Constructing acoustic timefronts using random matrix theory

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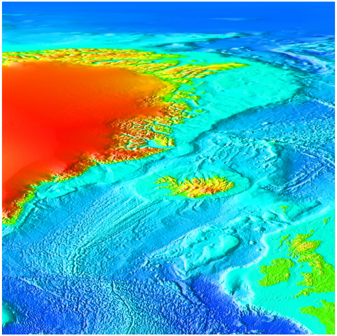
Collaboration

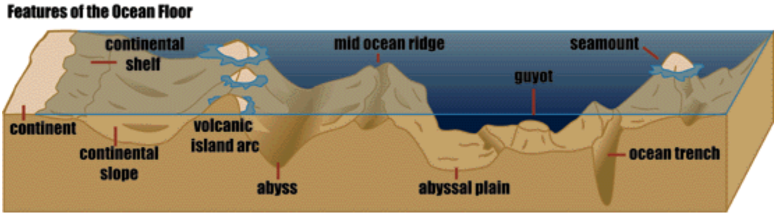
Katherine C. Hegewisch, Ph. D. thesis

Today's Thread of Logic

1) General considerations and experiments
   - The ocean
     - wave guide with disorder
   - Long range acoustic experiments
     - acoustic timefronts
2) Long range propagation models[1]
   - Wave equation
   - One way approximations
   - Paraxial optical approximations
   - Confinement and internal waves
3) Introducing Random Matrix Theory
   - Modes and mixing
   - Unitary propagation
   - Constructing acoustic time fronts
4) Concluding remarks

[1] Reviews: M. G. Brown et al., *J. Acoust. Soc. Amer.* **113** (5), 2533 (2003); F. J. Beron-Vera et al., *J. Acoust. Soc. Amer.* **114** (3), 1226 (2003); M. G. Brown and S. Tomsovic, in M. Wright and R. Weaver, editors, *New directions in linear acoustics and vibration: quantum chaos, random matrix theory, and complexity*, CUP, 2010





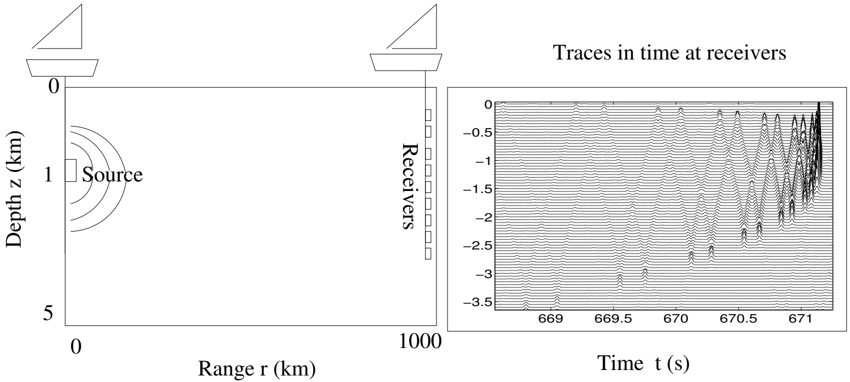
Ocean depth: 0-10 km (maximum)

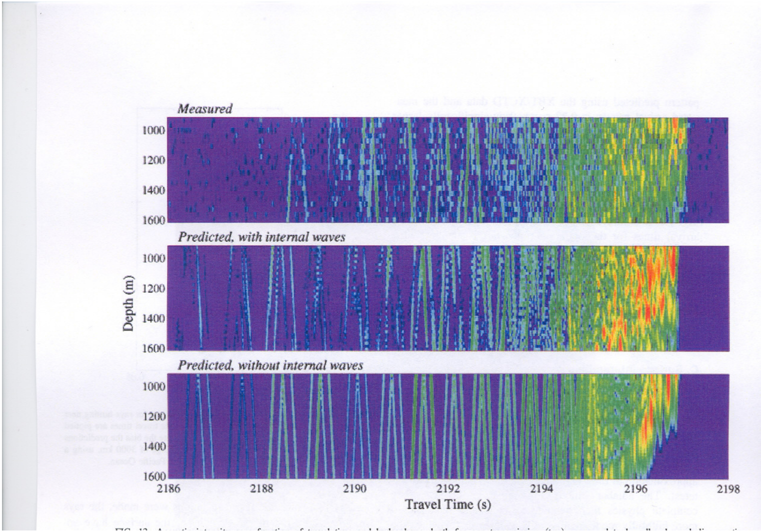
Considerations

1) Short range ocean acoustics
   - On continental shelves or inland seas
   - Typical ranges of tens of km at most
   - Frequencies up to a few kHz ($c_0 = 1.5\ km/s$)
   - Surface reflections
   - Dissipation quite important, especially bottom interactions
2) Long range ocean acoustics
   - Align with abyssal plain
   - Up to thousands of kilometers
   - Lower frequencies - 25 Hz to 250 Hz

   & Wave guide
     - Warm surface waters
     - Constant cold, high pressure waters at depths
     - Surface reflections, dissipation and bottom interactions largely avoided

Measurements





P. F. Worcester et al., *J. Acoust. Soc. Amer.* **105** , 3185 (1999)
J. A. Colosi et al., *J. Acoust. Soc. Amer.* **105** , 3202 (1999)

Some quantities of interest

- Mean ocean temperature
- Time front bias, wander, and spread - frequency dependence, range-dependence
- Intensity statistics
- Power distribution and infill
- Questions that have been asked:

  are ray methods applicable? if so where?
  how do they relate to mode methods of analysis?

Wave equation

1) A standard approach consists of beginning with the Helmholtz equation:

$$\begin{aligned} 0 &= \nabla^2 u(\mathbf{r};\omega) + \omega^2 c^{-2}(\mathbf{r}) u(\mathbf{r};\omega) \\ \omega &= \text{angular frequency} \\ c(\mathbf{r}) &= \text{position-dependent sound speed} \end{aligned}$$

2) The time-dependent wave equation solutions

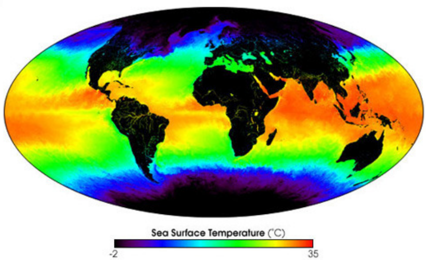
$$\frac{1}{c^2(\mathbf{r})}\frac{\partial^2 \phi(\mathbf{r};t)}{\partial t^2} = \nabla^2 \phi(\mathbf{r};t)$$

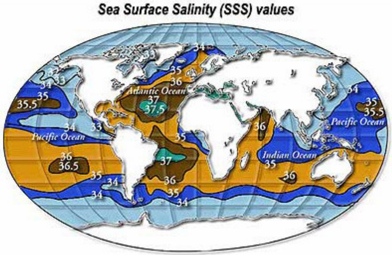
are built as weighted superpositions of eigenstates

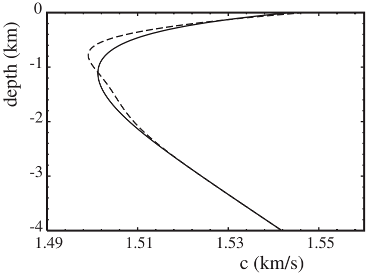
$$\phi(\mathbf{r};t) = \int_{-\infty}^{\infty} \mathrm{d}\omega \; \rho(\omega) \mathrm{e}^{-i\omega t} u(\mathbf{r};\omega)$$

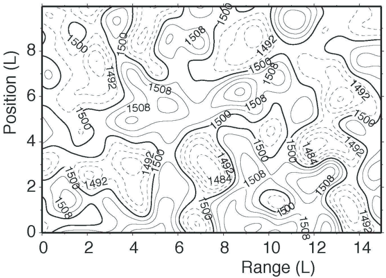
3) Boundary conditions:
determined by ocean surface, bottom, and acoustic source

Wave guide


Sea Surface Temperature (°C)
-2
35

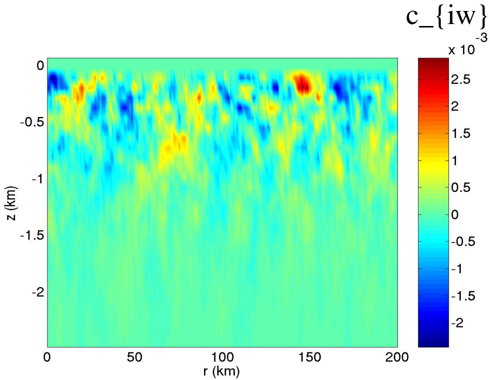

Sea Surface Salinity (SSS) values
Atlantic Ocean
Pacific Ocean
Pacific Ocean
Indian Ocean


depth (km)
c (km/s)
0
-1
-2
-3
-4
1.49
1.51
1.53
1.55


Position (L)
Range (L)
0
2
4
6
8
10
12
14

Buoyancy and temperature: internal waves

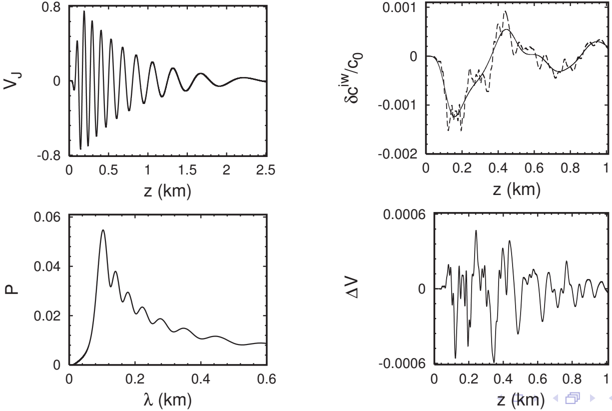
- Vertically displaced water undergoes restoring force
- Strongest force where temperature gradient is strongest
- In mid-latitudes, effect concentrated near surface
- Fluctuation scales range from meters to 100 km
- Vary on minutes to hours time scale
- Responsible for multiple scattering or wave chaos



Internal waves

Buoyancy modes:

$$\frac{\delta c_{iw}}{c_0} = \sum_{j=1}^{J_{max}} \sum_{k_r} e_{j,k_r} \exp\left(-\frac{3z}{2B}\right) \sin(j\pi\xi(z))$$



One way approximations

- Range can be used as the time-like variable if there is no backscattering
- In a semiclassical analysis, this leads to a Hamiltonian with a square root
- The quantized version is analogous to a Klein-Gordon equation
- Not a sufficient simplification, consider that the internal waves also can only scatter with small angle changes

Paraxial approximations - Tappert 1974

A good ansatz for forward-motion and small angle scattering is

$$u(\mathbf{r};\omega) \quad = \quad \Psi(z,\rho;\omega)\frac{e^{ik_0(\omega)\rho}}{\sqrt{\rho}}$$

Using Helmholtz and dropping small terms gives the parabolic equation

$$\frac{i}{k_0}\frac{\partial}{\partial\rho}\Psi(z,\rho;\omega) = -\frac{1}{2k_0^2}\frac{\partial^2}{\partial z^2}\Psi(z,\rho;\omega) + V(z,\rho)\Psi(z,\rho;\omega)$$

with $c(z,\rho) = c_0 + \delta c(z,\rho)$ and $\delta c(z,\rho) << c_0$, the potential is

$$V(z,\rho) = \frac{1}{2}\left(1 - \left(\frac{c_0}{c(z,\rho)}\right)^2\right) \approx \frac{\delta c(z,\rho)}{c_0}$$

Notes:

- $\rho \to t$ and $k_0 \to \hbar^{-1}$ gives the Schrödinger equation
- refraction naturally both range and depth dependent

Introducing random matrix theory for propagation

& So how does one go about constructing a random matrix theory for the propagation of ocean acoustic waves?

- Let's only consider the simplest problem, i.e. that of long range propagation
  - low, fixed frequency (Helmholtz to begin) - will use 75 Hz
  - no losses or dissipation
  - no surface or bottom interactions or absorption
  - no horizontal, out-of-vertical plane scattering
- There is a great deal of deterministic propagation that must be taken into account
- The internal waves create multiple scattering, but have non-zero correlation lengths and are a weak perturbation
- The time-dependent Schrödinger equation leads to unitary propagation

$$\Psi(z, \rho; \omega) = U(\rho; 0)\Psi(z, 0; \omega)$$

Random matrices

Imagine an N-dimensional Hermitian matrix, could be a Hamiltonian, but it has Gaussian random matrix elements,

$$\mathbf{H} = \begin{pmatrix} H_{11} & H_{12} & H_{13} & \cdots \\ H_{21} & H_{22} & H_{23} & \\ H_{31} & H_{32} & H_{32} & \\ \vdots & & & \ddots \end{pmatrix}$$

where ($j \neq k$ - diagonal elements get multiplied by a $\sqrt{2}$)

$$\beta = 1, \quad H_{jk} = x_{jk} \qquad \text{say } \rho(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$$

$$\beta = 2, \quad H_{jk} = x_{jk} + i x_{jk}^{(i)}$$

$$\beta = 4, \quad H_{jk} = x_{jk}\mathbf{1} + i\left[x_{jk}^{(1)}\sigma_1 + x_{jk}^{(2)}\sigma_2 + x_{jk}^{(3)}\sigma_3\right]$$

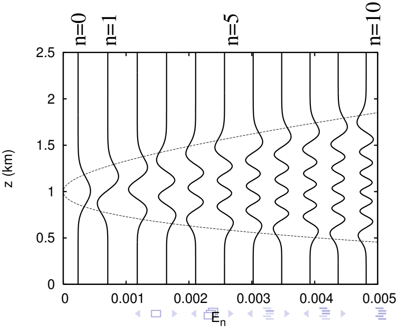
Modes

Mode picture of propagation (Dozier,Tappert, 1978)

– Modes ψ_m, energies E_m of unperturbed waveguide V_0

- Can also be defined adiabatically to account for mesoscale structure
- Somewhere around the 60^{th} mode, they begin to hit the surface - will ignore that
- They give a complete representation for the full propagation of the waves

$$-\frac{1}{2k_0^2}\frac{d^2\psi_m}{dz^2} + V_0(z)\psi_m = E_m\psi_m$$



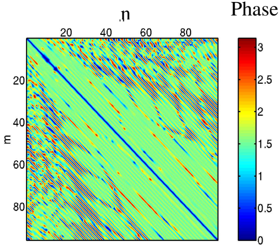
Mixing

- Unitary propagator coupling coefficients

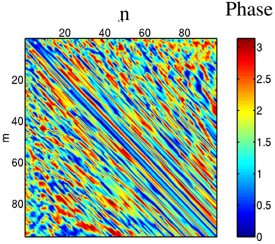
$$U_{m,n}(\rho;0) = \int \mathrm{d}z\ \psi_m^*(z) U(\rho;0) \psi_n(z)$$

gives probability amplitude of mode transition $n \to m$
  - Unperturbed propagation: $U_{mn}(\rho;0) = \Lambda_{mn} = e^{-ik_0 E_n \rho} \delta_{nm}$
  - Perturbed propagation: amplitude/phase deviations

$U' = \Lambda^{-1/2} U \Lambda^{-1/2}$ plotted



Propagation to 1 km



Propagation to 50 km

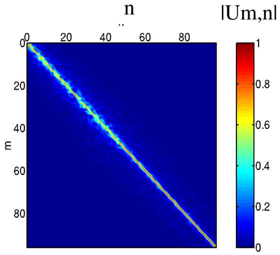
Mixing

- Unitary propagator coupling coefficients

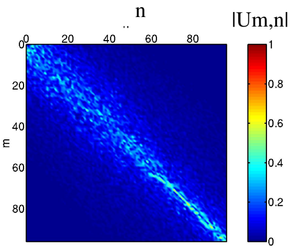
$$U_{m,n}(\rho;0) = \int \mathrm{d}z\ \psi_m^*(z) U(\rho;0) \psi_n(z)$$

gives probability amplitude of mode transition $n \to m$

  - Unperturbed propagation: $U_{mn}(\rho;0) = \Lambda_{mn} = e^{-ik_0 E_n \rho} \delta_{nm}$
  - Perturbed propagation: amplitude/phase deviations



Propagation to 50 km



Propagation to 1000 km

Unitary propagation with random matrices

Using building blocks for $\rho = 50$ km (similar to Perez et al, 2007, for quasi-1D electronic conductors) and a Cayley transformation (the matrix A Hermitian) for unitarity

$$U = \Lambda^{1/2}(I + i\epsilon A)^{-1}(I - i\epsilon A)\Lambda^{1/2}$$

- Unperturbed result: $\Lambda_{mn} = e^{-ik_0 E_n \rho}\delta_{mn}$
- Internal wave effects:

$$A_{mn}(k) = \frac{\sigma_{A_{mn}(k)}}{2} z_{mn} \qquad \text{i.e. } z_{mn}(k) \text{ perfectly correlated}$$

$$z_{m,n} = \begin{cases} \frac{G(0,1)+iG(0,1)}{\sqrt{2}} & \text{for } n \neq m \\ G(0,1) & \text{for } n = m \end{cases}$$

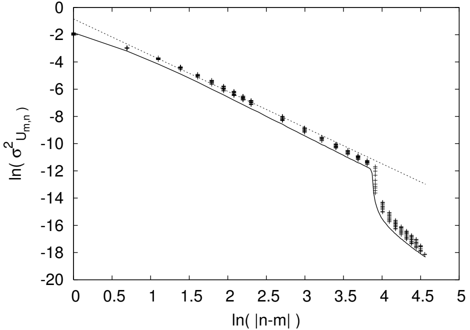
Long range propagation for $\rho = 50 \text{ x } N$ km

$$U(\rho; 0) = \prod_{i=1}^{N} U_i(\rho = 50)$$

Perturbation theory

- Using range-dependent (time-dependent) perturbation theory
$$A = \frac{k_0}{2} \int_0^{\rho=50\ km} \mathrm{d}\rho\ \hat{V}_I$$
where $\hat{V}_I$ is the operator corresponding to $\delta c(\mathbf{r})/c_o$ in the interaction picture
- The expectation value of squares of A matrix elements (variance) can be derived with internal wave formulation of Brown and Colosi, 1998
- They depend almost exclusively on the index difference $|n-m|$, i.e. A is banded with a width depending on $|n-m|$

- Variance at 75 Hz for 50 km



Approximate fit

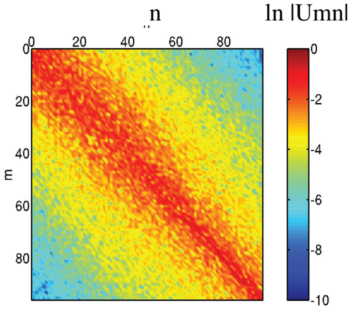
$$\sigma^2_{U_{m,n}} \approx |n-m|^{-2.6}$$

– Non-unitary, but similar ensemble exhibits localization and superdiffusion (Mirlin et al.,1996)

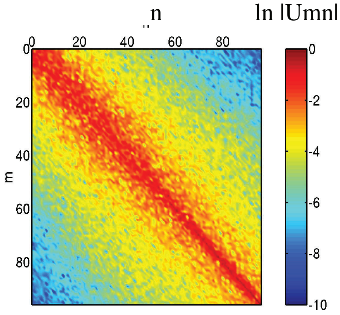
(line=pert. theory, plusses=simulations, dotted line=approx fit)

Paraxial propagation vs random matrix propagation

- Samples at 75 Hz



(a) wave eqn to 1000 km

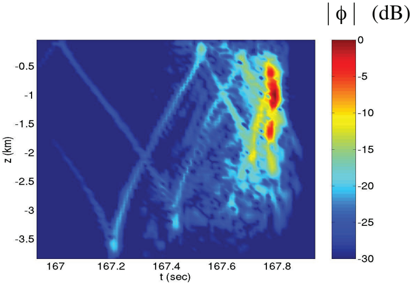


(b) RMT model to 1000 km

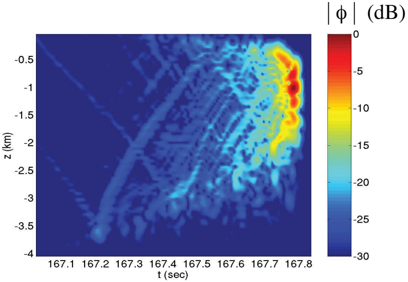
Acoustic time fronts

- A sample timefront $(k = \omega/c_0)$

$$\phi(z,\rho,t) = \frac{1}{\sqrt{2\pi\sigma_k^2}} \int e^{ikc_0(t-\rho/c_0)} u_k(z,\rho) \exp\left[-\frac{(k-k_0)^2}{2\sigma_k^2}\right] dk$$



(a) wave eqn to 250 km

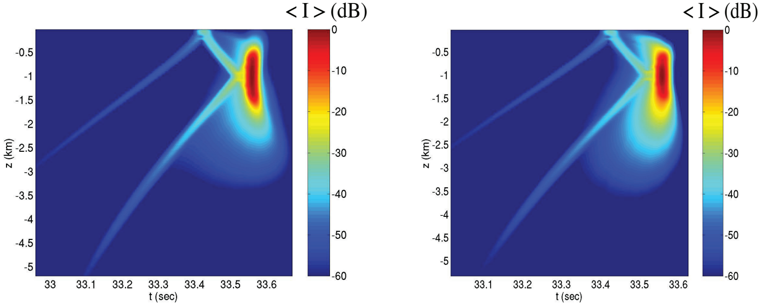


(b) RMT model to 250 km

Averaged timefronts

- Average timefront intensity

$$\langle I \rangle = \frac{1}{N} \sum_{i=1}^{N} |\phi(z, \rho, t)|^2$$

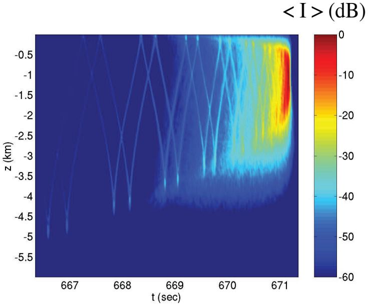


(a) wave eqn to 50 km (b) RMT model for 50 km

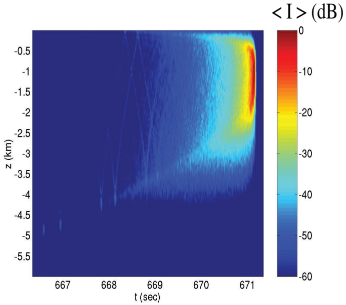
Averaged timefronts

- Average timefront intensity

$$\langle I \rangle = \frac{1}{N} \sum_{i=1}^{N} |\phi(z, \rho, t)|^2$$



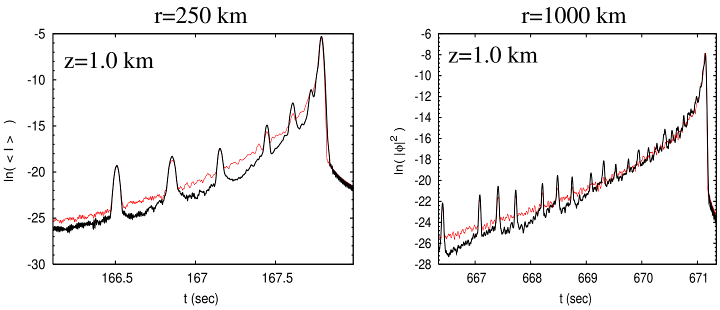
(a) wave eqn to 1000 km



(b) RMT model for 1000 km

The branches are there

Decay in time of $\langle I \rangle$ along sound axis



The branches are just enough slightly weaker that in our RMT they are not seen as clearly on the previous slide

Summary

- The minimum information which must be captured by random matrix ensembles is: i) unitarity; ii) a mean traveling phase for each mode; and iii) a variance decay rate with $|n - m|$ and k
- Items left out: i) neighboring matrix element correlations; ii) building block correlations; iii) k-dependence of A matrix elements; and iv) other ???
- Nevertheless, the ensemble goes a long way to capturing the statistical properties of experimental data
- It would be interesting to: i) investigate the general properties of power-law banded random unitary matrices; ii) understand what other information might yield more faithful RMT propagation (especially k-dependence of z matrix element correlations); and iii) learn how to apply RMT to a much broader class of ocean acoustic problems