# Non-commutative analysis — modern advances

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# Noncommutative analysis

Noncommutative analysis is an analysis of functions whose arguments do not commute and whose values do not commute. We will discuss

- differentiation,
- Taylor-type approximation,
- applications to mathematical physics /perturbation theory.

Functions are defined on operators acting in a separable Hilbert space (can think of finite matrices). In the first part of the talk, $H$ and $V$ are self-adjoint. $H$ is an initial operator and $V$ its perturbation.
Given a Borel function $f : \mathbb{R} \mapsto \mathbb{C}$ bounded on the spectrum on $H$, the operator function $f(H)$ is defined via the spectral theorem (functional calculus).

# Spectral theorem
(diagonalisation of an operator)

- Finite-dimensional case: $H = H^*$ is a finite matrix with eigenvalues $\{\lambda_k\}_{k=1}^n$. If $P_k$ is a projection onto the eigenspace corresponding to $\lambda_k$, then

$$H = \sum_{k=1}^{n} \lambda_k \cdot P_k.$$

- Infinite-dimensional case: $H = H^*$ is an operator. There is a unique spectral measure $E$ such that

$$H = \int_{\mathbb{R}} \lambda \, dE(\lambda) = \int_{\text{spectrum}(H)} \lambda \, dE(\lambda).$$

For Borel $f : \mathbb{R} \mapsto \mathbb{C}$ bounded on $\sigma(H)$, the function of $H$ is defined by

$$f(H) := \int_{\mathbb{R}} f(\lambda) \, dE(\lambda).$$

# I.M. Lifshits, *Physics Literature* '47 — *Uspehi Mat. Nauk* '52

If $H$ describes interactions between the atoms of a pure crystal and $H+V$ of a crystal with impurities, then the change in the free energy of a crystal equals (here, $\mathrm{tr}$ is the standard trace)

$$\mathrm{tr}\Big[f(H+V)-f(H)\Big],$$

for a small defect $V$ (mathematically, if the trace above is well defined). Lifshits was looking for efficient formulas to compute the change in the free energy.

# Non-commutative Lipschitz estimates

- In 1953 M. G. Krein proved that if $H$ and $V$ are self-adjoint and $V \in \mathfrak{S}^1$, then $f(H+V) - f(H) \in \mathfrak{S}^1$ for every $f \in C_c^\infty$. Here, $\mathfrak{S}^p$ is the Schatten-von Neumann class.
- M.G. Krein asked (in the case $p = 1$):
  Let $V \in \mathfrak{S}^p$, $1 \le p \le \infty$ and if $f \in C^1(\mathbb{R})$. Is it true that

$$f(H+V) - f(H) \in \mathfrak{S}^p?$$

- A positive answer was relatively simply found for $p = 2$.
- Yu. Farforovskaya (and later others, e.g. Davies, Peller) showed that the answer is negative for $p = 1$ and $p = \infty$ in 1972 and 1967, respectively.

**Theorem**

*The answer is positive if $1 < p < \infty$ ( D. Potapov & F.S., Acta Math. 2011).*

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# Perturbation argument, in its simplest form

Let $A = \sum_j \lambda_j E_j$ and $B = \sum_k \mu_k F_k$. We argue as follows

$$f(B) - f(A) =$$

$$\sum_{j,k} F_k \left(f(B) - f(A)\right) E_j =$$

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Thus, we obtained

$$f(B) - f(A) = T_{\psi_f}(A - B),$$

$$T_\phi(X) = \sum_{j,k} \phi(\lambda_j, \mu_k) F_k X E_j$$

$$\psi_f(x, y) = \frac{f(x) - f(y)}{x - y}$$

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# Schur multipliers and Fourier multipliers

As we have seen on the previous frame, the analysis of the difference

$$f(B) - f(A)$$

can be reduced to the question about the behavior of the Schur multiplier

$$T_{\psi_f}, \text{ where } \psi_f(x,y) = \frac{f(x) - f(y)}{x - y}$$

on the element $A - B \in \mathfrak{S}^p$, $1 \leq p < \infty$. The study of various classes of Schur multipliers on Schatten-von Neumann classes $\mathfrak{S}^p$ is one of the active areas of Noncommutative Analysis. This study is a noncommutative counterpart of the classical Fourier analysis. We shall exploit this connection for the case when $1 < p \neq 2 < \infty$.

# A $\mathfrak{S}^2$ estimate is simple

The following lemma is well known:

**Lemma (non-commutative Parseval's identity)**

*If $X \in \mathfrak{S}^2$, then*

$$\|X\|_2^2 = \sum_{j,k} \|F_k X E_j\|_2^2,$$

*where $\{E_j\}$ and $\{F_k\}$ are families of orthogonal projections.*

- This lemma ensures that $T_{\psi_f}$ is bounded on $\mathfrak{S}^2$ as long as

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# Vector-valued Harmonic analysis in UMD-spaces

- The new approach to $\mathfrak{S}^p$ have become possible due to recent (and not so recent) developments in the vector-valued Harmonic analysis.
- The key concept in this area is the concept of UMD (unconditional martingale differences) spaces introduced by Pisier and developed by Burkholder.
- One of the key results is the vector-valued Marcinkiewicz multiplier theorem due to J. Bourgain

**Theorem**

*If $X$ is a UMD Banach space, then the Fourier multiplier defined by*

$$(\widehat{T_m(f)})(k) = m_k \hat{f}(k), \quad k \in \mathbb{Z}$$

*is bounded on vector-valued Bochner space $L^p(\mathbb{T}, X)$ if $m$ is a bounded sequence and $m$ is of bounded variation over every dyadic interval $2^d \leq |k| < 2^{d+1}$ uniformly for $d \in \mathbb{N}$.*

# The new approach to $\mathfrak{S}^p$ with $1 < p < \infty$, $p \neq 2$

Unlike the simple case $p = 2$, the approach of D. Potapov & F.S. [Acta Math., 2011] is based on vector valued Marcinkiewicz multiplier theorem and the following ideas:

**Lemma**

*There is a rapidly decreasing function $h$ such that, for every $\|f'\|_\infty \leq 1$,*

$$\psi_f(x,y) = \frac{f(x) - f(y)}{x - y} = \int_{\mathbb{R}} h(\sigma)\, |f(x) - f(y)|^{i\sigma}\, |x - y|^{-i\sigma}\, d\sigma.$$

- The operator $R_\sigma = T_{w_\sigma}$, where $w_\sigma(x,y) = |x - y|^{i\sigma}$ is linked with the Calderon-Zygmund theory of vector-valued singular integral operators, in particular, with the Marcinkiewicz multiplier theorem.
- The representation above allows to write our Schur multiplier as follows

$$T_{\psi_f} = \int_{\mathbb{R}} h(\sigma)\, \tilde{R}_\sigma \cdot R_{-\sigma}\, d\sigma.$$

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computing the total variation of the sequence $\lambda = \{n^{is}\}_{n>0}$ over dyadic intervals via the fundamental theorem of the calculus, we have

$$\left|n^{is} - (n+1)^{is}\right| \leq \frac{|s|}{n}, \quad n \geq 1$$

and thus immediately

$$\sum_{2^k \leq n \leq 2^{k+1}} \left|n^{is} - (n+1)^{is}\right| \leq |s|, \quad k \geq 0.$$

Together with the vector valued Marcinkiewicz multiplier theorem and Transference Method (developed, in particular, by Berkson and Gillespie), we infer that $\|R_{-\sigma}\|_{\mathfrak{S}^p \to \mathfrak{S}^p} \leq (1 + |s|)$. A similar estimate also holds for $\tilde{R}_\sigma$. This allows us to conclude that $\|T_{\psi_f}\|_{\mathfrak{S}^p \to \mathfrak{S}^p} < \infty$. We are done.

# Spectral shift function of M.G. Krein

Answering Lifshits's question, computing

$$\mathrm{tr}\,(f(H+V)-f(H))\,,$$

M.G. Krein introduced an object known now as a spectral shift function of Krein (the function $\xi$ below).

**Theorem (M.G. Krein, 1953)**

*If $H$ and $V$ are self-adjoint and $V \in \mathfrak{S}^1$, then there is $L^1$-function $\xi = \xi_{H,V}$ such that*

$$\mathrm{tr}\,(f(H+V)-f(H)) = \int_{\mathbb{R}} f'(t)\,\xi(t)\,dt,$$

*for every $f \in C_c^\infty$.*

# Construction of the Krein's function in case of finite trace

Let $H, V \geq 0$, let $\tau$ be finite trace and let $n_H(t) := \tau(E_H(t,\infty)), \quad t \in \mathbb{R}$.
It follows from the functional calculus that $f(H) = -\int_0^\infty f(s)dE_H(s,\infty)$.
Taking the trace and integrating by parts, we obtain

$$\begin{aligned}\tau(f(H)) &= -\int_0^\infty f(s)dn_H(s)\\ &= -f(s)n_H(s)\Big|_0^\infty + \int_0^\infty f'(s)n_H(s)ds = \int_0^\infty f'(s)n_H(s)ds.\end{aligned}$$

Thus, we arrive at

$$\tau(f(H+V) - f(H)) = \int_0^\infty f'(s)(n_{H+V} - n_H)(s)ds.$$

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It follows from the functional calculus that $f(H) = -\int_0^\infty f(s)dE_H(s,\infty)$.
Taking the trace and integrating by parts, we obtain

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$$= -f(s)n_H(s)\Big|_0^\infty + \int_0^\infty f'(s)n_H(s)ds = \int_0^\infty f'(s)n_H(s)ds.$$

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# Spectral shift function explained

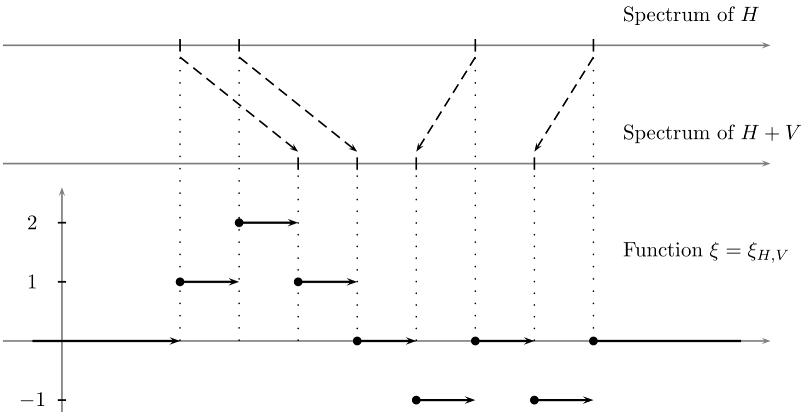


Figure: M.G. Krein spectral shift function explained

# L. Koplienko development of the trace formula

In 1984, L. Koplienko proved the following improvement of the M.G. Kreins trace formula:

**Theorem (Koplienko, 1984)**

*If $H$ and $V$ are self-adjoint and $V \in \mathfrak{S}^2$, then there is an $L^1$-function $\eta$ (the spectral shift function of Koplienko) such that*

$$\operatorname{tr}\left(R_2(f, H, V)\right) = \int_{\mathbb{R}} f''(t)\,\eta(t)\,dt, \quad \text{for every } f \in C_c^\infty,$$

$$\text{where } R_2(f, H, V) = f(H+V) - f(H) - \frac{d}{dt}\left[f\left(H_t\right)\right]\Big|_{t=0}.$$

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*Let $H$ and $V$ be self-adjoint operators and let $1 < p < \infty$, $n > 1$. There is $c_{n,p} > 0$ such that,*

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In 2008, F. Gesztesy, A. Pushnitski, and B. Simon conjectured that Koplienko result holds also for contractions. The conjecture was as follows (here and below $H$ and $V$ are not necessarily self-adjoint).

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# Applications and Connections, (aka Names dropping)

- Scattering theory [Birman, Krein, *Soviet Math. Dokl.* '62]: Krein's SSF = scattering phase
- Perturbation theory (connection with Fredholm perturbation determinant).
- More general perturbations have been studied:

$$H = (-\Delta)^{\frac{n}{4}+\epsilon}, \quad \epsilon > 0,$$

  where $V$ is a multiplication by a function in $L^\infty(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ (e.g., Koplienko, Yafaev, Azamov, Potapov, Skripka, F.S.).
- Inverse spectral problems for Schrödinger operators $-\Delta + V$, reconstruction of potentials from spectral data (e.g., Gesztesy, Simon, *Acta* '96)

# Applications and Connections, II

- Integrated density of states for some random operators (e.g., Combes, Hislop, Nakamura, *CMP* '01)
- (Physics) multichannel scattering problem and physical calculations for neutron scattering off heavy nuclei (e.g., Rubtsova, Kukulin, Pomerantsev, Faessler, *Physical Review C* '10)
- Noncommutative geometry (e.g., Azamov, A.L. Carey, & F. S., *CMP* '07)

  Krein's SSF = spectral flow (provided both exist)
- Super-symmetric quantum systems and connection with Witten index (Gesztesy, Tomilov, Carey, Potapov, F.S.)
- Q: what is a geometric meaning of Koplienko's SSF?