Extreme value statistics in random matrix theory: random and quantum chaotic states

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Outline

- A few basic concepts
  1) Random matrix theory

     spectra
     eigenvectors
     random matrices

  2) Extreme value statistics

     identically distributed, independent random variables
     correlated variables

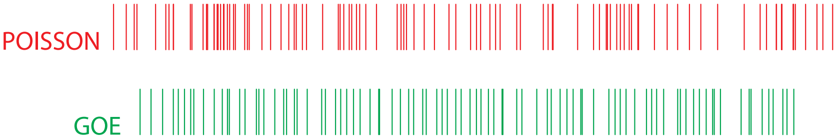
- Exact results
  1) Tracy-Widom distribution
  2) Eigenstate intensities

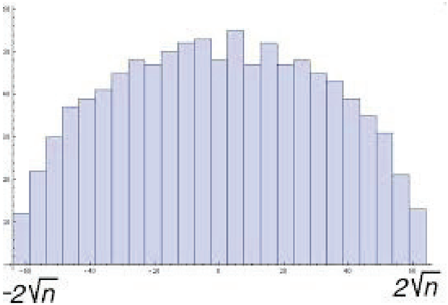
     random states
     kicked rotor eigenstates

Why do we care about random matrix theory?

- Eugene Wigner brought them into physics while thinking about slow neutron resonances, experiments motivated by nuclear reactor design
- Bohigas-Giannoni-Schmit conjecture - the properties of quantized strongly chaotic systems behave statistically in the same way as ensembles of random matrices
- Maybe we care about electron transport in nano-systems...
- Or maybe, connections to entanglement and localization...
- Or maybe, connections to number theory...
- Or maybe, certain analyses of financial markets...
- Or maybe, any of a broad range of engineering concerns...
- Or ... well, the list seems to just keep growing endlessly...

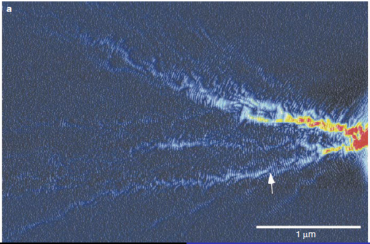
ENERGY LEVELS



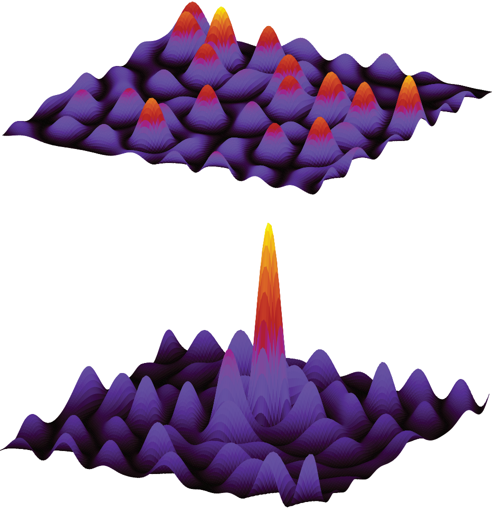


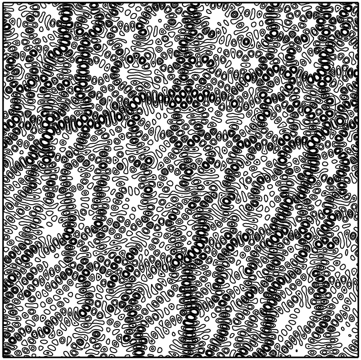
What are some basic eigenstate properties?

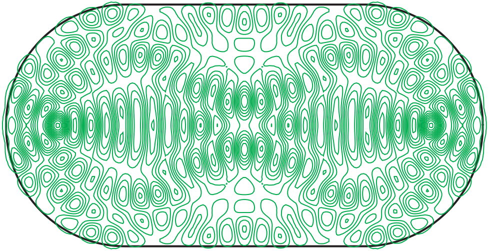
- Berry conjecture - asymptotically eigenstates of chaotic systems are like random waves
- Quantum ergodicity - probability densities associated with quantum eigenstates tend to uniform in a classical phase space (Schnirelman, Colin de Verdiére, Zelditch)
- Eigenstate scarring - short periodic orbits enhance features of chaotic quantum eigenstates (Heller)
- Coherent branching - scattering eigenstates or waves in weakly random media exhibit strong features along branches

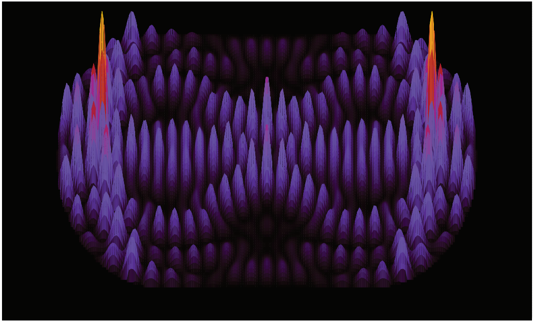


Random waves









Stadium eigenstate

Random matrices

Imagine an N-dimensional Hermitian matrix, could be a Hamiltonian, but it has Gaussian random matrix elements,

$$\mathbf{H} = \begin{pmatrix} H_{11} & H_{12} & H_{13} & \cdots \\ H_{21} & H_{22} & H_{23} & \\ H_{31} & H_{32} & H_{32} & \\ \vdots & & & \ddots \end{pmatrix}$$

where ($j \neq k$ - diagonal elements get multiplied by a $\sqrt{2}$)

$$\beta = 1, \quad H_{jk} = x_{jk} \qquad \text{say } \rho(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$$

$$\beta = 2, \quad H_{jk} = x_{jk} + i x_{jk}^{(i)}$$

$$\beta = 4, \quad H_{jk} = x_{jk}\mathbf{1} + i\left[x_{jk}^{(1)}\sigma_1 + x_{jk}^{(2)}\sigma_2 + x_{jk}^{(3)}\sigma_3\right]$$

Random matrices II

Then, the joint probability density for the ensemble of such matrices can be written (no correlations)

$$\rho(\mathbf{H})\mathrm{d}\mathbf{H} = \left(\frac{1}{\sqrt{2\pi}}\right)^{\frac{\beta N}{2}\left(N-1+\frac{2}{\beta}\right)} \exp\left[-\frac{1}{4}\mathrm{Tr}\left(\mathbf{H}^2\right)\right]\mathrm{d}\mathbf{H}$$

We care more about the eigenvalues and eigenvectors of $\mathbf{H}$ than the individual matrix elements. We are really interested more in something like

$$\begin{aligned}\rho(\mathbf{H})\mathrm{d}\mathbf{H} &= \rho(\lambda, \mathbf{U})\mathrm{d}\lambda\mathrm{d}\mathbf{U} \\ &= \rho(\lambda)\mathrm{d}\lambda\mathrm{d}\mathbf{U}\end{aligned}$$

where

$$\lambda = \mathbf{U}\mathbf{H}\mathbf{U}^{-1}$$

Random matrices III

Suddenly, there are very strong correlations introduced by the variable change, notice the Vandermonde determinant,

$$\rho(\lambda)\mathrm{d}\lambda\mathrm{d}\mathbf{U} = \mathcal{C}_\beta \prod_{k>j=1}^{N} |\lambda_j - \lambda_k|^\beta \exp\left[-\frac{1}{4}\sum_{j=1}^{N}\lambda_j^2\right] \mathrm{d}\lambda\mathrm{d}\mathbf{U}$$

- There are two very well-known spectral correlations, level repulsion and spectral rigidity.
- $\beta = 0$ in the Vandermonde determinant would correspond to Poisson eigenvalue statistics.
- The eigenstates behave statistically like uniformly random vectors according to the Haar measures for the special orthogonal $SO(N)$, unitary $SU(N)$, or symplectic $Sp(2N, \mathbf{R})$ groups.

Extreme value statistics

- In its simplest form, one considers the statistics of the largest or smallest of a set of random variable values - it puts the focuss on the tails of probability densities
- Mathematicians and engineers have been working on this since at least the turn of the 20^{th} century

  structural design for wind, floods, turbulence, ...
  finance issues, many other things ...

- Physicists have been rather slow coming to this by comparison, but now the interest is growing:

  disordered media - random lasers, minima in thermodynamic contexts, quantum information theory

Uncorrelated random variables

- Let $\{s_j\}$, $j = 1, ..., N$ be independent and identically distributed.
- A joint probability density can be defined and denoted

$$\rho(\vec{s}; N) = \rho(s_1, s_2, ..., s_N; N) = \prod_{j=1}^{N} \rho(s_j; N)$$

- The distribution function of the maximum is given by,

$$F_{max}(t; N) = \int_{-\infty}^{t} \mathrm{d}\vec{s}\ \rho(\vec{s}; N) = \left[\int_{-\infty}^{t} \mathrm{d}s_j\ \rho(s_j; N)\right]^N \quad \text{(any j)}$$

- Or for the minimum

$$F_{min}(t; N) = 1 - \int_{t}^{\infty} \mathrm{d}\vec{s}\ \rho(\vec{s}; N) = 1 - \left[1 - \int_{-\infty}^{t} \mathrm{d}s_j\ \rho(s_j; N)\right]^N$$

Universal distribution functions - Fisher/Tippett 1928

- The Weibull/Fréchet distribution function

$$F(t;N) = 1 - \exp[-(\pm t - a_N)^{\gamma_N}/b_N]$$

  is expected for uncorrelated random variables with compact support from above or below (or heavy tailed densities).

- The Gumbel distribution function

$$F(t;N) = \exp[-\mathrm{e}^{-(t-a_N)/b_N}]$$

  is expected for uncorrelated random variables with non-compact support whose tails decay at least exponentially fast.

- For example, consider the uniform density $\rho(t) = 1 \ (0 \leq t \leq 1)$:

$$\begin{aligned} F_{max}(t;N) &= t^N & \longrightarrow \mathrm{e}^{-N(1-t)} \qquad & \text{Weibull} \\ F_{min}(t;N) &= 1-(1-t)^N & \longrightarrow 1-\mathrm{e}^{-Nt} \qquad & \text{Fréchet} \end{aligned}$$

Complex and real Gaussian variables

- Absolute square of complex Gaussian variable:

$$\rho(t) = \mathrm{e}^{-t} \qquad (0 \le t \le \infty)$$

- and hence

$$\begin{aligned} F_{max}(t;N) &= \left(1-\mathrm{e}^{-t}\right)^N \to \exp\left(-\mathrm{e}^{-(t-\ln N)}\right) && \text{Gumbel} \\ F_{min}(t;N) &= 1-\left[1-\left(1-\mathrm{e}^{-t}\right)\right]^N \to 1-\mathrm{e}^{-Nt} && \text{Fréchet} \end{aligned}$$

- Square of real Gaussian variable:

$$\rho(t) = \sqrt{\frac{1}{2\pi t}}\mathrm{e}^{-t/2} \qquad (0 \le t \le \infty)$$

- and hence

$$\begin{aligned} F_{max}(t;N) &= \mathrm{erf}^N\left(\sqrt{t/2}\right) \to \exp\left(-\mathrm{e}^{-\frac{1}{2}(t-\ln\frac{2N}{\pi t})}\right) && \text{Gumbel} \\ F_{min}(t;N) &= 1-\left[1-\mathrm{erf}\left(\sqrt{t/2}\right)\right]^N \to 1-\mathrm{e}^{-N\sqrt{\frac{2t}{\pi}}} && \text{Fréchet} \end{aligned}$$

RMT example with strong correlations - largest eigenvalue

- Largest eigenvalue follows the Tracy-Widom distribution $F_\beta(s)$.
- Use semicircle radius R and matrix dimensionality to obtain scaled variable s

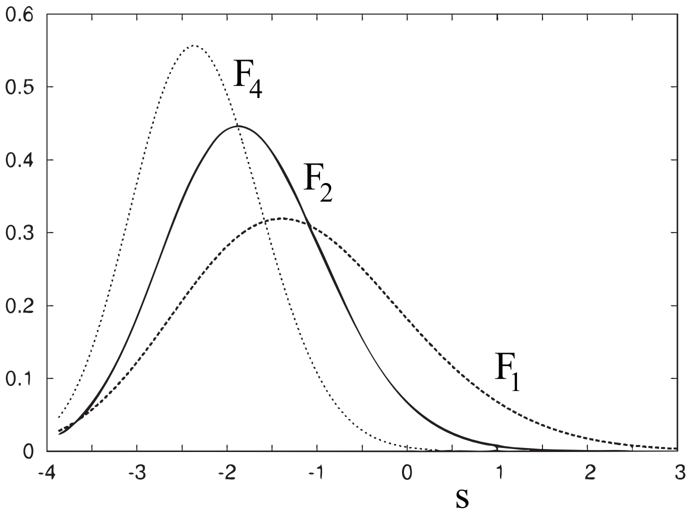
$$\begin{aligned} s &= \frac{R}{N^{1/3}}\left(\lambda_{max} - R\right) \\ R &= = 2\sqrt{\beta N - \beta + 2} \approx 2\sqrt{\beta N} \end{aligned}$$

Won't go through the math, but

$$\begin{aligned} q''(x) &= xq(x) + 2q(x)^3 \qquad \text{Painlevé equation} \\ F_2(s) &= \exp\left[-\int_s^\infty \mathrm{d}x\,(x-s)\,q^2(x)\right] \\ F_1(s) &= \exp\left[-\frac{1}{2}\int_s^\infty \mathrm{d}x\; q(x)\right] F_2(s)^{1/2} \end{aligned}$$

- Shows up in many unexpected ways, not just for λ_{max}.

Plot of Tracy-Widom distribution



Joint probability densities for intensities

- Norm constraint is naturally expressed in amplitude variables:

$$\rho_\beta(z_1, z_2, \ldots, z_N) = \frac{\Gamma\left(\frac{N\beta}{2}\right)}{\pi^{N\beta/2}} \, \delta\left(\sum_{j=1}^{N} |z_j|^2 - 1\right)$$

$\beta = 1, 2$ for real, complex respectively. Real $\longrightarrow$ orthogonal ensembles, complex $\longrightarrow$ unitary ensembles. Intensities:

$$\rho_\beta(\vec{s}; N) = \pi^{N(\beta/2-1)} \Gamma\left(\frac{N\beta}{2}\right) \left[\prod_{j=1}^{N} s_j^{\beta/2-1} \mathrm{d}s_j\right] \delta\left(\sum_{j=1}^{N} s_j - 1\right)$$

- Complex is equivalent to "broken stick problem" where $N-1$ cuts at uniformly random locations are made.
- Real is intimately connected to relationship between hyperspherical and cartesian coordinates.

An auxiliary function for *"decorrelating"* intensities

- The distribution function for the maximum is:

$$F^{\beta}_{max}(t;N) = \pi^{N(\beta/2-1)}\Gamma\left(\frac{N\beta}{2}\right)\left[\prod_{j=1}^{N}\int_0^t s_j^{\beta/2-1}\mathrm{d}s_j\right]\delta\left(\sum_{j=1}^{N}s_j - 1\right)$$

- Define auxiliary function $G^{\beta}(t,u;N)$; results by replacing unity in norm constraint with u, thus $F^{\beta}_{max}(t;N) = G^{\beta}(t,u=1;N)$.
- The Laplace transform of $G^{\beta}(t,u;N)$ renders the integrals over the N differentials $\mathrm{d}s_j$ into a product form and gives:

$$\int_0^{\infty} e^{-us}G_{\beta}(t,N,u)du = \begin{cases} \Gamma(\frac{N}{2})\left(\frac{\mathrm{erf}(\sqrt{st})}{\sqrt{s}}\right)^N & \text{real} \\ \Gamma(N)\left(\frac{1-e^{-st}}{s}\right)^N & \text{complex} \end{cases}$$

- The N integrals have been performed at the cost of now needing the inverse Laplace transforms of these expressions.

Exact results for unitary ensembles

- Distribution function for maximum follows by expanding the N^{th} power and using the inverse Laplace transform:

$$\mathcal{L}_s^{-1}\left(\frac{e^{-smt}}{s^N}\right) = \frac{1}{\Gamma(N)}(u-mt)^{N-1}\Theta(u-mt)$$

$$F_{max}^{\beta=2}(t;N) = \sum_{m=0}^{N}\binom{N}{m}(-1)^m(1-mt)^{N-1}\Theta(1-mt)$$

- Interestingly reduces to a piecewise smooth expression with the intervals $I_k = [1/(k+1), 1/k]$, where $k = 1, 2, \cdots, N-1$

$$F_{max}(t \in I_k; N) = \sum_{m=0}^{k}\binom{N}{m}(-1)^m(1-mt)^{N-1}$$

- Maximal distributions of correlated variables possessing unit norm constraints, satisfy this type of combinatoric form.

Exact results continued

- For the interval $I_1 = [1/2, 1]$:

$$F_{max}(t \in I_1; N) = 1 - N(1-t)^{N-1} = 1 - N\int_t^1 \mathcal{P}_1(s)\mathrm{d}s$$

and quite generally

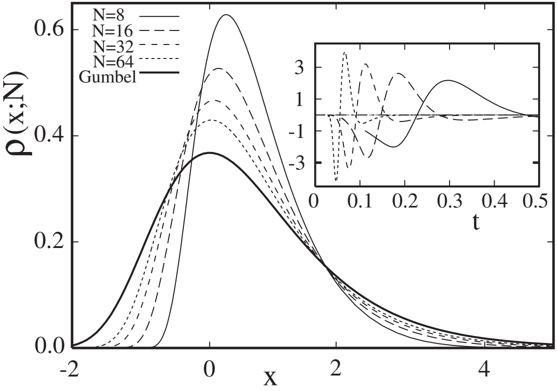
$$F_{max}(t \in I_k, N) = \sum_{m=0}^{k} \binom{N}{m} (-1)^m \int_{s_i \geq t} \mathcal{P}_m(s_1, \ldots, s_m)\,\mathrm{d}s_1 \cdots ds_m$$

- For the minima

$$F_{min}(t, N) = 1 - \Gamma(N)\left[\prod_{i=1}^{N} \int_t^1 ds_i\right] \delta\left(\sum_{i=1}^{N} s_i - 1\right)$$

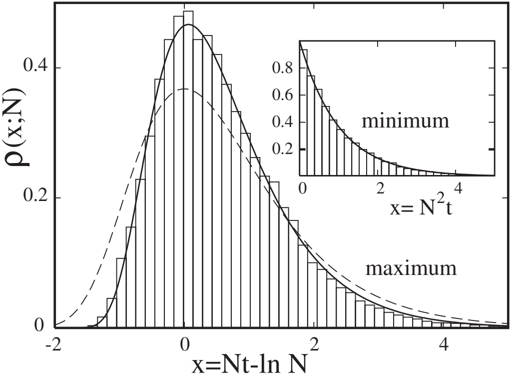
$$F_{min}(t, N) = \begin{cases} 1 - (1 - Nt)^{N-1} & 0 \leq t \leq 1/N \\ 1 & 1/N \leq t \leq 1 \end{cases}$$

Exact results continued again



Exact probability density vs asymptotic Gumbel density in scaled variable $x = N(t - \ln(N)/N)$. Inset shows difference between exact and Gumbel for same values of N, but unscaled.

Quantum kicked rotor as example of chaotic system



Probability densities (histograms) of scaled max and min (inset) of eigenfunctions in rotor position basis ($N = 32$) in parameter range $13.8 < K < 14.8$. Exact density for random states shown as continuous, dashed lines are respectively Gumbel and Fréchet.

Concluding eigenstate property remarks

- Extreme intensity statistics give alternate approach to understanding random states or quantum chaotic eigenstates.
- It is possible to derive some compact, exact results for unitary ensembles with any dimensionality and give excellent approximations for same quantities in orthogonal ensembles.
- Max intensities tend to ∞-dimensional limit very slowly and thus functional forms contain some information about system size. Means scale as $\ln a_\beta N/N$ for unitary and orthogonal cases.

- Min intensity statistics tend much more rapidly toward their ∞-dimensional Fréchet form. Mean minima scale as N^{-2} and πN^{-3} for unitary and orthogonal cases.
- These "extreme" measures provide new way to explore non-ergodicity. Localization and system dynamical features would generate deviations from the random matrix theory results.

Exact results for orthogonal ensembles?

- Exact results for interval $[1/2, 1]$ for ortho and unitary cases:

$$F_{max}(t \in I_1; N) = 1 - N(1-t)^{N-1} \qquad \text{unitary}$$

$$= \quad 1 - N\left[1 - \frac{2}{\pi}\sin^{-1}\sqrt{t} - \sqrt{t}\sum_{m=1}^{\frac{N-2}{2}} \frac{2^{2m}}{\pi m \binom{2m}{m}}(1-t)^{m-1/2}\right] \text{even}$$

$$= \quad 1 - N\left[1 - \sqrt{t} - \sqrt{t}\sum_{m=1}^{\frac{N-3}{2}} \frac{1}{2^{2m-1}}\binom{2m-1}{m}(1-t)^{m}\right] \qquad \text{odd}$$

- Reconsider the Laplace transform approach

$$F_{max}(t; N) = \Gamma\left(\frac{N}{2}\right)\sum_{m=0}^{N}(-1)^m\binom{N}{m}\mathcal{L}_s^{-1}\left[\frac{\mathrm{erfc}^m(\sqrt{st})}{s^{N/2}}\right]_{u=1}$$

A saddle point approximation

After playing around, divided and multiplied by complementary error function asymptotic form:

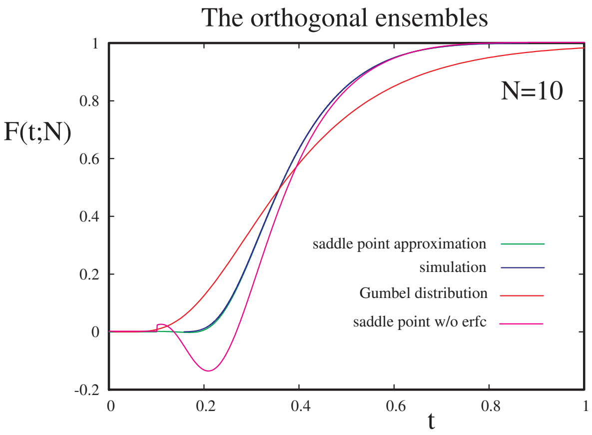
$$\mathcal{L}_s^{-1}\left((\pi st)^{m/2}\,\mathrm{e}^{mst}\mathrm{erfc}^m(\sqrt{st})\left(\frac{1}{\pi t}\right)^{m/2}\frac{e^{-smt}}{s^{\frac{N+m}{2}}}\right)\approx\left(\frac{N+m}{2-2mt}\right)^{m/2}$$

$$\times\quad \mathrm{e}^{\frac{m(N+m)t}{2-2mt}}\mathrm{erfc}^{\mathrm{m}}\left(\sqrt{\frac{\mathrm{(N+m)t}}{\mathrm{2-2mt}}}\right)\frac{1}{\Gamma\left(\frac{N+m}{2}\right)}(1-mt)^{\frac{N+m-2}{2}}\Theta(1-mt)$$

This gives

$$F_{max}^{\beta=1}(t\in I_k;N)\;=\;\sum_{m=0}^{k}\frac{\binom{N}{m}(-1)^m\left(\frac{N+m}{2}\right)^{\frac{m}{2}}\Gamma\left(\frac{N}{2}\right)(1-mt)^{\frac{N}{2}-1}}{\Gamma\left(\frac{N+m}{2}\right)}$$

$$\times\mathrm{e}^{\frac{m\left(\frac{N+m}{2}\right)t}{1-mt}}\mathrm{erfc}^{\mathrm{m}}\left(\sqrt{\frac{\left(\frac{\mathrm{N+m}}{2}\right)\mathrm{t}}{\mathrm{1-mt}}}\right)$$



Comparison of max intensity distribution functions for orthogonal ensembles. Full saddle point approximation improves considerably agreement with "exact" result vis-a-vis asymptotic Gumbel form. Simpler asymptotic form is an improvement only at larger t.

Distribution functions for maxima and minima intensities

- Suppose an ensemble of systems acts in an N-dimensional vector space, $\{|j\rangle\}$, $j = 1, ..., N$ with eigenvectors of a member system, $\{|\phi_n\rangle\}$, $n = 1, ..., N$. The intensities are $s_j = |\langle\phi_n|j\rangle|^2$.
- A joint intensity probability density can be defined

$$\rho(\vec{s}; N) = \rho(s_1, s_2, ..., s_N; N)$$

- Let the maximum intensity be $s = \max[s_j]$, $j = 1, ...N$
- The distribution function is given by,

$$F_{max}(t; N) = \int_{\frac{1}{N}}^{t} \mathrm{d}s\rho_{max}(s; N) = \int_0^t \mathrm{d}\vec{s}\rho(\vec{s}; N)$$

- Or for the minimum intensity $s = \min[s_j]$, $j = 1, ...N$

$$F_{min}(t; N) = 1 - \int_t^{\frac{1}{N}} \mathrm{d}s\ \rho_{min}(s; N) = 1 - \int_t^1 \mathrm{d}\vec{s}\ \rho(\vec{s}; N)$$

Distribution functions for maxima and minima intensities

- Let's pretend for a slide or two that the $\{s_j\}$, $j = 1, ..., N$ are similarly distributed independent random variables.
- A joint probability density can be defined and denoted

$$\rho(\vec{s}; N) = \rho(s_1, s_2, ..., s_N; N) = \prod_{j=1}^{N} \rho(s_j; N)$$

- The distribution function of the maximum is given by,

$$F_{max}(t; N) = \int^{t} \mathrm{d}\vec{s}\ \rho(\vec{s}; N) = \left[\int^{t} \mathrm{d}s_j\ \rho(s_j; N)\right]^{N} \quad \text{(any j)}$$

- Or for the minimum

$$F_{min}(t; N) = 1 - \int_{t} \mathrm{d}\vec{s}\ \rho(\vec{s}; N) = 1 - \left[1 - \int^{t} \mathrm{d}s_j\ \rho(s_j; N)\right]^{N}$$

2 relevant examples: complex and real Gaussian amplitudes

- Complex Gaussian amplitude with $\frac{1}{N}$-mean intensity density:

$$\rho(s) = N\mathrm{e}^{-Ns} \qquad (0 \leq s \leq \infty)$$

$$F_{max}(t;N) = \left(1 - \mathrm{e}^{-Nt}\right)^N \to \exp\left(-\mathrm{e}^{-N(t-\frac{1}{N}\ln N)}\right) \quad \text{Gu}$$

$$F_{min}(t;N) = 1 - \left[1 - \left(1 - \mathrm{e}^{-Nt}\right)\right]^N \to 1 - \mathrm{e}^{-N^2 t} \quad \text{Fr}$$

- Real Gaussian amplitude leads to $\frac{1}{N}$-mean intensity density:

$$\rho(s) = \sqrt{\frac{N}{2\pi s}}\mathrm{e}^{-Ns/2} \qquad (0 \leq s \leq \infty)$$

$$F_{max}(t;N) = \mathrm{erf}^N\left(\sqrt{Nt/2}\right) \to \exp\left(-\mathrm{e}^{-\frac{N}{2}(t-\frac{1}{N}\ln\frac{2N}{\pi t})}\right) \quad \text{Gu}$$

$$F_{min}(t;N) = 1 - \left[1 - \mathrm{erf}\left(\sqrt{Nt/2}\right)\right]^N \to 1 - \mathrm{e}^{-\sqrt{\frac{2N^3 t}{\pi}}} \quad \text{Fr}$$