Symbolic dynamics for surface diffeomorphisms

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Symbolic coding

- **Setup:** surface diffeomorphism $f : M \to M$
- **Orbits:** $f^n(x) := (f \circ \cdots \circ f)(x)$
- **Fundamental difficulty:** Direct calculation is difficult

Symbolic coding

A change of coordinates which transforms $f : M \to M$ to the action of the left shift on a space of sequences.

Easy to iterate!

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Itineraries

$\mathcal{R}$ a partition of M.

Itinerary of $x \in M$:

$(R_i)_{i\in\mathbb{Z}} \in \mathcal{R}^{\mathbb{Z}}$ s.t. $f^i(x) \in R_i$ for all i

Why this is useful:

f acts on itineraries by the left shift. Easy to iterate!

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What sequences arise as itineraries?

$\mathcal{R}$ is a partition

Every $\mathcal{R}$–itinerary is a walk on the dynamical graph of $\mathcal{R}$:

- Vertices: partition elements
- Edges: $R_1 \to R_2$ when $\exists x \in R_1$ s.t. $f(x) \in R_2$

But some paths on the graph may not be itineraries!

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Markov partitions

Special partitions s.t. every path is "almost" an itinerary

A Markov partition:

- Product structure
- Markov property

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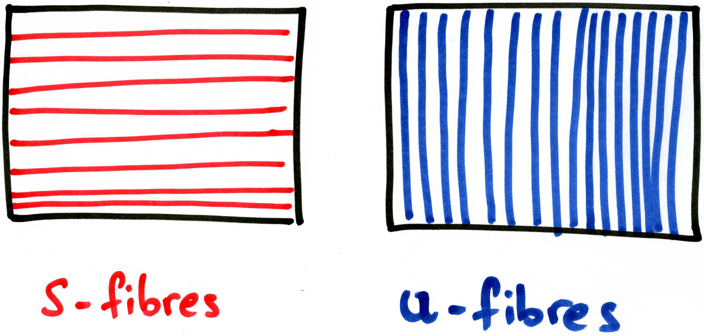
The definition of a Markov partition

Product structure



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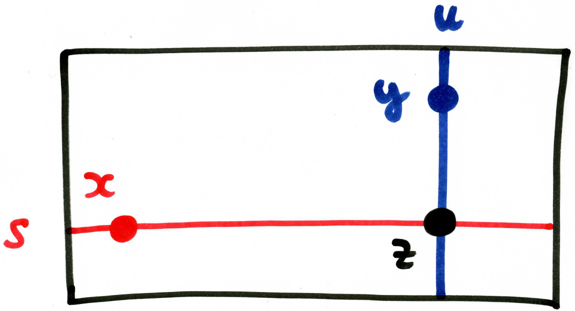

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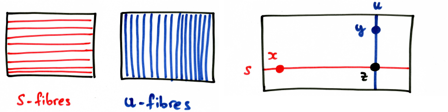

For every partition element R

- $R = \bigcup_{x \in R} W^u(x, R)$ and $R = \bigcup_{x \in R} W^s(x, R)$
- $W^u(\cdot, R)$ are equal or disjoint. Same for $W^s(\cdot, R)$
- $W^u(x, R) \cap W^s(x, R) = \{x\}$
- $W^u(x, R) \cap W^s(y, R) = \{z\}$, $z \in R$

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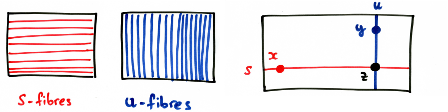

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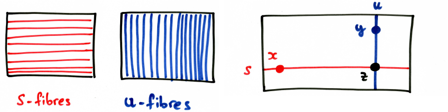

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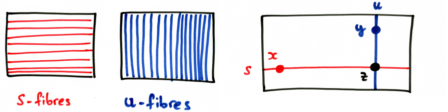

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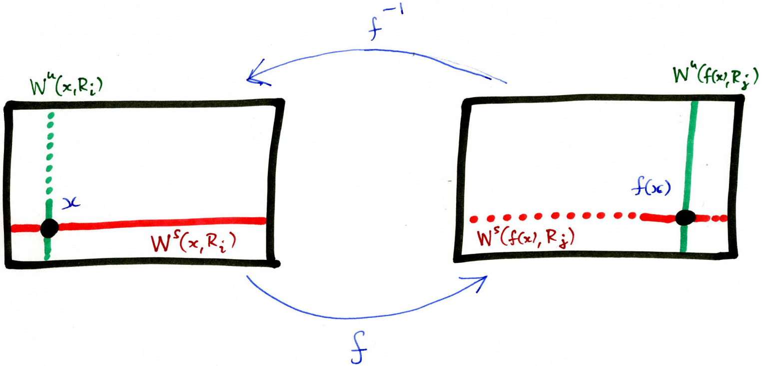

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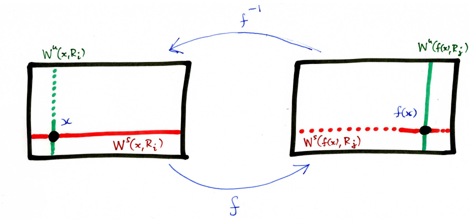

For any two partition elements R_i, R_j, if $x \in R_i, f(x) \in R_j$ then

- $f[W^s(x,R_i)] \subset W^s(f(x),R_j)$
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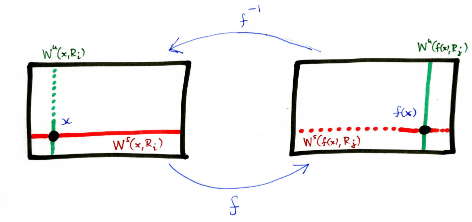

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Markov partitions: All paths are (almost) itineraries
Statement

- $f : M \to M$ homeo, M compact.
- $\mathcal{R}$ is a Markov partition
- $\mathcal{G}$: dynamical graph of $\mathcal{R}$

Theorem

For every path $(R_i)_{i\in\mathbb{Z}}$ on $\mathcal{G}$, $\exists x \in M$ s.t. $f^i(x) \in \overline{R_i} \quad (k \in \mathbb{Z})$.

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Proof

Cylinders:

$${}_m[R_m, R_{m+1}, \ldots, R_n] := \{x : f^i(x) \in R_i \ (m \le i \le n)\}$$

- **Must show:** for infinite paths $(R_k)_{k \in \mathbb{Z}}$,
$$\bigcap_{n=1}^{\infty} \overline{{}_{-n}[R_{-n}, \ldots, R_n]} \neq \varnothing$$
- **Enough to show:** ${}_{-n}[R_{-n}, \ldots, R_n] \neq \varnothing$
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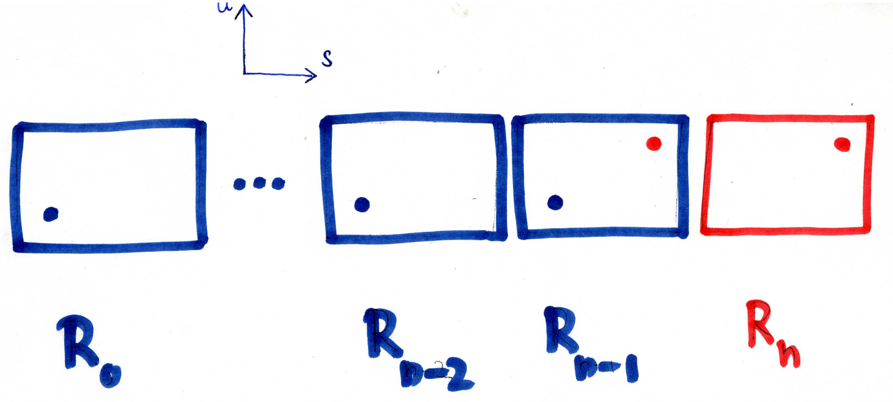

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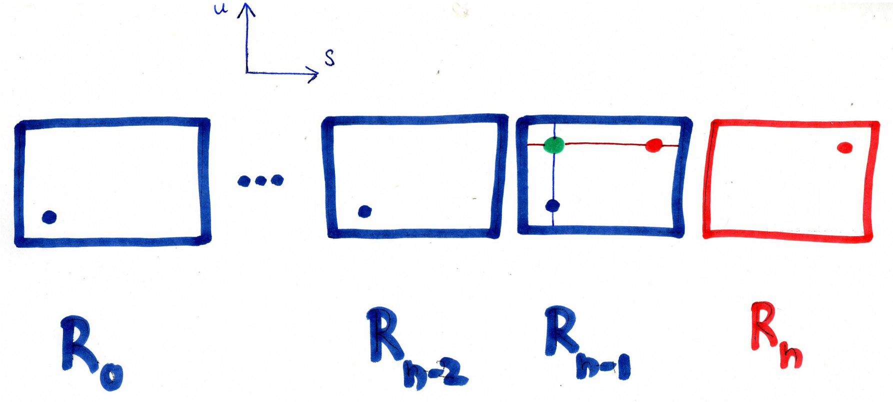

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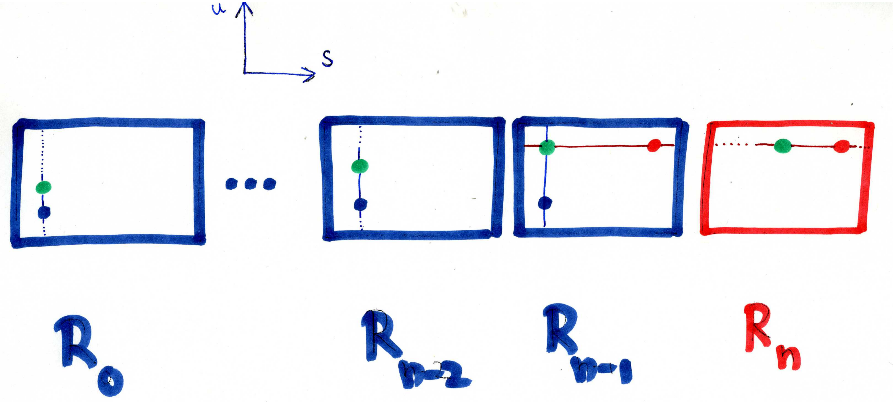

Existence of Markov partition?

When does f admit a Markov partition?
What's known

Special cases

- **Adler & Weiss, Berg:** Hyperbolic automorphisms of $\mathbb{T}^2$
- **Sinai:** Anosov diffeos
- **Bowen:** Axiom A diffeos
- **Fathi & Shub:** pseudo Anosov
- **Berger:** Hénon

General $C^{1+\epsilon}$ surface diffeos

A. Katok: $\exists$ "large" invariant sets K s.t. $f|_K$ has a finite MP.

"Large": $h_{top}(f|_K)$ arbitrarily close to $h_{top}(f)$.

Our result: Same as Katok, but "large"=of full measure.

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The result

Setup: $f : M \to M$ is a diffeomorphism on a compact smooth orientable surface M

- f is $C^{1+\epsilon}$
- $h_{top}(f) > 0$
- $\dim M = 2$

Theorem

For every $0 < \delta < h_{top}(f)$ there is an invariant Borel set E s.t.

1. *$f|_E$ has a countable Markov partition*
2. *$\mu(E) = 1$ for every ergodic invariant μ s.t. $h_\mu(f) > \delta$.*

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Setup: $f : M \to M$ is a diffeomorphism on a compact smooth orientable surface M

- f is $C^{1+\epsilon}$
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Let $\mathcal{G}$:=dynamical graph of the Markov partition

- $\Sigma(\mathcal{G}) := \{\text{paths on } \mathcal{G}\} = \{(R_i)_{i\in\mathbb{Z}} : R_i \to R_{i+1}\ (i \in \mathbb{Z})\}$
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Representation as a factor of a symbolic system

- **Tower constructions:** Takahashi, Hofbauer, Keller, Young, Buzzi, Pesin & Senti
- **Symbolic extensions:** Buzzi; Boyle, Fiebig & Fiebig, Boyle & Downarowicz, Burguet

Advantage of our coding: finiteness-to-one

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Applications

Application I: Counting periodic points

$$\mathrm{P}_n(f) := \#\{x \in M : f^n(x) = x\}$$

Theorem (Katok's conjecture)

Suppose f is a C^∞ surface diffeo with positive entropy, then $\exists p \in \mathbb{N}, C > 0$ s.t. for all $n \gg 1$ divisible by p, $P_n(f) \geq Ce^{nh_{top}(f)}$.

Katok's Theorem: $\mathrm{P}_n(f) \geq Ce^{n(h_{top}(f)-\epsilon)}$ along a subsequence

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2. B.M. Gurevich: Necessary & sufficient condition for the existence of measures of maximal entropy for countable Markov shifts

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A $C^{1+\epsilon}$–surface diffeo can have at most countably many different ergodic measures of maximal entropy.

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- **Data:** $\phi : M \to \mathbb{R}$ Hölder continuous
- **Equilibrium measure of ϕ:** Ergodic f–invariant measure μ which maximizes $h_\mu(f) + \int \phi d\mu$

Theorem

If $h_\mu(f) > 0$, then f equipped with μ is isomorphic to Bernoulli scheme $\times$ finite rotation.

Earlier related results: Pesin, Ledrappier, Ornstein & Weiss

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How to construct Markov partitions

Bowen's proof in the Anosov case

Setup: $f : \mathbb{T}^2 \to \mathbb{T}^2$ hyperbolic toral automorphism

Pseudo-orbits (Alexeev):

Sequences $(x_i)_{i\in\mathbb{Z}}$ s.t. $\forall i,\ d(f(x_i), x_{i+1}) < \epsilon$

- **Anosov Shadowing Lemma**
- **Finite alphabet suffices:** $\exists\mathcal{V}$ finite s.t. every orbit is shadowed by some p.o. in $\mathcal{V}^{\mathbb{Z}}$
- **Nearest neighbor property**

Directed graph representation:

Let $\mathcal{G}$ denote the graph with vertices $\mathcal{V}$ and edges $x \to y \Leftrightarrow d(f(x), y) < \epsilon$, then $\Sigma(\mathcal{G}) = \{\epsilon\text{–pseudo orbits in } \mathcal{V}^{\mathbb{Z}}\}$

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Main Step (Bowen–Sinai Refinement)

A procedure which refines a finite Markov collection with overlaps into a Markov partition.

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- **Markov Partition for** $\Sigma(\mathcal{G})$**:** $[v] = \{\underline{v} : v_0 = v\}$ $(v \in \mathcal{V})$
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Main Step (Bowen–Sinai Refinement)

A procedure which refines a finite Markov collection with overlaps into a Markov partition.

Bowen's construction of MP

Setup: $f : \mathbb{T}^2 \to \mathbb{T}^2$ hyperbolic toral automorphism

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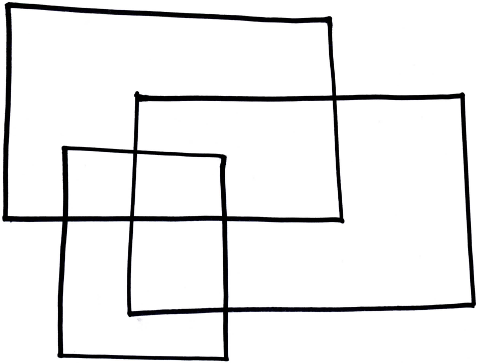

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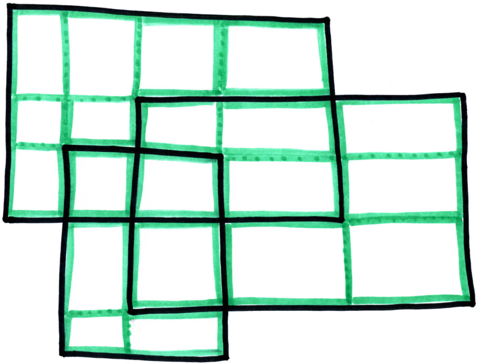

The non-uniformly hyperbolic case

What's easy to do:

To give a definition of "pseudo–orbits" in this context. But, we'll need an infinite alphabet.

Let's try Bowen's approach:

- $\Sigma(G) := \{$"generalized po"$\}$, where G is an infinite graph
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The difficulty

$\exists$ countable collections of sets without a countable refining partitions.

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Must have local finiteness!

Strategy of proof: Apriori local finiteness

Strategy of proof:

Come up with a definition of “pseudo–orbits" such that the Markov partition on $\Sigma(\mathcal{G})$ projects to a locally finite Markov collection in M. Then apply Bowen's construction.

What we need from the definition:

- Shadowing Lemma
- Countable alphabet suffices
- Nearest neighbor constraints
- $\rightarrow$ Inverse problem: Suppose a p.o. $(v_i)_{i\in\mathbb{Z}}$ shadows the orbit of x. Then we can “read” v_0 from x up to “bounded error”.

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- **Lecture 3:** shadowing lemma and inverse problem

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