Sequences of independent functions in symmetric spaces

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Summary

1. Astashkin S., Sukochev F. *Series of independent random variables in rearrangement invariant spaces: an operator approach.* Israel J. Math. **145** (2005), 125–156. Astashkin S., Sukochev F. *Independent functions and the geometry of Banach spaces.* Russian Math. Surveys **65** (2010), no. 6, 1003–1081
2. Junge M., Xu Q. *Noncommutative Burkholder/Rosenthal inequalities. II. Applications,* Israel J. Math. **167** (2008), 227–282. Junge M., Parcet J., Xu Q. *Rosenthal type inequalities for free chaos.* Ann. Probab. **35** (2007), no. 4, 1374–1437.
3. Nica A., Speicher R. *On the multiplication of free $N-$tuples of noncommutative random variables.* Amer. J. Math. **118** (1996), no. 4, 799–837.

A.Ya. Khintchine, *Über dyadische Brüche*", Math. Z., **18**:1 (1923), 109–116.

In 1923, A.Ya. Khintchine proved his famous inequalities, which we cite in the form accepted in function theory, using the Rademacher functions $r_n(t) = \mathrm{sign}\,\sin(2^n\pi t)$, $0 \le t \le 1$, $(n \in \mathbb{N})$.

Theorem (Khintchine inequalities)

For every $0 < p < \infty$ there are constants $A_p > 0$ and $B_p > 0$ such that for every $n \in \mathbb{N}$ and for arbitrary $a = (a_k)_{k=1}^n \in \mathbb{R}^n$ the following inequality holds

$$A_p\|a\|_2 \le \left\| \sum_{k=1}^{n} a_k r_k \right\|_p \le B_p\|a\|_2.$$

Here, $\|\cdot\|_p$ is the norm in $L_p[0,1]$ and $\|\cdot\|_2$ is the norm in l_2.

The most general form of the scalar Khintchine inequality

Rodin and Semenov (and independently, Pisier) managed to extend Khintchine inequality to general symmetric function spaces. Their objective was to describe the sharp condition on a space in which Khintchine inequality holds.

Theorem

Let E be a symmetric function space. The inequality

$$\|\sum_{k=1}^{n} a_k r_k\|_E \sim \|a\|_2$$

holds for an arbitrary $a \in l_2$ if and only if the space E contains a (separable part of the) Orlicz space $\exp(L_2)$.

Rosenthal inequality

In 1970, H. Rosenthal asked a more general question: what is the subspace in $L_p(0,1)$ generated by a sequence of independent random variables, thus extending Khintchine inequality.

Theorem (Rosenthal)

If $x_k \in L_p(0,1)$ are independent mean zero functions, then

$$\|\sum_{k=1}^{n} x_k\|_p \sim \|\bigoplus_{k=1}^{n} x_k\|_{(L_p \cap L_2)(0,\infty)}, \quad p > 2$$

$$\|\sum_{k=1}^{n} x_k\|_p \sim \|\bigoplus_{k=1}^{n} x_k\|_{(L_p + L_2)(0,\infty)}, \quad 1 < p < 2$$

Here, $\bigoplus$ denotes the sum of disjoint copies.

Johnson-Schechtman inequality (1989)

Johnson and Schechtman extended Rosenthal inequality to general symmetric function spaces as follows.

Theorem (Johnson-Schechtman)

Let $E \supset L_p$ (for some $p < \infty$) be a symmetric function space (on the interval $(0,1)$). If $x_k \in E$, $1 \le k \le n$, are independent mean zero random variables, then

$$\|\sum_{k=1}^{n} x_k\|_E \sim \|\bigoplus_{k=1}^{n} x_k\|_{E_2}.$$

Here, E_2 is the symmetric function space on the semi-axis with a quasi-norm (equivalent to a norm)

$$\|x\|_{E_2} = \|\mu(x)\chi_{(0,1)}\|_E + \|\min\{\mu(x), \mu(1,x)\}\|_2.$$

When does the Johnson-Schechtman inequality hold?

The question in the title of this frame is motivated by the Rodin-Semenov's result. The Johnson-Schechtman inequality is, in fact, valid outside of the L_p-scale. An answer to this question was given in 2005 by Astashkin and F.S. via so-called Kruglov operator.

Theorem

Let E be a symmetric function space (on the interval $(0,1)$) equipped with a Fatou norm. If $x_k \in E$, $1 \le k \le n$, are independent mean zero random variables, then the inequality

$$\|\sum_{k=1}^{n} x_k\|_E \sim \|\bigoplus_{k=1}^{n} x_k\|_{E_2}$$

holds if and only if the Kruglov operator $K : L_0(0,1) \to L_0(0,1)$ maps E into itself.

Singular value function

Let $\mathcal{M}$ be a von Neumann algebra equipped with a faithful normal semifinite trace τ. For a ($\tau-$measurable) operator A, define its singular value function $\mu(A) : (0,\infty) \to (0,\infty)$ by setting

$$\mu(t, A) = \inf\{\|A(1-p)\|_\infty : \ \tau(p) \leq t\}.$$

An equivalent definition involves a distribution function $d_{|A|} : (0,\infty) \to (0,\infty)$

$$d_{|A|}(t) = \tau(E_{|A|}(t,\infty)).$$

It can be proved that

$$\mu(t, A) = \inf\{s : \ d_{|A|}(s) \leq t\}.$$

Definition of a symmetric operator space

Definition

A Banach space $(E, \|\cdot\|_E)$ is said to be a symmetric operator space if

1. E consists of $\tau-$measurable operators affiliated with $\mathcal{M}$.
2. If operators $0 \le B \le A$ are such that $A \in E$, then $B \in E$ and $\|B\|_E \le \|A\|_E$.
3. If operators A, B are such that $\mu(A) = \mu(B)$ and $A \in E$, then $B \in E$ and $\|B\|_E = \|A\|_E$.

If $\mathcal{M} = L_\infty(0,1)$ or $\mathcal{M} = L_\infty(0,\infty)$, then E is a symmetric function space. If $\mathcal{M} = l_\infty$, then E is a symmetric sequence space.

J. von Neumann, *Some matrix inequalities and metrization of matric-space*, Rev. Tomsk Univ. **1** (1937), 286–300.

Suppose that $\mathcal{M}$ is an algebra of all $n \times n$ matrices. It is clear that $\mu(t, A)$, $t \in (n, n+1)$ is the $n-$th eigenvalues of the operator $|A|$ (eigenvalues are taken in the decreasing order).

Theorem (J. von Neumann, 1937)

Let $\|\cdot\|_E$ be a symmetric norm on $\mathbb{R}^n$. One can define a norm on an algebra of all $n \times n$ matrices by setting

$$\|A\|_E = \|(\mu(0, A), \ldots, \mu(n-1, A))\|_E.$$

Infinite dimensional generalisation of von Neumann's result

Given a symmetric function (respectively, sequence) space E and an atomless (respectively, atomic) von Neumann algebra $\mathcal{M}$, one can define

$$E(\mathcal{M}) = \{A \text{ is } \tau\text{−measurable} : \ \mu(A) \in E\}.$$

A priori, the mapping $E(\mathcal{M}) \to \mathbb{R}$ given by the formula $A \to \|\mu(A)\|_E$ does not have to be a norm. Even if it were a norm, it is not clear why it should be a Banach norm. This important question was resolved by Kalton and the speaker.

Theorem (N. Kalton & F.S., 2008)

Let E be a symmetric function (or sequence) space. The set $E(\mathcal{M})$ is a symmetric operator space.

Noncommutative random variables

Definition

Let $\mathcal{M}$ be a von Neumann algebra equipped with a faithful normal finite (normalized) trace τ.

1. A pair $(\mathcal{M}, \tau)$ is called noncommutative probability space.
2. Self-adjoint operators affiliated with $\mathcal{M}$ (that is commuting with all unitaries from $\mathcal{M}$) are called random variables.

Definition of noncommutative independence

We employ the noncommutative notion of independence due to Junge-Xu (1988) (p. 233). We then present some natural examples of noncommutative independent variables.

Definition

Let $(\mathcal{M}, \tau)$ be a noncommutative probability space and let $\mathcal{A}_k$, $k \in \mathbb{N}$, be von Neumann subalgebras of $\mathcal{M}$. We say that subalgebras $\mathcal{A}_k$ are independent if for every k, the equality

$$\tau(AB) = \tau(A)\tau(B)$$

holds for all $A \in \mathcal{A}_k$ and for all B in the von Neumann subalgebra generated by $\mathcal{A}_j$, $j \neq k$. Random variables A_k are called independent if they generate independent subalgebras.

Example: Tensor independence.

This independence is the most transparent generalisation of the classical one. Let $\mathcal{M}$ be a von Neumann algebra equipped with a faithful normal semifinite trace τ. Let $\mathcal{A}_k$, $k \in \mathbb{N}$, be a sequence of von Neumann subalgebras such that

$$\mathcal{M} = \overline{\otimes_{k\in\mathbb{N}}\mathcal{A}_k}.$$

Clearly, algebras $\mathcal{A}_k$, $k \in \mathbb{N}$, are independent. If all $\mathcal{A}_k$ are commutative, we recover the classical case.

Example: Free independence.

Let $\mathcal{M}$ be a von Neumann algebra equipped with a faithful normal semifinite trace τ. Let $\mathcal{M}_k$, $k \in \mathbb{N}$, be a sequence of von Neumann subalgebras such that

$$\mathcal{M} = \star_{k \in \mathbb{N}} \mathcal{M}_k.$$

Here, $\star$ denotes free product of von Neumann algebras.
Let $B_j \in \mathcal{M}_{n_j}$, $1 \leq j \leq m$, and let $\tau(B_j) = 0$ for every j. If $n_{j_1} \neq n_{j_2} \neq \cdots \neq n_{j_m}$, then $\tau(B_1 \cdots B_m) = 0$.
Let A_n, $n \in \mathbb{N}$, be a sequence of (noncommutative) random variables and let $\mathcal{A}_n$ be the subalgebra generated by A_n. Elements A_n, $n \in \mathbb{N}$, are said to be freely independent if the family of subalgebras $\mathcal{A}_n$, $n \in \mathbb{N}$, possesses the property above.

Classical independence versus Free independence.

Classical independence may be stated as follows

$$\tau(A_1 A_2 \dots A_n) = 0$$

whenever $A_k \in \mathcal{A}_{j_k}$ and $j_1, j_2, \dots, j_n$ are ALL distinct and $\tau(A_k) = 0$ for all $1 \le k \le n$.
Free independence requires only

$$\tau(A_1 A_2 \dots A_n) = 0$$

whenever $A_k \in \mathcal{A}_{j_k}$ and $j_1 \neq j_2 \neq j_3 \dots \neq j_n$ and $\tau(A_k) = 0$ for all $1 \le k \le n$.
The freeness is related to independence only by analogy, since independence is about commuting subalgebras and freeness is highly noncommutative.

Rosenthal inequality

Theorem (Junge-Xu)

Let $(\mathcal{M},\tau)$ be a noncommutative probability space and let $A_k \in L_p(\mathcal{M})$, $1 \leq k \leq n$, be independent random variables such that $\tau(A_k) = 0$ for $1 \leq k \leq n$. We have

$$\|\sum_{k=1}^{n} A_k\|_p \sim \|\bigoplus_{k=1}^{n} A_k\|_{(L_p\cap L_2)(\mathcal{M}\otimes l_\infty)}, \quad p > 2.$$

$$\|\sum_{k=1}^{n} A_k\|_p \sim \|\bigoplus_{k=1}^{n} A_k\|_{(L_p+L_2)(\mathcal{M}\otimes l_\infty)}, \quad 1 < p < 2.$$

Johnson-Schechtman inequality

The following generalisation of the Junge-Xu's result from the preceding frame is due to Dirksen, de Pagter, Potapov and F.S. (2011).

Theorem

Let $(\mathcal{M}, \tau)$ be a noncommutative probability space and let $A_k \in \mathcal{M}$, $1 \leq k \leq n$, be independent random variables such that $\tau(A_k) = 0$ for $1 \leq k \leq n$. If E is an $(L_p, L_q)-$interpolation space for some $2 < p < q < \infty$, then

$$\|\sum_{k=1}^{n} A_k\|_{E(\mathcal{M})} \sim \|\bigoplus_{k=1}^{n} A_k\|_{E_2(\mathcal{M}\otimes l_\infty)}.$$

Johnson-Schechtman inequality in the free probability (mean zero case)

Remarkably, Johnson-Schechtman inequality holds in the free probability setting **without any restrictions** on the symmetric operator space. The following results are due to F.S. and Zanin.

Theorem

Let $(\mathcal{M},\tau)$ be a noncommutative probability space and let $A_k \in \mathcal{M}$, $1 \le k \le n$, be freely independent random variables such that $\tau(A_k) = 0$ for $1 \le k \le n$. If E is a symmetric function space on $(0,1)$ equipped with a Fatou norm, then

$$\|\sum_{k=1}^{n} A_k\|_{E(\mathcal{M})} \sim \|\bigoplus_{k=1}^{n} A_k\|_{E_2(\mathcal{M}\otimes l_\infty)}.$$

Johnson-Schechtman inequality in the free probability (positive case)

Theorem

Let $(\mathcal{M}, \tau)$ be a noncommutative probability space and let $A_k \in \mathcal{M}$, $1 \leq k \leq n$, be freely independent positive random variables. If E is a symmetric function space on $(0,1)$ equipped with a Fatou norm, then

$$\|\sum_{k=1}^{n} A_k\|_{E(\mathcal{M})} \sim \|\bigoplus_{k=1}^{n} A_k\|_{E_1(\mathcal{M}\otimes l_\infty)}.$$

Here, E_1 is the symmetric function space on the semi-axis with a quasi-norm (equivalent to a norm)

$$\|x\|_{E_1} = \|\mu(x)\chi_{(0,1)}\|_E + \|\min\{\mu(x), \mu(1,x)\}\|_1.$$

Semicircular random variables

Definition

A random variable $A \in \mathcal{M}$ is called semicircular if its distribution is absolutely continuous with the density given by the formula

$$\frac{1}{2\pi}\sqrt{4-t^2}\chi_{(-2,2)}(t)dt.$$

In the free probability theory it plays a role similar to that of Gaussian random variables in the classical probability theory.

Nica-Speicher result

The proof of the latter theorem depends crucially on the following construction due to Nica and Speicher (1996) (see p. 806).

Theorem

Let $(\mathcal{M}, \tau)$ be a noncommutative probability space and let $\mathcal{M}_1$, $\mathcal{M}_2$ be freely independent von Neumann subalgebras of $\mathcal{M}$. If $A_k \in \mathcal{M}_1$, $1 \leq k \leq n$, are pairwise orthogonal random variables and if $B \in \mathcal{M}_2$ is semicircular, then BA_kB, $1 \leq k \leq n$, are freely independent random variables.

Free Kruglov operator

Let $(\mathcal{M}, \tau)$ be a noncommutative probability space and let $\mathcal{M}_1$, $\mathcal{M}_2$ be freely independent von Neumann subalgebras of $\mathcal{M}$. Identify $L_\infty(0,1)$ with a von Neumann subalgebra of $\mathcal{M}_1$. If $B \in \mathcal{M}_2$ is a semicircular random variable, then

$$K : L_0(0,1) \to L_0(\mathcal{M}), \quad Kx = BxB$$

is a linear operator mapping disjointly supported functions into freely independent random variable. Since the commutative Kruglov operator maps disjointly supported functions into tensor-independent ones, the analogy is very clear.
Since B is a bounded random variable, it follows that $K : E \to E(\mathcal{M})$ for every symmetric function space E.

Idea of the proof 1

If freely independent positive random variables A_k, $1 \leq k \leq n$, are such that

$$\sum_{k=1}^{n} \tau(\mathrm{supp}(A_k)) \leq 1,$$

then the inequality

$$\|\sum_{k=1}^{n} A_k\|_{E(\mathcal{M})} \leq \mathrm{const} \cdot \|\bigoplus_{k=1}^{n} A_k\|_{E(\mathcal{M})}$$

follows from the boundedness of the free Kruglov operator $K : E \to E(\mathcal{M})$.

Idea of the proof 2

If $E = L_\infty(0,1)$, then

$$\|\sum_{k=1}^{n} A_k\|_\infty \leq 64\|\bigoplus_{k=1}^{n}\|_{(L_\infty \cap L_2)(\mathcal{M}\otimes l_\infty)}$$

is a Voiculescu inequality.

The inequality $\leq$ in our theorem follows from the particular case stated in the previous slide combined with Voiculescu inequality.

The proof of the inequality $\geq$ is much more involved. In particular, this is the place where we use the Fatou norm.

In the special case $E = L_p$, $1 \leq p \leq \infty$, this result was proved by Junge, Parcet and Xu (see [Theorem A]).