Symbolic dynamics for surface diffeomorphisms

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Symbolic coding

• Setup: surface diffeomorphism $f: M \to M$

• **Orbits:** $f^n(x) := (f \circ \cdots \circ f)(x)$

• Fundamental difficulty: Direct calculation is difficult

Symbolic coding

A change of coordinates which transforms $f: M \rightarrow M$ to the action of the left shift on a space of sequences.

Easy to iteratel

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Symbolic coding
Markov partitions

Itineraries

\mathcal{R} a partition of M.

Itinerary of $x \in M$:

 $(\mathbf{R}_i)_{i\in\mathbb{Z}}\in\mathcal{R}^{\mathbb{Z}}$ s.t. $f^i(\mathbf{x})\in\mathbf{R}_i$ for all i

Why this is useful:

f acts on itineraries by the left shift. Easy to iterate!

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The result	Symbolic coding
Applications	Markov partitions
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What sequences arise as itineraries?

 $\ensuremath{\mathcal{R}}$ is a partition

Every \mathcal{R} --itinerary is a walk on the dynamical graph of \mathcal{R} :

- Vertices: partition elements
- Edges: $R_1 \rightarrow R_2$ when $\exists x \in R_1$ s.t. $f(x) \in R_2$

But some paths on the graph may not be itineraries!

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Symbolic coding Markov partitions The result

Markov partitions

Special partitions s.t. every path is "almost" an itinerary

A Markov partition:

- Product structure
- Markov property

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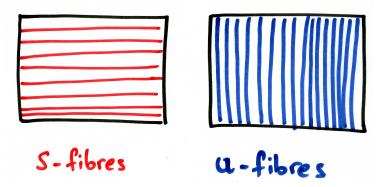
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Symbolic coding Markov partitions The result

The definition of a Markov partition Product structure

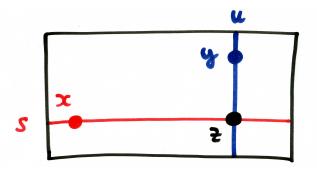


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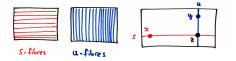


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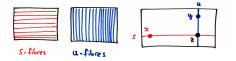
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For every partition element R

- $R = \bigcup_{x \in R} W^u(x, R)$ and $R = \bigcup_{x \in R} W^s(x, R)$
- $W^{u}(\cdot, R)$ are equal or disjoint. Same for $W^{s}(\cdot, R)$
- $W^{u}(x,R) \cap W^{s}(x,R) = \{x\}$
- $W^{u}(x, R) \cap W^{s}(y, R) = \{z\}, z \in R$

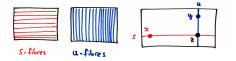
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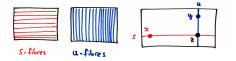
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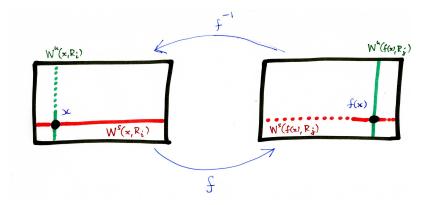
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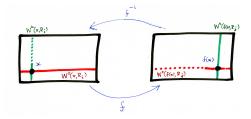
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The definition of a Markov partition Markov property



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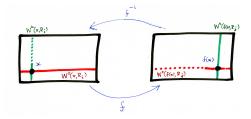
For any two partition elements R_i, R_j , if $x \in R_i, f(x) \in R_j$ then • $f[W^s(x, R_i)] \subset W^s(f(x), R_j)$

• $f^{-1}[W^u(f(x), R_j)] \subset W^u(x, R_j)$

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Symbolic coding Markov partitions The result

Markov partitions: All paths are (almost) itineraries

• $f: M \to M$ homeo, M compact.

- \mathcal{R} is a Markov partition
- \mathcal{G} : dynamical graph of \mathcal{R}

Theorem

For every path $(R_i)_{i \in \mathbb{Z}}$ on \mathcal{G} , $\exists x \in M \text{ s.t. } f^i(x) \in \overline{R_i} \ (k \in \mathbb{Z})$.

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Markov partitions: All paths are (almost) itineraries

Cylinders:

$$m[R_m, R_{m+1}, \ldots, R_n] := \{x : f^i(x) \in R_i \ (m \le i \le n)\}$$

• Must show: for infinite paths $(R_k)_{k \in \mathbb{Z}}$,

$$\bigcap_{n=1}^{\infty} \overline{-n[R_{-n},\ldots,R_n]} \neq \emptyset$$

• Enough to show: $_{-n}[R_{-n}, ..., R_n] \neq \emptyset$ Equivalently, $_0[R_0, ..., R_{2n}] \neq \emptyset$

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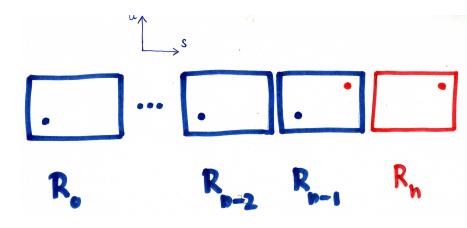
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Markov partitions: All paths are (almost) itineraries

Induction hypothesis: $_0[R_0, \ldots, R_{n-1}] \neq \emptyset$, and $R_{n-1} \rightarrow R_n$

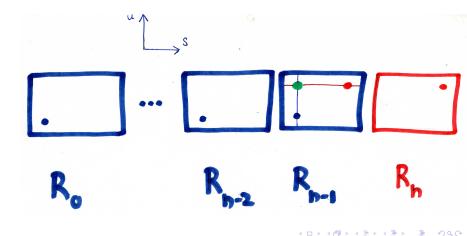


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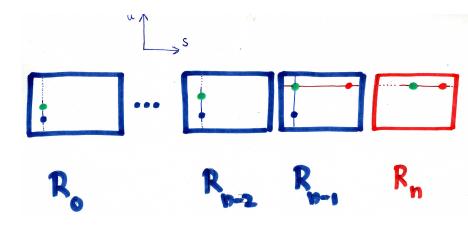
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Applications	Symbolic coding Markov partitions The result
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Existence of Markov partition?

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The resultSApplicationsNIdea of proofT

Symbolic coding Markov partitions The result

When does *f* admit a Markov partition?

Special cases

- Adler & Weiss, Berg: Hyperbolic automorphisms of T²
- Sinai: Anosov diffeos
- Bowen: Axiom A diffeos
- Fathi & Shub: pseudo Anosov
- Berger: Hénon

General $C^{1+\epsilon}$ surface diffeos

A. Katok: \exists "large" invariant sets K s.t. $f|_K$ has a finite MP.

"Large": $h_{top}(f|_{\mathcal{K}})$ arbitrarily close to $h_{top}(f)$.

Our result: Same as Katok, but "large"=of full measure.

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"**Large**": $h_{top}(f|_{\mathcal{K}})$ arbitrarily close to $h_{top}(f)$.

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When does *f* admit a Markov partition?

Special cases

- Adler & Weiss, Berg: Hyperbolic automorphisms of T²
- Sinai: Anosov diffeos
- Bowen: Axiom A diffeos
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	The result Applications Idea of proof	Symbolic coding Markov partitions The result	
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- *f* is C^{1+ϵ}
 *h*_{top}(*f*) > 0
- dim M = 2

Theorem

For every $0 < \delta < h_{top}(f)$ there is an invariant Borel set E s.t.

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Let $\mathcal{G} :=$ dynamical graph of the Markov partition

- $\Sigma(\mathcal{G}) := \{ \text{paths on } \mathcal{G} \} = \{ (R_i)_{i \in \mathbb{Z}} : R_i \to R_{i+1} \ (i \in \mathbb{Z}) \}$
- Metric: $d(\underline{v}, \underline{w}) := \exp[-\min\{|i| : v_i \neq w_i\}]$
- Left shift map: $\sigma : (v_i)_{i \in \mathbb{Z}} \mapsto (v_{i+1})_{i \in \mathbb{Z}}$.

Theorem

There is a Hölder continuous $\pi : \Sigma(\mathcal{G}) \to M$ s.t.

- $\bigcirc \pi \circ \sigma = f \circ \pi$
- $\bigcirc \pi[\Sigma(\mathcal{G})]$ is δ -large
- $\{x \in M : \text{ finitely many preimages}\}$ is δ -large

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The result Applications Idea of proof Symbolic coding Markov partition The result

Other results in the literature

Representation as a factor of a symbolic system

- **Tower constructions:** Takahashi, Hofbauer, Keller, Young, Buzzi, Pesin & Senti
- **Symbolic extensions:** Buzzi; Boyle, Fiebig & Fiebig, Boyle & Downarowicz, Burguet

Advantage of our coding: finiteness-to-one

- Principal extension property
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The result	Symbolic coding
Applications	Markov partitions
Idea of proof	The result

Applications

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Application I: Counting periodic points

$\mathsf{P}_n(f) := \#\{x \in M : f^n(x) = x\}$

Theorem (Katok's conjecture)

Suppose *f* is a C^{∞} surface diffeo with positive entropy, then $\exists p \in \mathbb{N}, C > 0 \text{ s.t. for all } n \gg 1$ divisible by $p, P_n(f) \ge Ce^{nh_{top}(f)}$.

Katok's Theorem: $P_n(f) \ge Ce^{n(h_{top}(f) - \epsilon)}$ along a subsequence

Key tools

- $igodoldsymbol{0}$: Membourse: $G^{(n)}$ diffeos have measures of maximal entropy (
- 8 B.H. Guenelek: Nacionary & sufficient condition for the existence of measuring of machinel entropy for countable Markov al-file.

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Application II: The measure of maximal entropy

Theorem (Buzzi's Conjecture)

A $C^{1+\epsilon}$ -surface diffeo can have at most countably many different ergodic measures of maximal entropy.

Buzzi's Theorem: Finitely many for piecewise monotonic interval maps, piecewise linear affine homeos

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• **Data:** $\phi : M \to \mathbb{R}$ Hölder continuous

• Equilibrium measure of ϕ : Ergodic *f*-invariant measure μ which maximizes $h_{\mu}(f) + \int \phi d\mu$

Theorem

If $h_{\mu}(f) > 0$, then f equipped with μ is isomorphic to Bernoulli scheme \times finite rotation.

Earlier related results: Pesin, Ledrappier, Ornstein & Weiss

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The result	Bowen's construction in the UH case
Applications	The difficulty in the NUH case
Idea of proof	Strategy of proof

How to construct Markov partitions

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Setup: $f: \mathbb{T}^2 \to \mathbb{T}^2$ hyperbolic toral automorphism

Pseudo-orbits (Alexee

Sequences $(x_i)_{i \in \mathbb{Z}}$ s.t. $\forall i, d(f(x_i), x_{i+1}) < \epsilon$

- Anosov Shadowing Lemma
- Finite alphabet suffices: ∃V finite s.t. every orbit is shadowed by some p.o. in V^Z
- Nearest neighbor property

Directed graph representation:

Let \mathcal{G} denote the graph with vertices \mathcal{V} and edges $x \to y \Leftrightarrow d(f(x), y) < \epsilon$, then $\Sigma(\mathcal{G}) = \{\epsilon \text{-pseudo orbits in } \mathcal{V}^{\mathbb{Z}}\}$

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The result	Bowen's construction in the UH case
Applications	The difficulty in the NUH case
dea of proof	Strategy of proof

Setup: $f : \mathbb{T}^2 \to \mathbb{T}^2$ hyperbolic toral automorphism

Pseudo-orbits (Alexeev):

Sequences $(x_i)_{i \in \mathbb{Z}}$ s.t. $\forall i, d(f(x_i), x_{i+1}) < \epsilon$

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- Finite alphabet suffices: ∃V finite s.t. every orbit is shadowed by some p.o. in V^Z
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Bowen's construction of MP

Setup: $f : \mathbb{T}^2 \to \mathbb{T}^2$ hyperbolic toral automorphism

- $\Sigma(\mathcal{G}) = \{ \text{pseudo-orbits in } \mathcal{V}^{\mathbb{Z}} \}$
- π : pseudo-orbit → real orbit defines a map π : Σ(G) → M
 s.t. π ∘ shift = f ∘ π.
- Markov Partition for $\Sigma(\mathcal{G})$: $[v] = \{\underline{v} : v_0 = v\} \ (v \in \mathcal{V})$
- Project this partition to M: π[ν] (ν ∈ V). Markov, but has overlaps.

A procedure which refines a finite Markov collection with overlaps into a Markov parition.

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Main Step (Bowen–Sinai Refinement)

A procedure which refines a finite Markov collection with overlaps into a Markov partition.

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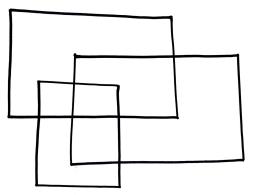
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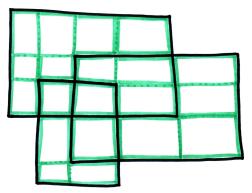
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Bowen Sinai Refinement



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 The result
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 Idea of proof
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The non-uniformly hyperbolic case

What's easy to do:

To give a definition of "pseudo-orbits" in this context. But, we'll need an infinite alphabet.

Let's try Bowen's approach

- $\mathbb{Z}(\mathcal{G}) = \{$ "generalized p.o." $\}$, where \mathcal{G} is an infinite graph.
- Σ(G) has a countable Markov partition
- projects to countable Markov collection in M.

The difficulty

∃ countable collections of sets without a countable refining partitions.

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The result	Bowen's construction in the UH case
Applications	The difficulty in the NUH case
Idea of proof	Strategy of proof

Must have local finiteness!

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Strategy of proof: Apriori local finiteness

Strategy of proof:

Come up with a definition of "pseudo–orbits" such that the Markov partition on $\Sigma(\mathcal{G})$ projects to a locally finite Markov collection in *M*. Then apply Bowen's construction.

What we need from the definition:

- Shadowing Lemma.
- Countable alphabet suffices
- Nearest neighbor constraints.
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Plan		

• Lecture 2: definition of "generalized pseudo-orbits"

• Lecture 3: shadowing lemma and inverse problem

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