of the Deleted Product Criterion.
- Minimal counterexample: Almost-6-embedding $\sigma^{65} \to \mathbb{R}^{12}$.

Sufficiency of DelProdCrit: Structure of the Proof

Structured along the same lines as proof of classical (VK1):

1. r-fold Van Kampen obstruction represented by r-fold intersection number cocycle: For arbitrary $f : K \to \mathbb{R}^d$ in general position, $\mathfrak{o}(K^r_\Delta) = [\varphi_f]$,

$$\varphi_f(\sigma_1 \times \cdots \times \sigma_r) = \underbrace{f(\sigma_1) \bullet \ \ldots \bullet f(\sigma_r)}_{r\text{-fold algebraic intersection number}}$$

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2. If $\mathfrak{o}(K^r_\Delta) = 0$, then we can modify arbitrary initial f by r-fold Finger Moves to obtain $g\colon K \to \mathbb{R}^d$ with $\varphi_g = 0$ as a cocycle, i.e., for every disjoint $\sigma_1, \ldots, \sigma_r$, $\sum_i \dim \sigma_i = d(r-1)$, $g(\sigma_1) \cap \cdots \cap g(\sigma_r)$ consists of pairs of r-fold points of **opposite sign**

Sufficiency of DelProdCrit: Structure of the Proof

Structured along the same lines as proof of classical (VK1):

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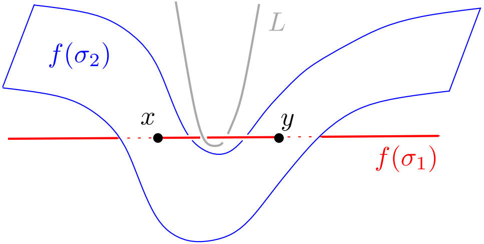
3. Use r-fold generalization of the Whitney trick to modify g and eliminate these pairs without introducing new r-fold points

The Classical Whitney Trick

Classical PL **Whitney trick** [Weber]:

- Eliminate a pair of isolated double points of *opposite sign* of a PL map by an ambient isotopy fixed outside a small ball, provided the codimension is at least 3.



- Idea: "push" $f(\sigma_2)$ upwards until the two intersections points x and y disappear, keeping the boundary of $f(\sigma_2)$ fixed.
- In low codimensions, doing this might require passing over some obstacles and/or introducing new double points, but if $d - \dim(\sigma_i) \geq 3$, $i = 1, 2$ this can be avoided.

r-Fold Whitney Trick

Theorem (**Higher-Multiplicity Whitney Trick**)

Let $r \geq 2$, and let $\sigma_1, \ldots, \sigma_r$ simplices[3], $\dim \sigma_i = m_i$, *such that* $\sum_{i=1}^{r} m_i = d(r-1)$ *and* $d - m_i \geq 3$, $1 \leq i \leq r$. *Let*

$$f : \sigma_1 \sqcup \cdots \sqcup \sigma_r \to \mathbb{R}^d$$

be a PL map in general position.
Suppose that $f(\sigma_1) \cap f(\sigma_2) \cap \cdots \cap f(\sigma_r) = \{x, y\}$ *consists of two r-fold points of opposite signs.*
Then there exist ambient isotopies $H^i : \mathbb{R}^d \times [0,1] \to \mathbb{R}^d \times [0,1]$, $2 \leq i \leq r$ *such that*

$$f(\sigma_1) \cap H_1^2(f(\sigma_2)) \cap \cdots \cap H_1^r(f(\sigma_r) = \emptyset$$

Isotopies can be chosen to be **local**: *Given any closed polyhedron* $L \subset \mathbb{R}^d$, $\dim L \leq d-3$, $x, y \notin L$, *there exists a PL ball* $B^d \subset \mathbb{R}^d$ *disjoint from L such that* H^i *is fixed outside of* $\mathring{B}^d$, $2 \leq i \leq r$.

[3] More generally, connected, orientable PL manifolds.

r-Fold Whitney Trick, cont'd

- A triple Whitney trick in codimension 3 was independently discovered by Melikhov (unpublished) and used to classify ornaments $S^{2k-1} \sqcup S^{2k-1} \sqcup S^{2k-1} \to \mathbb{R}^{3k-1}$ up to ornament homotopy.
- For codimension $k = 2$ and multiplicity $r \geq 3$, we only have a partial analogue of the Whitney trick: We can eliminate global r-fold points in pairs of opposite signs, but we may introduce local r-fold points (e.g., self-intersections of the $f(\sigma_i)$ in the process.

Ongoing and Future Work / Open Questions

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4. **Complexity of Almost-*r*-Embeddings.** For $r = 2$ and $m \geq 3$, there are m-complexes with $\mathfrak{o}(K^2_\Delta) = 0$ and n simplices, s.t. any PL embedding into $\mathbb{R}^{2m}$ requires subdivision with at least C^n simplices [Freedman–Krushkal].

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Thank you for your attention!