

Lecture 3: Shadowing Theory

Symbolic dynamics for surface diffeomorphisms

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What we want to do

Setup

$C^{1+\beta}$ -surface diffeo with topological entropy $> \chi$

Aim

Construct a countable Markov partition

Strategy

Define **generalized pseudo-orbits** (gpos).....

..... so that can apply Bowen's method for constructing MP

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Generalized pseudo-orbits

Pesin charts

Maps $\Psi_x : [-Q(x), Q(x)]^2 \rightarrow M$ s.t.

$\Psi_{f(x)}^{-1} \circ f \circ \Psi_x \approx$ linear hyperbolic map

Generalized pseudo-orbits

Sequences $(\Psi_{x_i}^{q_i^u, q_i^s})_{i \in \mathbb{Z}}$ of “double charts”

$\Psi_{x_i}^{q_i^u, q_i^s} := (\Psi_{x_i}|_{[-q_i^u, q_i^u]^2}, \Psi_{x_i}|_{[-q_i^s, q_i^s]^2})$, $0 < q_i^u, q_i^s \leq Q(x_i)$

satisfying certain nearest neighbor conditions.

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Nearest neighbor conditions

$\Psi_x^{p^u, p^s} \rightarrow \Psi_y^{q^u, q^s}$ if

- ① $\Psi_{f(x)}^{q^u \wedge q^s} \stackrel{\epsilon}{\approx} \Psi_y^{q^u \wedge q^s}$ and $\Psi_{f^{-1}(y)}^{p^u \wedge p^s} \stackrel{\epsilon}{\approx} \Psi_x^{p^u \wedge p^s}$ where ($a \wedge b := \min\{a, b\}$)
- ② $q^u = \min\{e^\epsilon p^u, Q(y)\}$ (\rightarrow)
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To apply Bowen's method, we must show

- ① **Shadowing Lemma:** Every gpo “shadows” a real orbit
- ② **Inverse Theorem:** Suppose a p.o. $(v_k)_{k \in \mathbb{Z}}$ shadows the orbit of x . Then we can “read” v_k from x “approximately”. *

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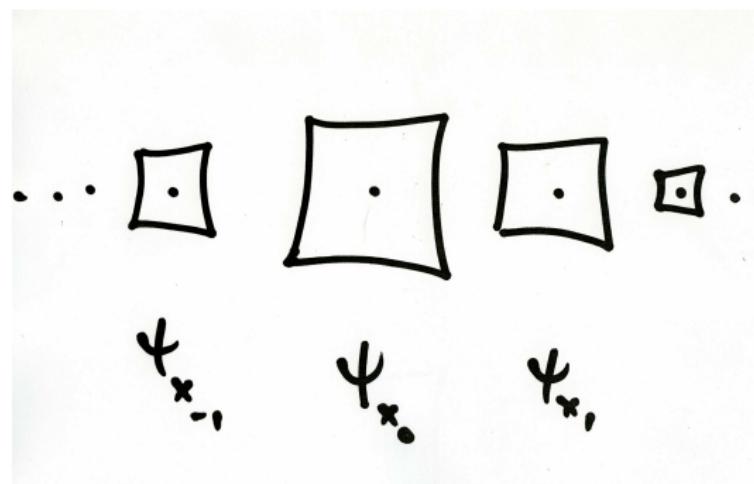
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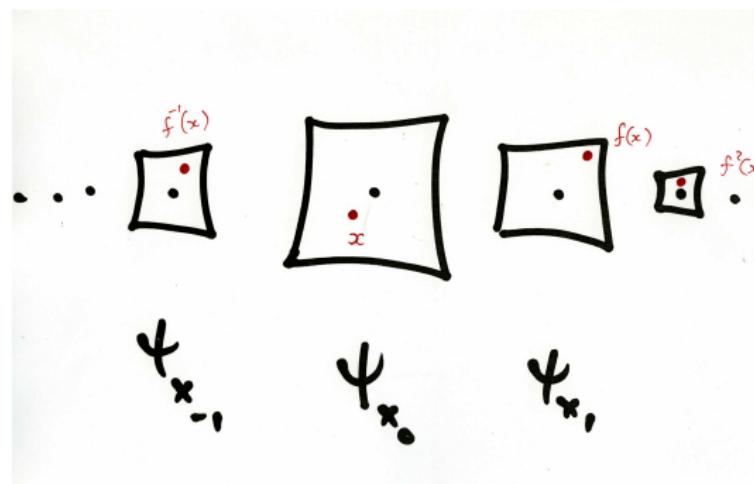
Shadowing Lemma

Definition of shadowing



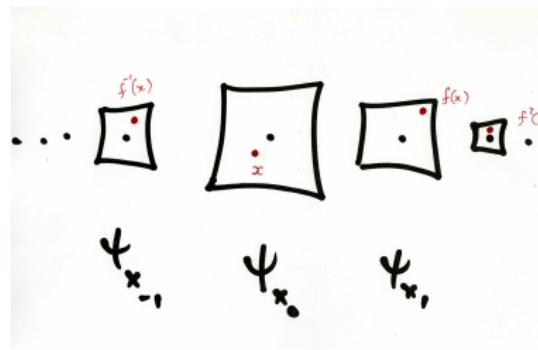
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A gpo $(\Psi_{x_i}^{q_i^u, q_i^s})_{i \in \mathbb{Z}}$ shadows a real orbit $\{f^i(x)\}_{i \in \mathbb{Z}}$ if $f^i(x) \in \Psi_{x_i}([-Q(x_i), Q(x_i)]^2)$ for all $i \in \mathbb{Z}$.

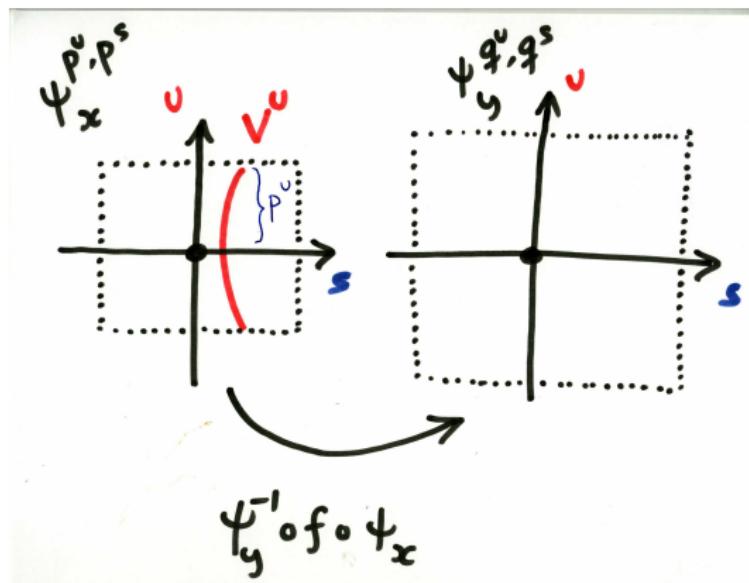


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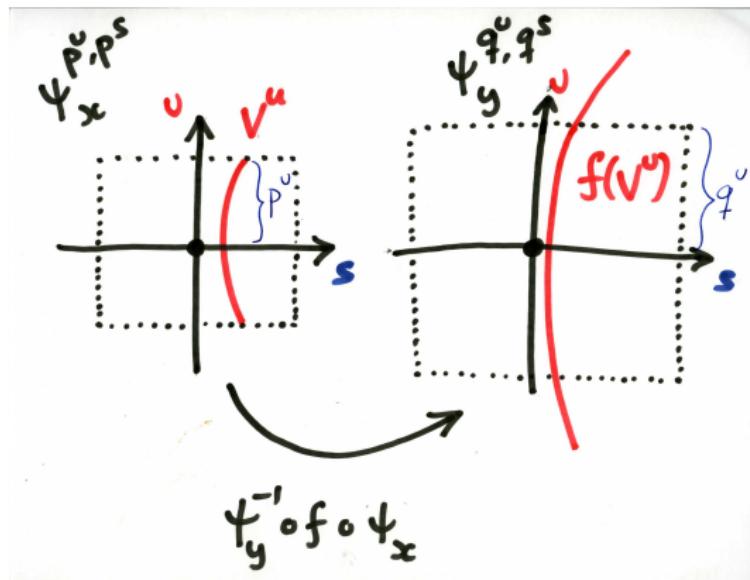
Shadowing Lemma

Every gpo shadows a unique orbit.

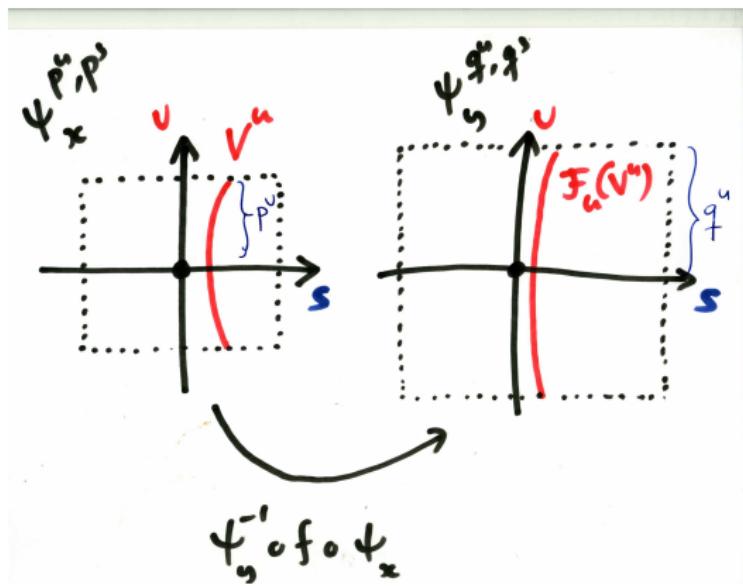
Graph transform



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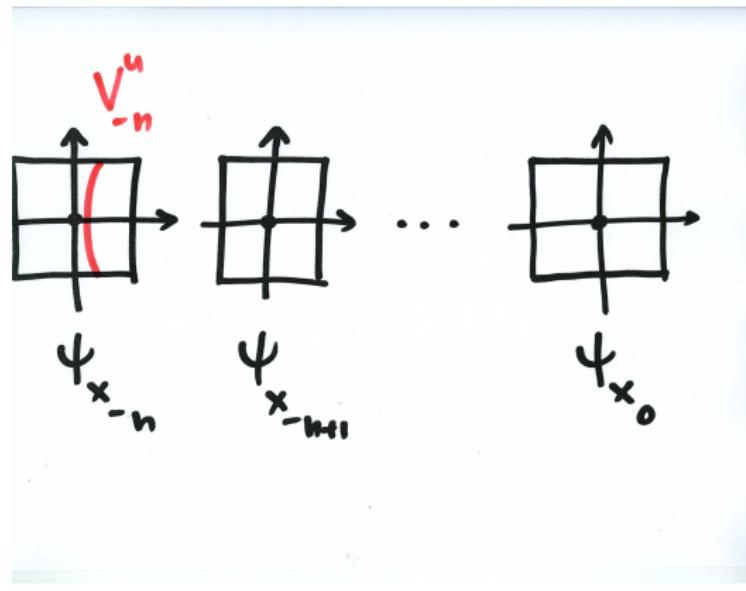
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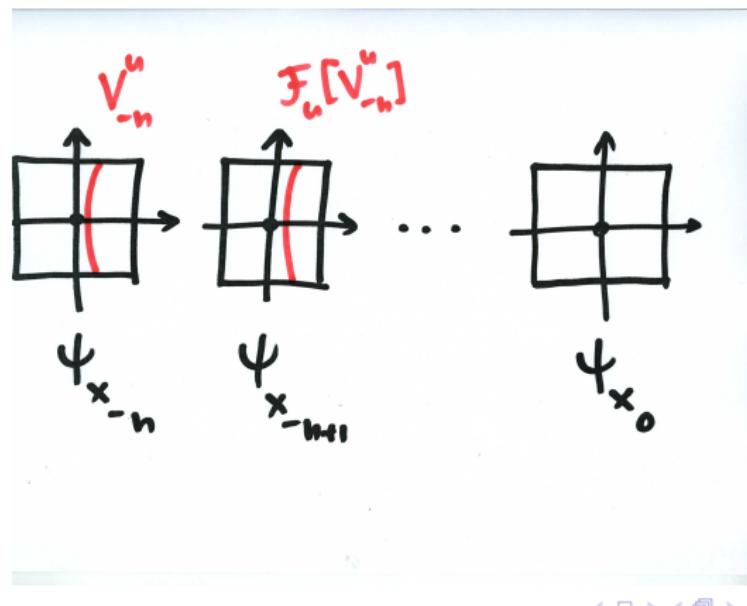
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$\{\Psi_{x_i}^{p_i^u, p_i^s}\}_{i \in \mathbb{Z}}$ is a gpo



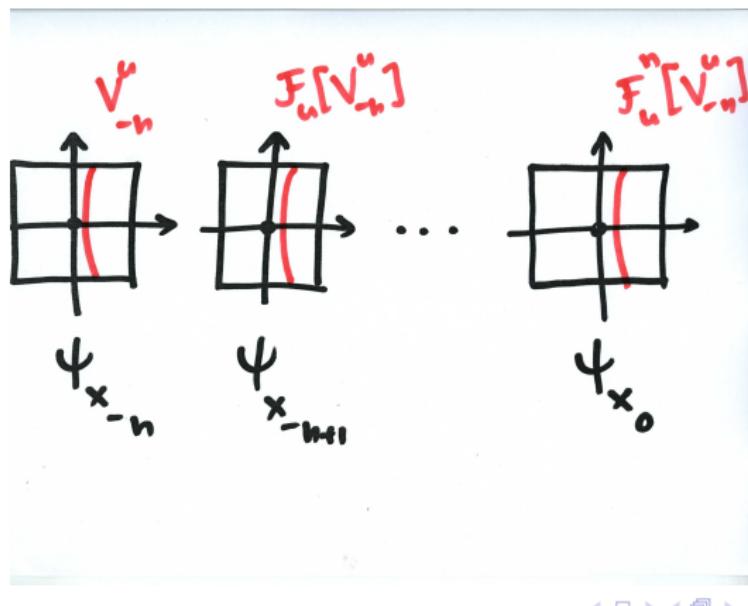
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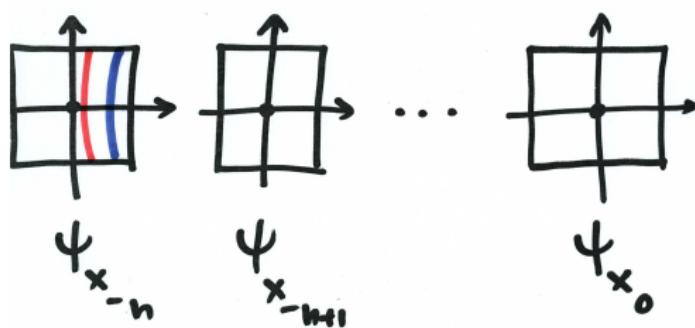
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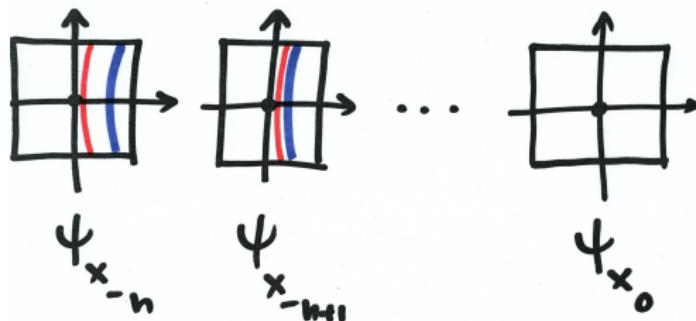
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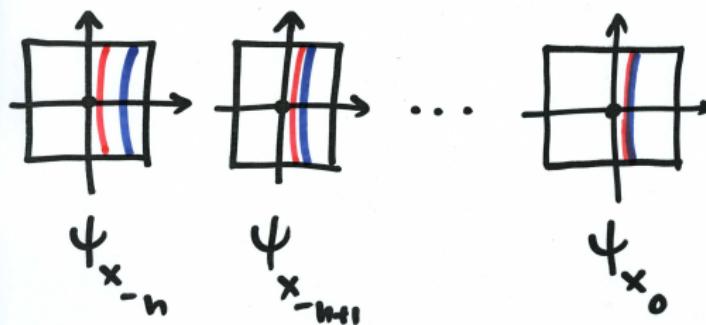
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Unstable manifold of a gpo

The following limit exists: $V^u[\{\Psi_{x_i}^{p_i^u, p_i^s}\}_{i \leq 0}] = \lim_{n \rightarrow \infty} \mathcal{F}_u^n[V_{-n}^u]$ for some (any) choice of "u-manifolds" V_{-n}^u in $\Psi_{x_{-n}}^{p_n^u, p_n^s}$.

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Invariance properties of stable/unstable manifolds

Suppose $(\psi_i)_{i \in \mathbb{Z}}$ is a gpo, where $\psi_i = \psi_{x_i}^{q_i^u, q_i^s}$. Then

Invariance properties

- $f \circ V^s \subset V^s \circ \sigma$:

$$f(V^s[(\psi_i)_{i \geq 0}]) \subset V^s[(\psi_i)_{i \geq 1}]$$

- $f^{-1} \circ V^u \subset V^u \circ \sigma^{-1}$:

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Analytic properties of stable/unstable manifolds

Represent $V^{s/u} = V^{s/u}[\underline{\psi}]$ in Ψ_{x_0} -coordinates:

$$V^s = \Psi_{x_0} (\{(t, F(t)) : |t| \leq q_0^s\})$$

$$V^u = \Psi_{x_0} (\{(\underline{F}(t), t) : |t| \leq q_0^u\})$$

The representing function F satisfies

- $|F(0)| \leq 10^{-3}(q^u \wedge q^s)$
- $|F'(0)| \leq \frac{1}{2}(q^u \wedge q^s)^{\beta/3}$
- $\|F'\|_{C^{\beta/3}} \leq \frac{1}{2}$

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Proof of the shadowing lemma

Suppose $(\Psi_i)_{i \in \mathbb{Z}}$ is a gpo, $\Psi_i = \Psi_{x_i}^{q_i^u, q_i^s}$.

- ➊ Construct $V^u = V^u[(\Psi_i)_{i \leq 0}]$ and $V^s = V^s[(\Psi_i)_{i \geq 0}]$
- ➋ V^u, V^s intersect at a unique point x (analytic properties)
- ➌ the orbit of the intersection point is shadowed by $(\Psi_i)_{i \in \mathbb{Z}}$:
 - $f^n(x) \in f^n(V^s[(\Psi_i)_{i \geq 0}])$

Goal: Show that $f^n(x) \in f^n(V^s[(\Psi_i)_{i \geq 0}])$

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- ➌ the orbit of the intersection point is shadowed by $(\Psi_i)_{i \in \mathbb{Z}}$:
 - $f^n(x) \in f^n(V^s[(\Psi_i)_{i \geq 0}]) \subset V^s[(\Psi_i)_{i \geq n}] \subset \Psi_{x_n}([-Q(x_n), Q(x_n)]^2)$
 - $f^{-n}(x) \in f^{-n}(V^u[(\Psi_i)_{i \leq 0}]) \subset V^u[(\Psi_i)_{i \leq -n}] \subset \Psi_{x_{-n}}([-Q(x_{-n}), Q(x_{-n})]^2)$

Inverse Theorem

Statement of the Inverse Theorem

Imprecise

Theorem

Any two regular gpos $(\Psi_i)_{i \in \mathbb{Z}}, (\Phi_i)_{i \in \mathbb{Z}}$ which shadow the same orbit have “nearly the same” coordinates: $\Psi_i \approx \Phi_i$.

Regular gpos Ψ_i, Φ_i which shadow the same orbit

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Regular gpos \Rightarrow $\Psi_i = \Phi_i$

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- ① repeats some symbol infinitely often in the future
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The following is true for all ϵ small enough.

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- 1 $d(x_i, y_i) < \epsilon$
- 2 $\text{dist}_{C^1}(\Psi_{y_i}^{-1} \circ \Psi_{x_i}, \pm \text{Id}) < \sqrt[3]{\epsilon}$ on $[-\epsilon, \epsilon]^2$
- 3 $p_i^u/q_i^u, p_i^s/q_i^s \in [e^{-\sqrt[3]{\epsilon}}, e^{\sqrt[3]{\epsilon}}]$

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- assume that two regular gpos shadow the same orbit
- compare the **parameters** of the corresponding coordinates

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Parameters of Pesin charts

$\Psi_x := \exp_x \circ C_\chi(x)$:

- $x \in M$
- $C(x) : \mathbb{R}^2 \rightarrow T_x M$ is the linear map s.t.

$$C(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = s(x)^{-1} \underline{e}^s(x), \quad C(x) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = u(x)^{-1} \underline{e}^u(x)$$

- $\underline{e}^{s/u}(x)$ are unit vectors in the stable/unstable direction at x
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Parameters of double charts

$$\Psi_{x_i}^{p_i^u, p_i^s} = (\Psi_{x_i}|_{[-p_i^u, p_i^u]^2}, \Psi_{x_i}|_{[-p_i^s, p_i^s]^2})$$

- ➊ Position parameters: x_i
- ➋ Axes parameters: $e^s(x_i), e^u(x_i), \alpha(x_i) := \angle(e^s(x_i), e^u(x_i))$
- ➌ Scaling parameters: $s(x_i), u(x_i)$
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Strategy

Compare the parameters of gpos which shadow the same point.

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Position parameters

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Proof

$$\begin{aligned} d(x_i, y_i) &\leq d(x_i, f^i(x)) + d(f^i(x), y_i) \\ &< \text{diam}(\Psi_{x_i}) + \text{diam}(\Psi_{y_i}) < \epsilon \end{aligned}$$

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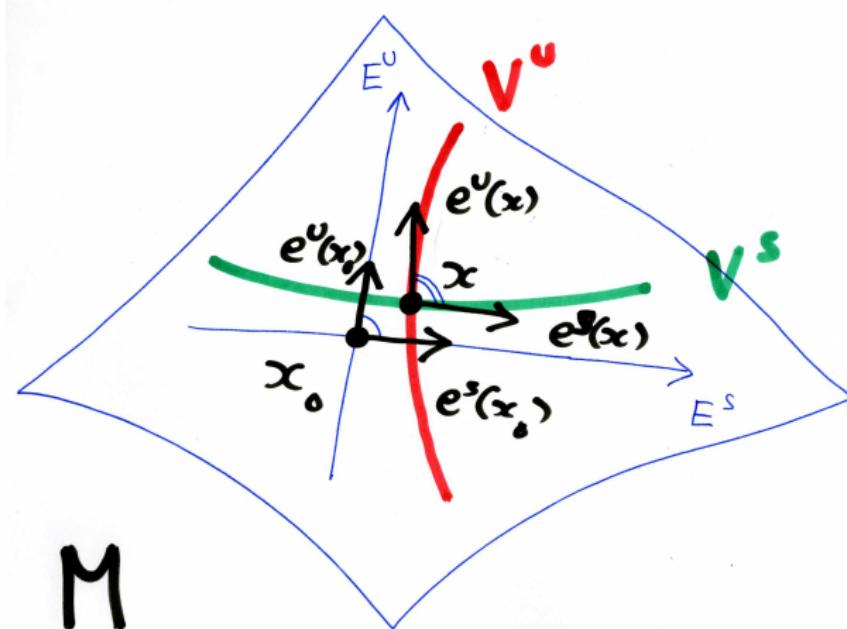
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Delicate point:

Trivial in Pesin charts. The point is that the estimate hold in the Riemannian metric uniformly.

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If $(\Psi_{x_i}^{p_i^u, p_i^s})_{i \in \mathbb{Z}}$ shadows the orbit of x , then

- ① $\underline{e}^s(x_0) \approx \pm \text{direction of } V^s[\Psi]$
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Scaling parameters $s(\cdot), u(\cdot)$

$$s(x) = \sqrt{2} \left(1 + \sum_{n=1}^{\infty} e^{2nx} \| df_x^n e^s(x) \|^2 \right)^{\frac{1}{2}}$$
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Proposition

If two **regular** gpos $(\psi_{x_i}^{p_i^u, p_i^s})_{i \in \mathbb{Z}}, (\psi_{y_i}^{q_i^u, q_i^s})_{i \in \mathbb{Z}}$ shadow the same orbit, then

$$\frac{s(x_i)}{s(y_i)}, \frac{u(x_i)}{u(y_i)} \in [e^{-\sqrt[3]{\epsilon}}, e^{\sqrt[3]{\epsilon}}] \text{ for all } i \in \mathbb{Z}.$$

Window parameters

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Informal reason why this is true

Recall:

- $Q(x_i)$ = maximal possible size of $\text{dom}(\Psi_{x_i})$
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Greedy algorithm!

$p_i^{u/s}$ are almost maximal subject to the following conditions:

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Formal statement

Definition

A regular half-gpo $(\Psi_i)_{i \leq 0}$ is **almost maximal** if for any

- regular gpo extension $(\Psi_i)_{i \in \mathbb{Z}}$, and
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Proof that regular gpos are almost maximal

Suppose $(\Psi_i)_{i \in \mathbb{Z}}, (\Phi_i)_{i \in \mathbb{Z}}$ are **regular** gpos which shadow the same orbit.

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- ① $Q(x_i)/Q(y_i) \in [e^{-\sqrt[3]{\epsilon}}, e^{\sqrt[3]{\epsilon}}]$ for all i
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We have proved

Two regular gpos which shadow the same orbit have nearly the same parameters at every coordinate.

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- ➊ Discretize the set of double charts into a locally finite sufficient countable collection \mathcal{A}
- ➋ Rectangles: $Z(\Psi) = \{x : x \text{ is shadowed by a regular gpo } (\Psi_i)_{i \in \mathbb{Z}} \text{ s.t. } \Psi_0 = \Psi\}$
- ➌ Inverse theorem: $\{Z(\Psi) : \Psi \in \mathcal{A}\}$ is locally finite
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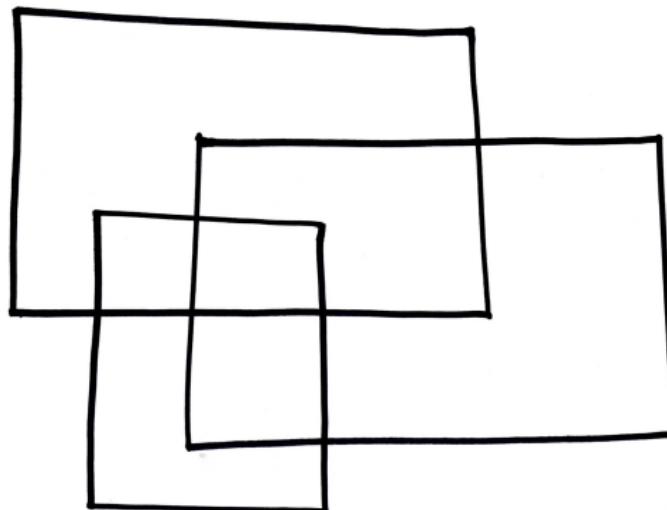
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Bowen Sinai Refinement



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