Structure of Unital Maps and the Asymptotic Quantum Birkhoff Conjecture

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The Superposition Principle (Physicists):

If a quantum system can be in one of two mutually distinguishable states $|\,\mathbf{A}\rangle$ and $|\,\mathbf{B}\rangle$, it can be both these states at once. Namely, it can be in the *superposition* of states

$$\alpha\,|\,\mathbf{A}\rangle + \beta\,|\,\mathbf{B}\rangle$$

where α and β are both complex numbers and $|\alpha|^2 + |\beta|^2 = 1$.
If you look at the system, the chance of seeing it in state $|\,\mathbf{A}\rangle$ is $|\alpha|^2$ and in state $|\,\mathbf{B}\rangle$ is $|\beta|^2$.

The Superposition Principle (Mathematicians):

The state of a quantum system is a unit vector in a complex Hilbert space ($\mathbb{C}^d$ for finite dimensional systems.
Measuring the system projects the vector onto one of a set of orthonormal basis vectors, with probability proportional to the squared length of the projection.

We call a two-dimensional quantum system a *qubit*.

Example: If you have a polarized photon, there can only be two distinguishable states, for example, vertical $|\updownarrow\rangle$ and horizontal $|\leftrightarrow\rangle$ polarizations.

All other states can be made from these two.

$$|\nearrow\!\!\!\!\swarrow\rangle = \frac{1}{\sqrt{2}}|\leftrightarrow\rangle + \frac{1}{\sqrt{2}}|\updownarrow\rangle$$

$$|\nwarrow\!\!\!\!\searrow\rangle = \frac{1}{\sqrt{2}}|\leftrightarrow\rangle - \frac{1}{\sqrt{2}}|\updownarrow\rangle$$

$$|\circlearrowleft\rangle = \frac{1}{\sqrt{2}}|\leftrightarrow\rangle + \frac{i}{\sqrt{2}}|\updownarrow\rangle$$

$$|\circlearrowright\rangle = \frac{1}{\sqrt{2}}|\leftrightarrow\rangle - \frac{i}{\sqrt{2}}|\updownarrow\rangle$$

If you have two qubits, their joint state space is the tensor product of their individual state spaces (e.g., $\mathbb{C}^4$).

Two qubits can be in any superposition of the four states

$$|\updownarrow\updownarrow\rangle \quad |\updownarrow\leftrightarrow\rangle \quad |\leftrightarrow\updownarrow\rangle \quad |\leftrightarrow\leftrightarrow\rangle$$

This includes states such as an EPR (Einstein-Podolsky-Rosen) pair of photons,

$$\frac{1}{\sqrt{2}}(|\updownarrow\leftrightarrow\rangle - |\leftrightarrow\updownarrow\rangle) = \frac{1}{\sqrt{2}}(|\swarrow\!\!\!\nearrow\ \nwarrow\!\!\!\searrow\rangle - |\nwarrow\!\!\!\searrow\ \swarrow\!\!\!\nearrow\rangle),$$

where neither qubit alone has a definite state. Such states are called *entangled* states.

If you have n qubits, their joint state is described by a 2^n dimensional vector.

Let's label basis vectors for each qubit by $|\,0\rangle$ and $|\,1\rangle$.

The basis states of this vector space are:

$$|\,000\ldots00\rangle \quad |\,000\ldots01\rangle \quad \cdots \quad |\,111\ldots11\rangle$$

where

$$|\,0100\ldots1\rangle = |\,0\rangle \otimes |\,1\rangle \otimes |\,0\rangle \otimes |\,0\rangle \otimes \ldots \otimes |\,1\rangle$$

This high dimensional tensor product spaces is where quantum information theory (as well as quantum computation) lives.

Density Matrices

In quantum mechanics, the fundamental objects are often taken to be pure quantum states (unit vectors in $\mathbb{C}^d$). These are analogous to deterministic states of classical systems.

For quantum information theory, we need to work with probabilistic ensembles of quantum states. These are represented by *density matrices.*

A *density matrix* ρ is an $d \times d$ *Hermitian trace-1 positive semi-definite matrix.*

Density Matrices, Continued

A rank one density ρ corresponds to the pure quantum state v (sometimes denoted $|\, v\rangle$) with $\rho = vv^\dagger$ (or $\rho = |\, v\rangle \langle v\,|$).

Density matrices arise naturally from pure states in two ways:

1. probabilistic ensembles of pure quantum states.
2. states of subsystems of pure quantum states.

Density Matrices I

Suppose we have a probabilistic quantum system which is in state v_i with probability p_i.

The corresponding density matrix is

$$\rho = \sum_i p_i v_i v_i^\dagger$$

The density matrix ρ gives as much information as possible about the outcomes of experiments performed on the system, so two systems with the same density matrix ρ are indistinguishable.

Density Matrices II

Suppose you have a joint quantum system on $\mathbb{C}^a \otimes \mathbb{C}^b$ in the state ρ_{AB}. If you only consider the first part of the system, it is effectively in the state

$$\rho_A = \mathrm{Tr}_B \ \rho_{AB}$$

Here, Tr_B is the partial trace over the second quantum space. That is, if we have a tensor product state

$$\rho_{AB} = \rho_A \otimes \rho_B,$$

then

$$\mathrm{Tr}_B \ \rho_{AB} = (\mathrm{Tr} \rho_B) \ \rho_A,$$

and we extend this linearly to define the partial trace on entangled states.

POVM Measurements

(Positive Operator Valued Measurements).

We are given a set of positive semidefinite matrices E_i satisfying $\sum_i E_i = I$.

The probability of the i'th outcome is

$$p_i = \mathrm{Tr}(E_i \rho)$$

For von Neumann measurements, take a basis e_i and let $E_i = e_i e_i^\dagger$

Then, the probability of the i'th outcome is

$$p_i = \mathrm{Tr} \rho_i e_i e_i^\dagger = e_i^\dagger \rho e_i$$

Quantum Channels

A (memoryless) quantum channel (or quantum operation) Φ is a completely positive trace-preserving linear map.

This is the most general physical reasonable map on quantum states (i.e., density matrices).

- Trace-preserving: Φ takes trace 1 matrices to trace 1 matrices.

- Positive: Φ takes positive semidefinite matrices to positive semidefinite matrices.

- Completely positive: Even if Φ is tensored with the identity map, $\Phi \otimes I$ remains positive.

Another characterization of quantum channels

Krauss operator sum representation:

Any quantum channel Φ on a finite dimensional space can be represented as

$$\Phi(\rho) = \sum_i A_i \rho A_i^\dagger$$

where the A_i are matrices satisfying

$$\sum_i A_i{}^\dagger A_i = I.$$

This second condition is required for Φ to be trace preserving.

A third characterization of quantum channels.

Theorem (Stinespring dilation theorem)

Any quantum channel $\rho \to \Phi(\rho)$ can be implemented by first embedding the input space into a larger Hilbert space; this takes $\rho \to \rho \otimes \sigma$; next by applying a unitary trasformation U, and finally taking a partial trace (e.g. discarding part of the larger Hilbert space)

$$\begin{tikzcd} \rho \arrow[r, "\Phi"] \arrow[d] & \Phi(\rho) \\ \rho \otimes \sigma \arrow[r, "U"] & U(\rho \otimes \sigma)U^\dagger \arrow[u, "\mathrm{Tr}_2"'] \end{tikzcd}$$

Birkhoff's Theorem

Every doubly stochastic matrix is a convex combination of permutation matrices.

$$\begin{pmatrix} \frac{5}{6} & \frac{1}{6} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \frac{1}{6}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

What this says is that any stochastic map which takes the uniform random distribution can be represented as a probabilistic coin flip, followed by a deterministic permuation map.

No Quantum Birkhoff Theorem

We can ask whether this is true of quantum channels. Can every unital map be represented as the convex combination of unitary maps?

This is not true. The unital map $\rho \to \sum_i A_i \rho A_i^\dagger$ cannnot be represented this way, where

$$A_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \ A_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \ A_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

is not a convex combination of unitaries.

The Asymptotic Quantum Birkhoff Conjecture

Suppose that we have a quantum map Φ. Can Φ^n be arbitrarily well approximated by unitaries as n goes to ∞? The Asymptotic Quantum Birkhoff (AQB) Conjecture is that it can be.

We can show that the 3-dimensional map given above is a counterexample to the AQB conjecture.

Since this process involves the limit of $\Phi^{\otimes n}$ as $n \to \infty$, it matters which norm we use in the approximation. The correct one is the diamond norm, known as the *cb*-norm by analysts.

Exactly Factorizable Maps

Recall that any quantum map can be written as

$$\Phi(\rho) = \mathrm{Tr}_2 U(\rho \otimes \sigma)U^\dagger$$

where σ is some quantum state (this is the Stinespring dilation theorem).

An exactly factorizable map is one that can be written

$$\Phi(\rho) = \mathrm{Tr}_2 U(\rho \otimes I_n/n)U^\dagger$$

for some n.

Exactly Factorizable Maps and the Asymptotic Quantum Birkhoff Property

Theorem: If $\Phi^{\otimes n}$ is a convex combination of unitaries with rational coefficients, then Φ is exactly factorizable.

Proof: Suppose that

$$\Phi^{\otimes n}(\rho) = \frac{1}{m}\sum_{j=1}^{m} U_j \rho U_j^\dagger$$

Then define a unitary map V as follows

$$V(|w\rangle|j\rangle) = (U_j|w\rangle)|j\rangle$$

We get (where $\mathrm{Tr}_{\bar{1}}$ is the trace on all but the first system)

$$\begin{aligned}\mathrm{Tr}_{\bar{1}} V(\rho_1 \otimes I/d^{n-1} \otimes I/m)V^\dagger &= \mathrm{Tr}_{\bar{1}} \frac{1}{m}\sum_{j=1}^{m} U_j(\rho \otimes I/d^{n-1})U_j^\dagger \\ &= \mathrm{Tr}_{\bar{1}} \Phi^{\otimes n}(\rho \otimes I/d^{n-1}) = \Phi(\rho).\end{aligned}$$

Strongly Factorizable Maps

Call the closure of the set of exactly factorizable maps the set of set of *strongly factorizable maps*.

Theorem: If Φ satisfies the asymptotic Birkhoff conjecture, then Φ is strongly factorizable.

Proof Sketch: Use the previous theorem and take limits.

A Non-Strongly-Factorizable Map

We start the proof. We will show that the map Φ given above which was not a mixture of unitaries is not exactly factorizable. It is easy to calculate that

$$\Phi\left(\rho_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Choosing U to match Φ If $\Phi(\rho) = \mathrm{Tr} U \rho U^\dagger$, and

$$U = \begin{pmatrix} A & B & C \\ D & E & F \\ G & H & J \end{pmatrix}$$

then

$$U(\rho_1 \otimes I_n)U^\dagger = \begin{pmatrix} AA^\dagger & AD^\dagger & AG^\dagger \\ DA^\dagger & DD^\dagger & DG^\dagger \\ GA^\dagger & GD^\dagger & GG^\dagger \end{pmatrix}$$

By matching with the reult of $\Phi(\rho)$ from the previous slide, this shows that $\mathrm{Tr} AA^\dagger = 0$, so $A = 0$. We also have that $\frac{1}{n}\mathrm{Tr} DD^\dagger = \frac{1}{n}\mathrm{Tr} GG^\dagger = 1/2$.

Proof Continued

We have shown that

$$U = \begin{pmatrix} 0 & B & C \\ D & 0 & F \\ G & H & 0 \end{pmatrix}$$

This must be unitary, so $UU^\dagger = U^\dagger U = I$. This says that

$$BB^\dagger + CC^\dagger = DD^\dagger + FF^\dagger = GG^\dagger + HH^\dagger = I$$

$$CF^\dagger = BH^\dagger = DG^\dagger = 0$$

$$D^\dagger D + G^\dagger G = B^\dagger H + F^\dagger H = C^\dagger C + F^\dagger F = I$$

$$G^\dagger H = D^\dagger F = B^\dagger C = 0$$

Proof Continued

We have

$$GG^\dagger + HH^\dagger = I \tag{1}$$

We also have $G^\dagger H = 0$. Multiplying both sides by H, we get

$$HH^\dagger H = H$$

This shows that the eigenvalues of $H^\dagger H$ are either 0 or 1. Recall that $\mathrm{Tr} H^\dagger H = \frac{n}{2}$, so we have exactly $\frac{n}{2}$ eigenvalues of 0 and of 1.

By (1), the column space of G is perpendicular to the column space of H.

Proof Continued

Recall that $GG^\dagger$ had $\frac{n}{2}$ eigenvalues of 0 and $\frac{n}{2}$ eigenvalues of 1.

By the singular value theorem, this means that

$$G = \sum_{k=1}^{n/2} |\, v_k \rangle \, \langle w_k \,|$$

where $|\, v_k\rangle$ are orthonormal vectors. The same is true of H.

Since $GG^\dagger = \sum |\, v_k\rangle\langle v_k \,|$ and $GG^\dagger + HH^\dagger = I$ we see that the column space of G is orthogonal to the column space of H.

Proof Concluded

By looking at inputs other than ρ_1 into the map Φ, we can show that

$$\mathrm{Tr} C G^\dagger = n/2$$

By Cauchy-Schwarz, this can only happen if the column (and row) spaces of C and G are equal. This gives

$$\mathrm{col}(C) = \mathrm{col}(G) \perp \mathrm{col}(H) = \mathrm{col}(F) \perp \mathrm{col}(D) = \mathrm{col}(B) \perp \mathrm{col}(C)$$

a contradiction.

How About Strongly Factorizable?

We need to show that we cannot approximate this unital map with an exactly factorizable map. Thie proof goes along the same lines, except that we need to make lots of approximations.

We won't go into the details in this talk.

Related Work

After we did this research, we discovered that Haagerup and Musat had done similar work.

Definition A factorizable map is one which can be represented as

$$\Phi(\rho) = \mathrm{tr}_2 U(\rho \otimes \tau)U^\dagger$$

where U is an automorphism of an operator algebra $M_d(\mathcal{V})$, the set of $d \times d$ matrices over a von Neumann algebra $\mathcal{V}$, τ is a tracial state, and tr is a normalized trace.

The difference between factorizable and strongly factorizable is that for strongly factorizable, $\mathcal{V}$ has to be M_n (and we are allowed to take the limit $n \to \infty$). For von Neumann algebras, one can show that no limit is necessary.

A Non-Factorizable Map

Haagerup and Musat showed that the map $\Phi(\rho) = \sum_i A_i \rho A_i^\dagger$ is not factorizable, where the A_i are

$$A_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; A_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} [A_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

They also have a number of other results showing that different maps were not factorizable (some based on previous work).

Unital

Factorizable

Strongly Factorizable

Exactly Factorizable

AQBP

Mixtures of Unitaries

Unital

$| \neq$

Factorizable

$|$

Strongly Factorizable

$|$

Exactly Factorizable

$|$

AQBP

$| \neq$

Mixtures of Unitaries

Unital

$\quad | \neq$

Factorizable

$\quad | =$ iff Connes embedding conjecture is true

Strongly Factorizable

$\quad |$

Exactly Factorizable

$\quad |$

AQBP

$\quad | \neq$

Convex Combination of Unitaries

Convex Combination of Unitaries $\subsetneq$ AQBP

Mendl and Wolf, 2009, computationally investigated the set of unital channels, and found cases where $\Phi \otimes \Phi$ was a convex combination of unitaries, when Φ was not.

Schur contractions

A subclass of quantum operation are Schur contractions. In these maps, an input matrix ρ_{ij} goes to $\alpha_{ij}\rho_{ij}$, where α_{ij} is a Hermitian matrix with diagonal 1 (so $\alpha_{ii} = 1$).

Recall from Choi's theorem that a trace-preserving linear map Φ is a quantum operation if and only if, when $\Phi \otimes I$ is applied to the maximally entangled state $\frac{1}{d}\sum_{i=1}^{d} |\, ii\rangle \sum_{i=1}^{d} \langle ii\,|$, the result is positive semidefinite.

Thus, for a Schur contration to be a quantum operation, we require that the matrix

$$\frac{1}{d}\sum_{i,j} \alpha_{ij}\,|\, ii\rangle\,\langle jj\,|$$

is positive semidefinite. This is equivalent to the matrix $[\alpha_{ij}]$ being positive semidefinite.

More on Schur Contractions

Suppose we have a Schur contraction that is factorizable.

Then

$$U \,|\, i\rangle \,|\, \phi\rangle = |\, i\rangle\, U_i \,|\, \phi\rangle$$

We have

$$\mathrm{tr}_2 U(|\, i\rangle \langle j \,| \otimes I)U^\dagger = |\, i\rangle \langle j \,| \,\mathrm{tr} U_i U_j^\dagger .$$

Thus $\alpha_{ij} = \mathrm{tr} U_i U_j^\dagger$.

Here, tr_2 is a normalized trace, and in dimension n should be thought of as $\frac{1}{n}\mathrm{Tr}_2$.

Connes' Embedding Conjecture

Let $\mathcal{E}_d$ be the set of exactly factorizable Schur contractions in dimension d (i.e. $[\alpha_{ij}]$ where where $\alpha_{ij} = \mathrm{Tr} U_i U_j^\dagger$).

Let $\mathcal{G}_d$ be the set of factorizable Schur contractions in dimension d.

Theorem (Dykema and Juschenko)
Closure$(\mathcal{E}_d) = \mathcal{G}_d$ iff Connes' embedding conjecture is true.

Connes' Embedding Conjecture

Theorem: Connes' embedding conjecture is true iff the set of strongly factorizable maps is the set of factorizable maps.

$\leftarrow$ By the results of Dykema and Juschenko, we only need to prove that every strongly factorizable Schur contraction is the limit of exactly factorizable Schur contractions.

$\rightarrow$ This follows from the statement of Connes' embedding conjecture and theorems about von Neumann algebras.