Lecture 2: Generalized Pseudo-Orbits (gpos)

Symbolic dynamics for surface diffeomorphisms

O. Sarig

Weizmann Institute of Science, Israel

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What we want to do

Setup

$C^{1+\beta}$–surface diffeo with positive topological entropy

Aim

Construct a countable Markov partition

Strategy

Define generalized pseudo-orbits (gpos)......

...... so that can apply Bowen's method for constructing MP

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What the definition of gpos must satisfy:

1. nearest neighbor conditions
2. Every gpo “shadows" a real orbit
3. Countable alphabet suffices
4. **Inverse problem:** Suppose a p.o. $(v_k)_{k\in\mathbb{Z}}$ shadows the orbit of x. Then we can “read" v_k from x “approximately". *

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Today: describe a definition which works.

Preparations ⟶ Review of Pesin Theory

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Pesin charts

Setup: $f : M \to M$ is a $C^{1+\beta}$ surface diffeomorphism with positive entropy

Theorem (Pesin)

"Almost every" x has a neighborhood with a system of coordinates $\Psi_x : [-Q(x), Q(x)]^2 \to M$ *s.t.*

$$\Psi_{f(x)}^{-1} \circ f \circ \Psi_x : [-Q, Q]^2 \to \mathbb{R}^2 \simeq \textit{linear hyperbolic map}$$

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Surface diffeos with $h_{top}(f) > 0$ are non-uniformly hyperbolic

Fix $0 < \chi < h_{top}(f)$ (as small as you wish)

Theorem (Oseledets Ergodic Theorem, Ruelle Inequality)

The following invariant set $NUH_\chi(f)$ has full measure w.r.t. any ergodic invariant μ s.t. $h_\mu(f) > \chi$: The set of $x \in M$ s.t. $T_x M = E^s(x) \oplus E^u(x)$ where

1. $E^{u/s}(x)$ *are one-dimensional*
2. $\lim_{n\to\infty} \frac{1}{n} \log \|df_x^n \underline{v}\|_{f^n(x)} < -\chi$ *on* $E^s(x) \setminus \{\underline{0}\}$
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Diagonalizing the differential $df : TM \to TM$

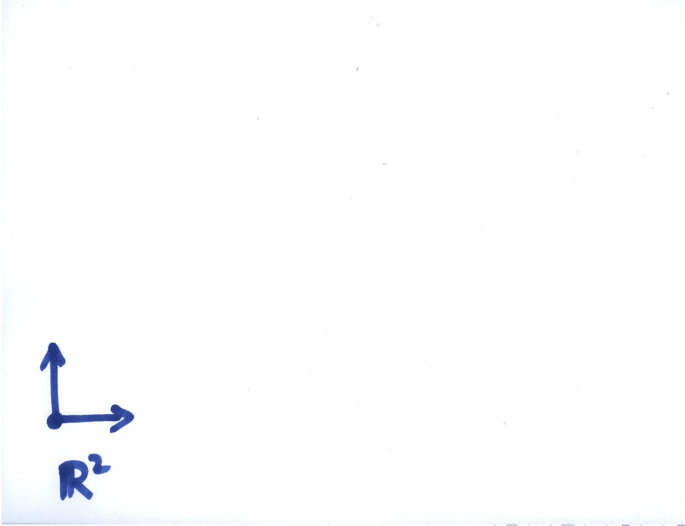
Diagonalizing the differential

Choose unit vectors $e^s(x) \in E^s(x)$, $e^u(x) \in E^u(x)$ and scalars $s(x), u(x) \geq \sqrt{2}$

Oseledets-Pesin transformation

Linear $C(x) : \mathbb{R}^2 \to T_x M$ which maps the standard basis (e^1, e^2) to $(s(x)^{-1} e^s(x), u(x)^{-1} e^u(x))$

Diagonalizing the differential



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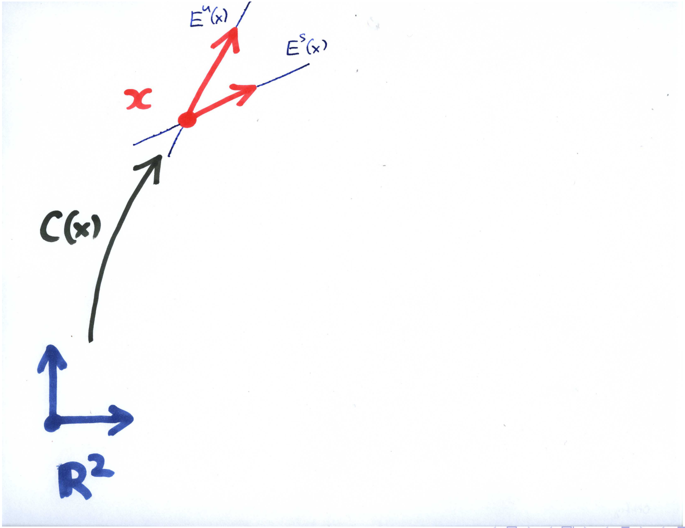

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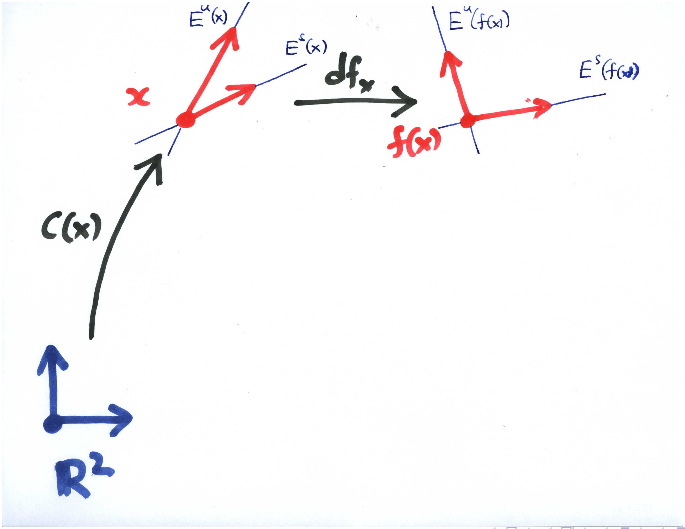

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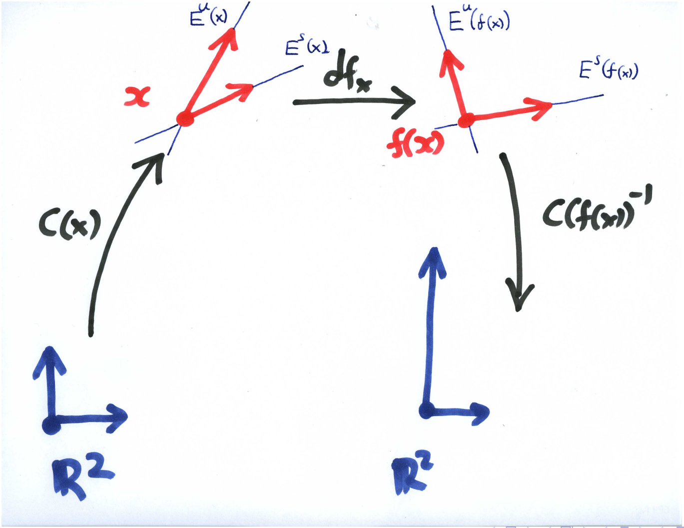

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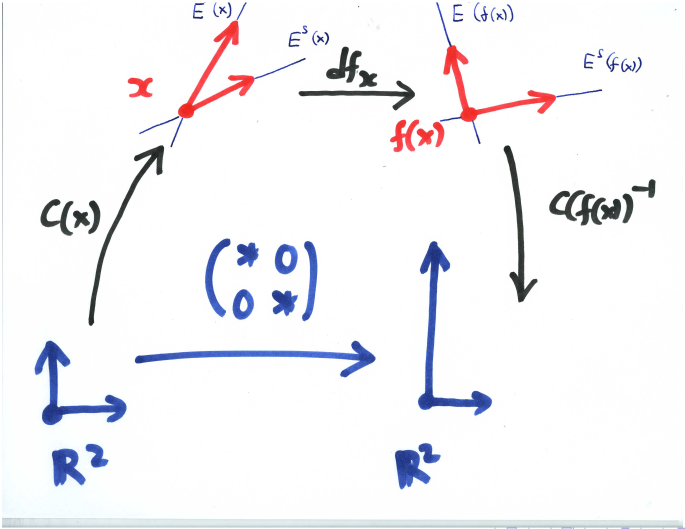

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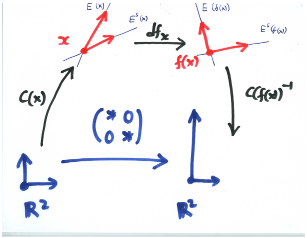

Choice of $s(x), u(x)$

There is a choice of $s(x), u(x)$ s.t. $\|C(x)\| \leq 1$ and

$$C(f(x))^{-1} df_x C(x) = \begin{pmatrix} \lambda(x) & 0 \\ 0 & \mu(x) \end{pmatrix}$$

where $|\lambda(x)| < e^{-\chi}$, $|\mu(x)| > e^{\chi}$ are bounded away from $0, \infty$

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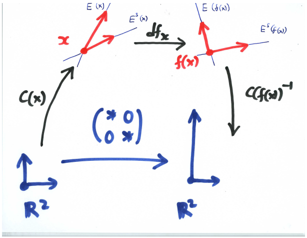

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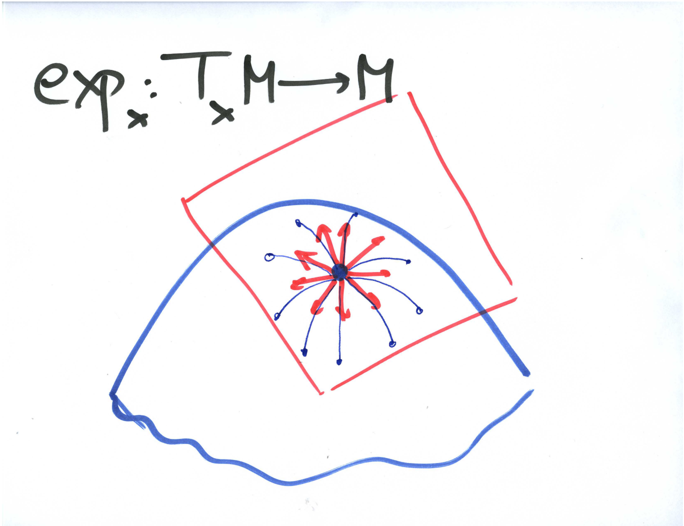
Diagonalizing the map $f : M \to M$

Reminder: the exponential map



Fix constants

- $0 < \chi < h_{top}(f)$ ("hyperbolicity bound")
- $0 < \epsilon \ll \chi$ ("closeness bound")

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$\Psi_x : \mathbb{R}^2 \to M$, $\Psi_x : \binom{\xi}{\eta} \mapsto \exp_x\left[C(x)\binom{\xi}{\eta}\right]$

- $\Psi_{f(x)}^{-1} \circ f \circ \Psi_x : \mathbb{R}^2 \to \mathbb{R}^2$
- $\underline{0} \mapsto \underline{0}$
- Derivative at the origin: $C(f(x))^{-1} \circ df_x \circ C(x) = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$
- So $\Psi_{f(x)}^{-1} \circ f \circ \Psi_x \approx$ linear hyperbolic map near the origin

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$\approx$ *means ϵ–close in $C^{1+\frac{\beta}{2}}$*

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The size of the chart

f is $C^{1+\beta}$

The size of the chart

$$Q(x) = \epsilon^{3/\beta}\left(\frac{\sqrt{u(x)^2+s(x)^2}}{|\sin\alpha(x)|}\right)^{-\frac{12}{\beta}}$$

- $\alpha(x) = \measuredangle(E^s(x), E^u(x))$
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$\frac{1}{n}\log Q(f^n(x)) \xrightarrow[n\to\pm\infty]{} 0$ *a.e. w.r.t. any ergodic invariant measure with entropy* $> \chi$.

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$Q(x) \geq q(x)$ *where* $e^{-\frac{1}{3}\epsilon} \leq \frac{q(f(x))}{q(x)} \leq e^{\frac{1}{3}\epsilon}$

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where $\lfloor Q \rfloor_\epsilon := \max\{q \in I_\epsilon : q \leq Q\}$ and $I_\epsilon = \{e^{-\frac{1}{3}\ell\epsilon} : \ell \geq 1\}$

The definition of “generalized pseudo-orbits"

Basic idea

	exact
Orbits	$(x_i)_{i\in\mathbb{Z}}$ s.t. for all i $x_{i+1} = f(x_i)$

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Defining $\Psi_x \approx \Psi_y$ (intuitive)

$\Psi_x : [-Q(x), Q(x)]^2 \to M$ is a Pesin chart, $Q(x)$ =size

- **Notation:** Ψ_x^p denotes $\Psi_x : [-p, p]^2 \to M$ $(0 < p \leq Q(x))$
- $\Psi_x^p \overset{\epsilon}{\approx} \Psi_y^q$ (**ϵ–overlap**): Intuitive definition
  - $\Psi_x([-p, p]^2) \approx \Psi_y([-q, q]^2)$
  - $\Psi_y^{-1} \circ \Psi_y \approx Id$
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Defining $\Psi_x \approx \Psi_y$ (formal definition)

$\Psi_x^p \overset{\epsilon}{\approx} \Psi_y^q$ (ϵ–overlap)

1. $e^{-\epsilon} < \frac{p}{q} < e^{\epsilon}$
2. **Similar charted areas:** $\Psi_x(e^{-2\epsilon} \cdot [-p,p]^2) \subset \Psi_y([-q,q]^2)$ $\Psi_y(e^{-2\epsilon} \cdot [-q,q]^2) \subset \Psi_x([-p,p]^2)$
3. **Similar chart maps:** $\Psi_x^{-1} \circ \Psi_y$, $\Psi_y^{-1} \circ \Psi_x$ are $\epsilon p^2 q^2$–close to *id* in $C^{1+\frac{\beta}{2}}$.

Designed to achieve:

If $\Psi_y^q \overset{\epsilon}{\approx} \Psi_{f(x)}^q$, then $\Psi_y^{-1} \circ f \circ \Psi_x \approx$ linear hyperbolic map where $\approx$ is ϵ–closeness in $C^{1+\frac{\beta}{3}}$.

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Generalized pseudo-orbits

Ψ_x^p denotes $\Psi_x : [-p, p]^2 \to M$ $(0 < p \leq Q(x))$

Symbols

Double charts $\Psi_x^{p^u, p^s} := (\Psi_x^{p^u}, \Psi_x^{p^s})$, where $0 < p^u, p^s \leq Q(x)$, $p^u, p^s \in I_\epsilon = \{e^{-\frac{1}{3}\ell\epsilon} : \ell \geq 1\}$.

Nearest neighbor conditions

$\Psi_x^{p^u, p^s} \to \Psi_y^{q^u, q^s}$ if

1. $\Psi_{f(x)}^{q^u \wedge q^s} \overset{\epsilon}{\approx} \Psi_y^{q^u \wedge q^s}$ and $\Psi_{f^{-1}(y)}^{p^u \wedge p^s} \overset{\epsilon}{\approx} \Psi_x^{p^u \wedge p^s}$ (where $a \wedge b := \min\{a, b\}$)
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$\{\Psi_{x_i}^{p_i^u, p_i^s}\}_{i\in\mathbb{Z}}$ s.t. $\Psi_{x_i}^{p_i^u, p_i^s} \to \Psi_{x_{i+1}}^{p_{i+1}^u, p_{i+1}^s}$ for all $i \in \mathbb{Z}$.

Shadows a real orbit $\{f^i(x)\}_{i\in\mathbb{Z}}$ if

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Existence of gpos

Question

When are there p_i^u, q_i^s s.t. $\{\Psi_{f^i(x)}^{p_i^u, p_i^s}\}_{i\in\mathbb{Z}}$ is a gpo?

- $0 < p_i^{u/s} \leq Q(f^i(x))$
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Whenever $\frac{1}{n}\log Q_n(x) \xrightarrow[|n|\to\infty]{} 0$.

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Pesin's Temperdness Lemma: $\frac{1}{n}\log Q(f^n(x)) \xrightarrow[n\to\infty]{} 0$ a.e. for every ergodic μ s.t. $h_\mu(f) > \chi$

Pesin Temperdness Lemma

There is a function $0 < q \leq Q$ on $\{x : \frac{1}{n}\log Q\circ f^n \to 0\}$ s.t.

$$e^{-\epsilon} < \frac{q\circ f}{q} < e^{\epsilon}$$

Construction

1. $p_i^s := \max\{t \in I_\epsilon : Q(f^{i+k}(x)) \geq e^{-k\epsilon} t \text{ for all } k \geq 0\}$
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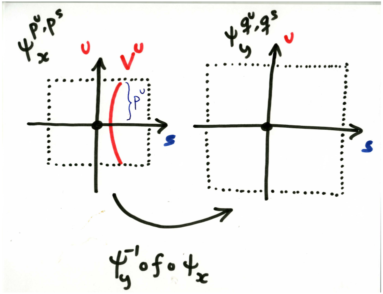
What the nearest neighbor conditions mean

If $\Psi_x^{p^u,p^s} \to \Psi_y^{q^u,q^s}$, then

- $\Psi_y^{-1} \circ f \circ \Psi_x \approx \Psi_{f(x)}^{-1} \circ f \circ \Psi_x \approx \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$
  where $|\lambda| < e^{-\chi}$, $|\mu| > e^{\chi}$.
- $q^u = \min\{e^{\epsilon}p^u, Q(y)\} \le e^{\epsilon}p^u <$ expansion $\times\ p^u$



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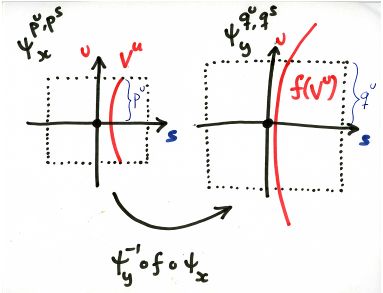

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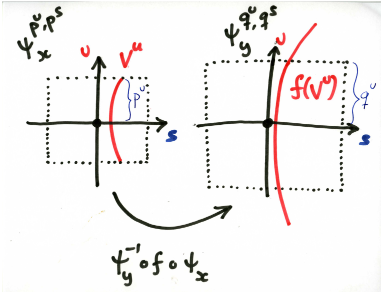

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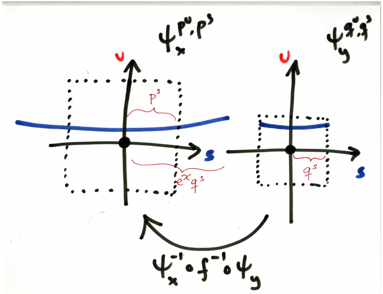

Graph transform

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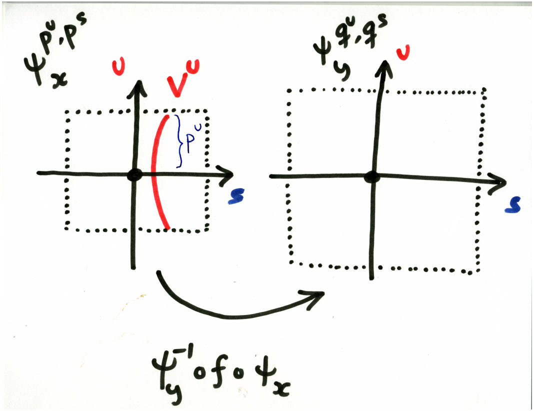

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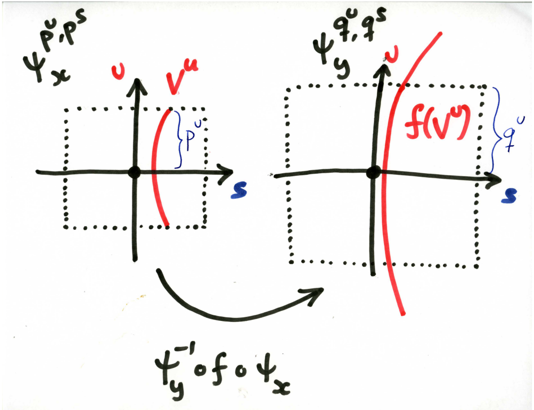

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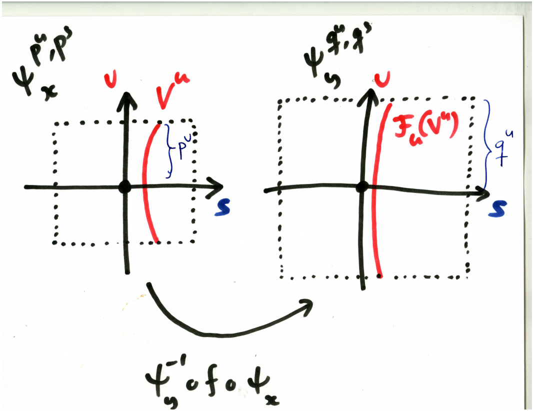
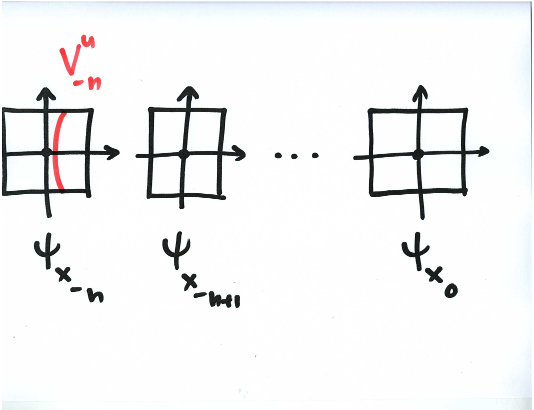


Graph transform

Application: Pesin's Unstable Manifold Theorem

The unstable manifold of a gpo

$\{\Psi_{x_i}^{p_i^u, p_i^s}\}_{i\in\mathbb{Z}}$ is a gpo



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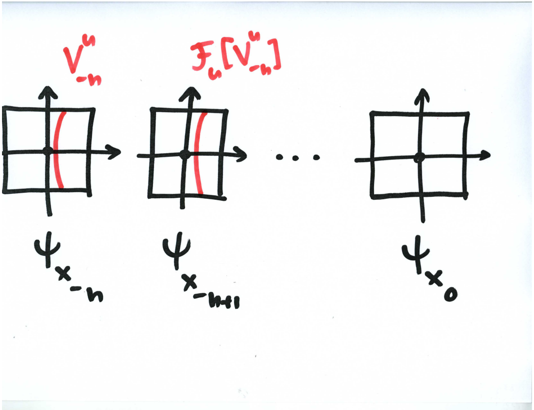

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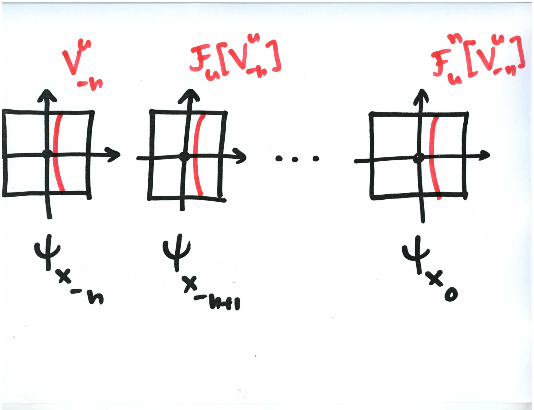

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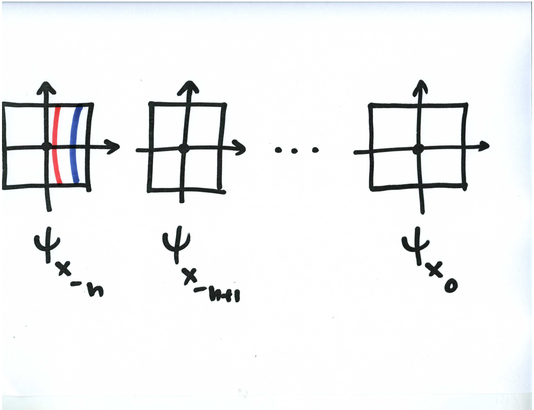

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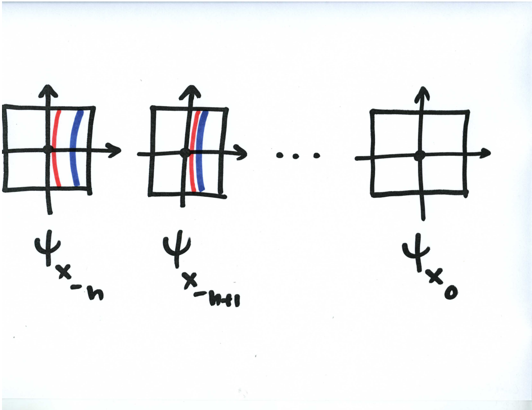

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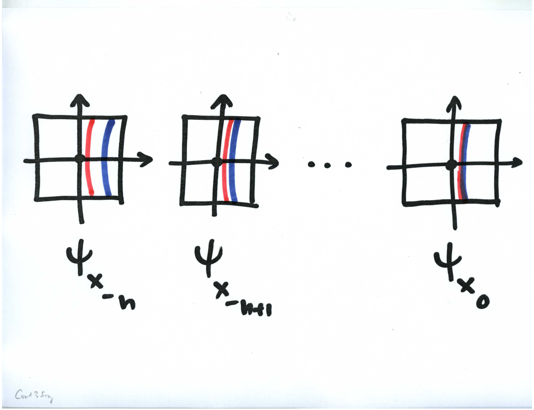

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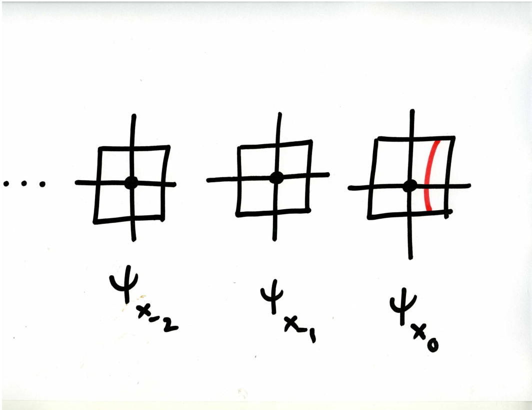
Unstable manifold of a gpo

The following limit exists:

$$V^u[\{\Psi_{x_i}^{p_i^u, p_i^s}\}_{i\leq 0}\}_{i\leq 0}] = \lim_{n\to\infty} \mathcal{F}_u^n[V_{-n}^u]$$

for some (any) choice of "u–manifolds" V_{-n}^u in $\Psi_{x_{-n}}^{p_n^u, p_n^s}$.

"Staying in windows"



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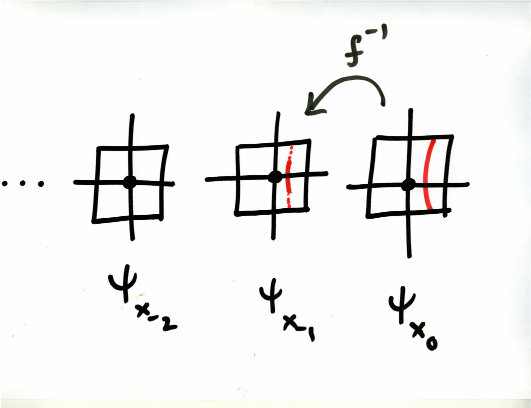

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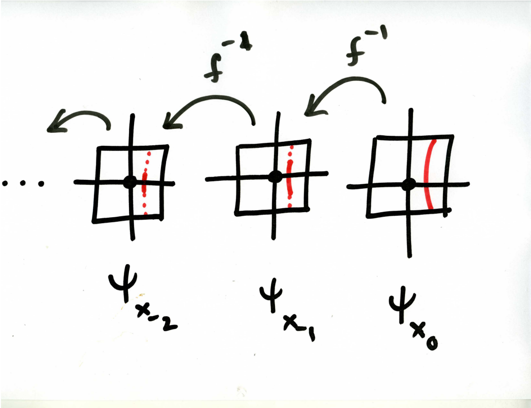

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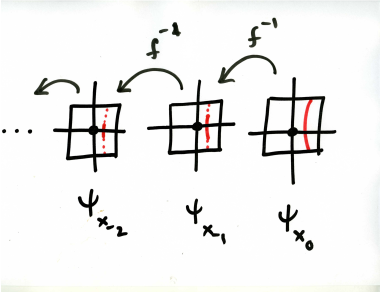

Corollary: $V^u[\{\Psi_{x_i}^{p_i^u,p_i^s}\}]$ is a local unstable manifold

- it is tangent to E^u
- f^{-1} contracts exponentially on $V^u[\{\Psi_{x_i}^{p_i^u,p_i^s}\}]$

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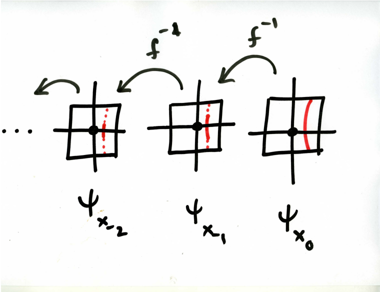

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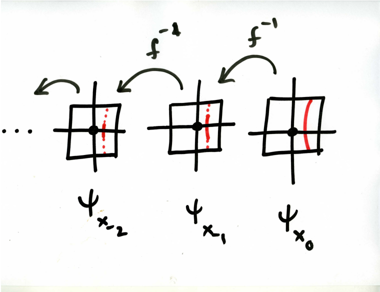

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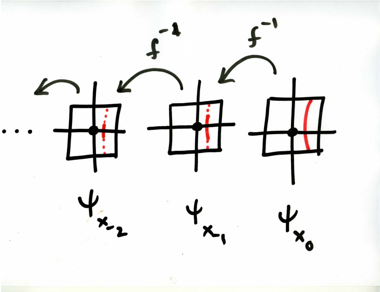

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Pesin's Unstable Manifold Theorem

Suppose μ is an ergodic measure with positive entropy for a $C^{1+\beta}$–surface diffeo.

Pesin's Unstable Manifold Theorem

A.e. $x \in M$ lies on a one dimensional manifold $V^u(x)$ s.t.

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