Maximal Range of Polynomials: A Survey and Open Problems

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I. The basics

Definition Let Ω be a domain in $\mathbb{C}$ with $1 \in \Omega$. Let

$$\mathcal{P}_n(\Omega) := \{P \in \mathcal{P}_n \ : \ P(0) = 1; P(\mathbb{D}) \subset \Omega\}.$$

Then

$$\Omega_n := \bigcup_{P \in \mathcal{P}_n(\Omega)} P(\mathbb{D})$$

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Maximal Range problem: Describe the maximal range Ω_n for any given domain Ω and, in particular, describe the extremal polynomials for Ω_n.

Definition A polynomial $P \in \mathcal{P}_n(\Omega)$ is called *extremal polynomial* for Ω_n if

$$P(1) \in (\partial\Omega_n \setminus \partial\Omega) \, .$$

Example:

Let $\Omega := RH$, so that $\mathcal{P}_n(\Omega) = \{P \in \mathcal{P}_n \ : \ P(0) = 1; P(\mathbb{D}) \subset \Omega\}$.

Then, for $P \in \mathcal{P}_n(\Omega)$, (L. Fejér 1916)

$$\operatorname{Re} P(z) < n + 1, \ z \in \mathbb{D}.$$

This is sharp for the Fejér polynomials

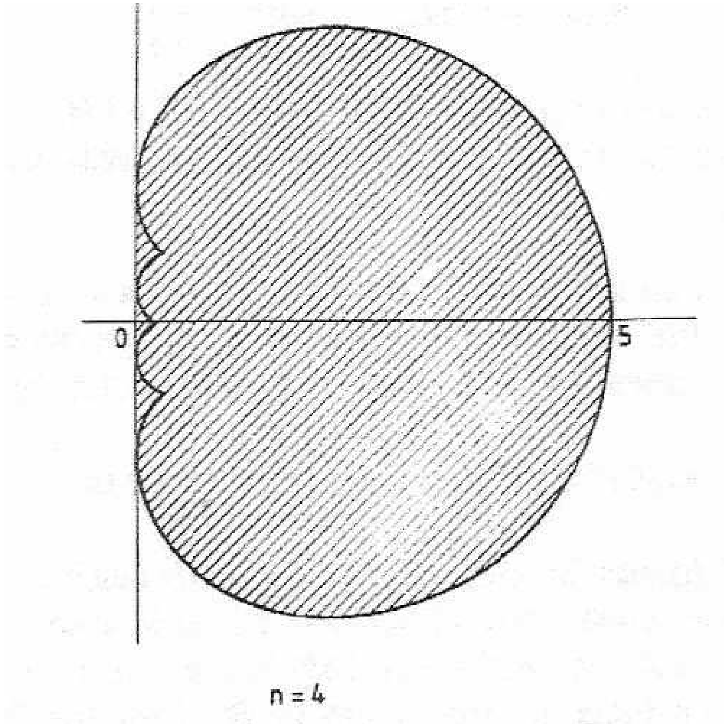
$$P(z) := F_n(z) = 1 + 2 \sum_{k=1}^{n} \frac{n+1-k}{n+1} z^k \ ,$$

where $F_n(1) = n + 1$.

Theorem (A. Córdova, S.R. 1990):

$$\Omega_n := \bigcup_{P \in \mathcal{P}_n(\Omega)} P(\mathbb{D}) \ = \text{co}\, F_n(\mathbb{D}).$$

Here is the case $n = 4$. The thick line is the border of $F_4(\mathbb{D})$, the hatched area is Ω_4.



Note that F_n is univalent in $\mathbb{D}$ with all zeros of its derivative on $\partial\mathbb{D}$.

Theorem (A. Córdova, C. Genthner, L. Salinas, S.R. (1990-2003))

Let Ω be a simply connected domain with $1 \in \Omega$ and let $n \in \mathbb{N}$. Then every extremal polynomial $P \in \mathcal{P}_n(\Omega)$ (with $P(1) = \omega \in \partial\Omega_n \setminus \partial\Omega$) is

1) univalent in $\mathbb{D}$ with all zeros of its derivative on $\partial\mathbb{D}$, say

$$z_j = e^{i\theta_j} : \quad 0 < \theta_1 < \cdots < \theta_{n-1} < 2\pi.$$

2) The θ_j are interlaced by

$$\varphi_j : \quad 0 < \varphi_0 \leq \theta_1 \leq \varphi_1 \leq \ldots \theta_{n-1} \leq \varphi_{n-1} < 0$$

so that $P(e^{i\varphi_j}) \in \partial\Omega$ ('points of contact').

3) If Ω is a convex domain then P satisfies the "arc-conjecture" i.e. P is uniquely determined and the arc $\{P(e^{i\varphi}) : \varphi_{n-1} - 2\pi < \varphi < \varphi_0\}$ belongs to $\partial\Omega_n$.

2. The univalence of the extremal polynomials

Surprisingly, the most complicated part of the proof of the previous Theorem seems to be the univalence of the extremal polynomials. It is based on the following situation.

We are using the notion of "almost extremal polynomials", which are defined essentially by the conditions 1) and 2) in the previous Theorem (without the univalence):

Definition *For some simply-connected domain Ω let $P \in \mathcal{P}_n(\Omega)$ be such that:*

1) All the zeros of its derivative are on $\partial\mathbb{D}$, say

$$z_j = e^{i\theta_j} : \quad 0 < \theta_1 < \cdots < \theta_{n-1} < 2\pi.$$

2) The θ_j are interlaced by

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so that $P(e^{i\varphi_j}) \in \partial\Omega$ ('points of contact').

Then P is called **"almost extremal"***.*

Theorem (C. Genthner, L. Salinas, S.R. 2003) *Every almost extremal polynomial is univalent in $\mathbb{D}$.*

History

1) "Proof" by A.C. and S.R. (1990): 1 page, false...(unpublished)

2) Proof by C. Genthner (PhD-Thesis) (1996): 123 pages... (unpublished)

3) Proof by C. Genthner, L. Salinas and S.R. (2003): 11 pages (CMFT vol. 2) (still complicated)

3. Approximation Questions

A slightly different approach to the maximal range problem has been used by V. Andrievskii and S.R.

Let Ω be a simply connected domain with $1 \in \Omega$, and let $f : \mathbb{D} \to \Omega$ be a conformal mapping onto Ω with $f(0) = 1$. For $0 < s < 1$ define $f_s(z) := f((1-s)z)$, $z \in \mathbb{D}$.

Theorem (V. Andrievskii, S.R. 1998) *There exists a universal constant $c_0 > 1$ with the following property: for each simply connected Ω as above and every $n \geq 2c_0$ there exists a univalent $p \in \mathcal{P}_n(\Omega)$ such that*

$$f_{\frac{c_0}{n}} \prec p \prec f.$$

In particular,

$$f_{\frac{c_0}{n}}(\mathbb{D}) \subset \Omega_n \subset \Omega.$$

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Here $\prec$ stands for the $subordination$ of analytic functions in $\mathbb{D}$.

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The proof of this result is constructive, however a first quantitative study based on the proof led to the weak estimate

$$c_0 \leq 10^{73}.$$

This has later been improved by R. Greiner (1994) to

$$\pi \leq c_0 < 73,$$

where the value π occurs for the slit domain $\mathbb{C} \setminus (-\infty, -1/4)$. The correct value for c_0 is still unknown.

More precise information we get for special domains.

Theorem (R. Greiner, S.R.)

Let Ω *be a convex domain. Then for* $n \geq 4$ *there exists a univalent* $p \in \mathcal{P}_n(\Omega)$ *such that*

$$f_{\frac{2}{n}} \prec p \prec f.$$

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There are better estimates than c_0/n for domains with special boundary conditions.

Application of special cases (domains)

For quite a number of special domains the maximal ranges have been found explicitely. Knowing those will generally lead to new estimates for polynomials subject to certain restrictions. We give a few examples for circular domains.

Theorem *Let $P \in \mathcal{P}_n$ satisfy $P(0) = 1$ and $|P(z)| > \rho, z \in \mathbb{D}$. Then*

$$|\arg P(z)| \leq (n+1) \arccos\left(\rho^{1/(n+1)} \cos \frac{\pi}{2n+2}\right) - \frac{\pi}{2}, \quad z \in \mathbb{D}.$$

This estimate is sharp for $P = Q_n^\rho$.

Theorem *Let $P \in \mathcal{P}_n$ satisfy $P(0) = 1$ and $|P(z)| > \rho, z \in \mathbb{D}$. Then*

$$||P|| \leq \rho \, T_{n+1}\left(\rho^{-1/(n+1)}\right).$$

This estimate is sharp for $P = Q_n^\rho$.

Theorem *Let* $n > 1$, $P \in \mathcal{P}_n$, $P(0) = 1$, *and* $||P|| \leq \rho$. *Then, for* $z \in \mathbb{D}$, *we have*

$$\operatorname{Re} P(z) > \begin{cases} \rho T_{n+1}\left(\rho^{-1/(n+1)}\right), & 1 \leq \rho \leq \left[\cos \frac{\pi}{n+1}\right]^{-n-1}, \\ -\rho, & \textit{elsewhere} . \end{cases}$$

These bounds are sharp.

There are many other special cases for slit-domains, sectors, strip domains, squares and rectangles etc., too many to mention them all.

5. A general special case

Theorem (A. Cordova, S.R. 1991)

Let $P \in \mathcal{P}_n$ be an 'almost extremal' polynomial and let $Q \in \mathcal{P}_n$ be such that $P(0) = Q(0)$ and $Q(\overline{\mathbb{D}})$ does not meet any of the critical values of P. Then $Q \prec P$ and, in particular, $Q(\mathbb{D}) \subset P(\mathbb{D})$.

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Example: Let $P(z) = z - \frac{z^n}{n}$, and $Q \in \mathcal{P}$ with $Q(0) = 0$ and $\frac{n-1}{n} e^{2\pi i j/(n-1)} \notin Q(\overline{\mathbb{D}})$ for $j = 1, \ldots, n-1$. Then $Q(\mathbb{D}) \subset P(\mathbb{D})$ and, in particular, $||Q|| \leq \frac{n+1}{n}$.

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Thank you for your attention !!