Estimates for Polynomials in the Unit Disk With Varying Constant Terms

Stephan Ruscheweyh
Würzburg University, Germany

Far Eastern Federal University

Vladivostok, August 28, 2012

A. Two methods for producing inequalities for polynomials in $\mathbb{D}$

A.1. Bound-preserving operators

Let $\mathcal{P}_n$ be the set of complex polynomials of degree $\leq n$.

A function (formal power series) $f(z) = \sum_{k=0}^{\infty} c_k z^k$ is called *bound-preserving of degree* n ($f \in B_n$) if for all polynomials $P(z) = \sum_{k=0}^{n} a_k z^k \in \mathcal{P}_n$ we have

$$||f * P|| = ||\sum_{k=0}^{n} c_k a_k z^k|| \leq ||P|| \qquad (||.|| := \sup_{z \in \mathbb{D}} |.|).$$

If $f \in B_n$ and $f(0) = 1$ we say that $f \in B_n^0$. Note that

$$f(z) = \sum_{k=0}^{n} c_k z^k \in B_n^0 \quad \Rightarrow \quad \tilde{f}(z) := z^n f(1/z) = \sum_{k=0}^{n} c_{n-k} z^k \in B_n$$

Theorem 1 (Carathéodory-Toeplitz) $f(z) = \sum_{k=0}^{n} c_k z^k \in B_n^0$ if and only if the Hermite-Toeplitz matrix

$$H_n := \begin{pmatrix} 1 & c_1 & c_2 & \cdots & c_n \\ \overline{c_1} & 1 & c_1 & \cdots & c_{n-1} \\ \overline{c_2} & \overline{c_2} & 1 & \cdots & c_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \overline{c_n} & \overline{c_{n-1}} & \overline{c_{n-2}} & \cdots & 1 \end{pmatrix}$$

is positive semi-definite.

Example: For $|\varepsilon| \leq 1$ we have

$$Q_n(\varepsilon) := \sum_{k=0}^{n-1} \frac{n-k}{n} z^k + \frac{2\varepsilon}{n+2} z^n \in B_n^0.$$

Using $\widetilde{Q_n}$ this example leads to a classical result and its refinement:

Theorem ($\varepsilon = 0$, S. Bernstein) *For $P \in \mathcal{P}_n$ we have*

$$||P'|| \leq n||P||.$$

Using $\widetilde{Q_n}$ this example leads to a classical result and its refinement:

Theorem ($\varepsilon = 0$, S. Bernstein) *For $P \in \mathcal{P}_n$ we have*

$$||P'|| \leq n||P||.$$

Theorem ($|\varepsilon| = 1$, S.R. (1982)) *For $P \in \mathcal{P}_n$ we have*

$$||P'|| \leq n||P|| - \frac{2n}{n+2}\,|P(0)|.$$

The bound $\frac{2n}{n+2}$ is best possible.

Many improvements and new proofs of classical results via this method have been obtained (R. Fournier, C. Frappier, Q.I. Rahman, S.R. and others). This always deals with the (sometimes tedious) study of the semi-definiteness of special Hermite-Toeplitz matrices.

Question: For which polynomials are the inequalities generated by this method best possible?

This question is open in general.

For Bernstein's inequality, however, the answer has been known for long. For the generalization given above, the (identical) answer was established by R. Fournier, namely

$$P(z) = cz^n, \quad c \in \mathbb{C},$$

are the only polynomials for which these inequalities are sharp.

Recently the following answer for at least certain cases has been obtained:

Theorem (Fournier, S.R.) *Let H be a Hermite-Toeplitz $(n+1) \times (n+1)$-matrix with its first row $(1, c_1, \ldots, c_n)$ and the following properties*

1. $|c_n| < 1$.
2. *All principal minors of H of order $\leq n$ are positive,*
3. $\det H = 0$.

If $P(z) = \sum_{k=0}^{n} a_k z^k$ satisfies

$$||\sum_{k=0}^{n} a_k c_k z^k|| = ||P||,$$

then $P \equiv$ const.

A problem and the consequences of its solution

In 2008 Gerard Letac (Toulouse) posed the following

Problem[1] *For* $n \geq 2$ *and* $x \in \mathbb{C}$ *let*

$$H_n(x) := \begin{pmatrix} 1 & x & x & \dots & x \\ \overline{x} & 1 & x & \cdots & x \\ \overline{x} & \overline{x} & 1 & \cdots & x \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \overline{x} & \overline{x} & \overline{x} & \cdots & 1 \end{pmatrix}$$

be a Hermite-Toeplitz $n \times n$ *matrix. Describe the set* $\Delta_n \subset \mathbb{C}$ *of those* x *for which* $H_n(x)$ *is positive semi-definite.*

Clearly, the solution to this problem must have applications to polynomial inequalities.

[1] Amer. Math. Monthly, Problem 11396 (2008)

Theorem (R. Fournier, G. Letac, S.R. (2010)) *For $n \geq 2$ and $x \in \mathbb{C}$ let*

$$H_n(x) := \begin{pmatrix} 1 & x & x & \dots & x \\ \overline{x} & 1 & x & \cdots & x \\ \overline{x} & \overline{x} & 1 & \cdots & x \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \overline{x} & \overline{x} & \overline{x} & \cdots & 1 \end{pmatrix}$$

be a Hermite-Toeplitz $n \times n$ matrix. Let $\Delta_n \subset \mathbb{C}$ be the set of those x for which $H_n(x)$ is positive semi-definite. Then Δ_n is the convex hull of the Jordan curve

$$\left\{ -e^{i\varphi} \frac{\sin \frac{\varphi}{n}}{\sin(\frac{n-1}{n}\varphi)} \ : \ |\varphi| \leq \pi \right\}.$$

For all $n \geq 2$ we have $1 \in \partial\Delta_n$. Furthermore, for all $x \in \partial\Delta_n \setminus \{1\}$ we have

$$\det H_k(x) \begin{cases} = 0, & k = n, \\ > 0, & 2 \leq k < n. \end{cases}$$

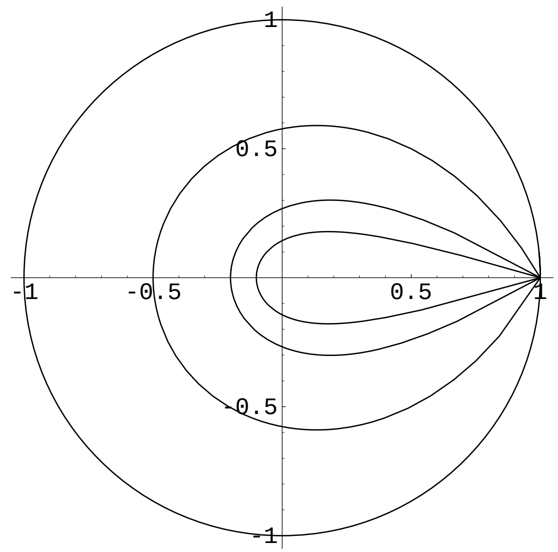


Figure 1: $\partial\Delta_n$ for $n = 2, 3, 6, 11$

In terms of polynomial inequalities this result has the following interpretation:

Theorem (Fournier, Letac, S.R. (2010)) *For $P \in \mathcal{P}_{n-1}$ and $\alpha \in \mathbb{C}$ we have*

$$||P|| \leq n\,(||z\,P(z) + \alpha|| - |\alpha|)\,.$$

For no $n \in \mathbb{N}$ and no $\alpha > 0$ the constant n can be replaced by anything smaller without violating the conclusion. On the other hand, the only polynomial for which we have equality is $P \equiv 0$.

In terms of polynomial inequalities this result has the following interpretation:

Theorem (Fournier, Letac, S.R. (2010)) *For* $P \in \mathcal{P}_{n-1}$ *and* $\alpha \in \mathbb{C}$ *we have*

$$||P|| \le n\,(||z\,P(z) + \alpha|| - |\alpha|)\,.$$

For no $n \in \mathbb{N}$ *and no* $\alpha > 0$ *the constant* n *can be replaced by anything smaller without violating the conclusion. On the other hand, the only polynomial for which we have equality is* $P \equiv 0$.

Clearly, the following is an equivalent version of this inequality:

$$\forall Q \in \mathcal{P}_n \;:\; ||Q \,-\, Q(0)|| \;\le\; n(||Q|| - |Q(0)|).$$

In terms of polynomial inequalities this result has the following interpretation:

Theorem (Fournier, Letac, S.R. (2010)) *For $P \in \mathcal{P}_{n-1}$ and $\alpha \in \mathbb{C}$ we have*

$$||P|| \leq n\,(||z\,P(z) + \alpha|| - |\alpha|)\,.$$

For no $n \in \mathbb{N}$ and no $\alpha > 0$ the constant n can be replaced by anything smaller without violating the conclusion. On the other hand, the only polynomial for which we have equality is $P \equiv 0$.

Because of the structure of this result it is clear that the estimating factor n cannot depend on α. This changes if we look only at polynomials in the class

$$\mathcal{P}^*_{n-1} := \{P \in \mathcal{P}_{n-1} : ||P|| = 1\}.$$

Problem *For $\alpha \geq 0$ and $n \in \mathbb{N}$ determine the numbers*

$$M_n(\alpha) := \inf_{P \in \mathcal{P}^*_{n-1}} (||zP(z) + \alpha|| - \alpha).$$

Note that from the matrix approach derived above we readily obtain

$$\inf_{\alpha>0} M_n(\alpha) = \frac{1}{n},$$

but the methods described so far do not help with finding the actual values of $M_n(\alpha)$.

A.2. The Maximal Range method.

Examples

1. **Well known:** Let $\Omega := \mathbb{C} \setminus \{0\}$, and

$$\mathcal{P}_n(\Omega) := \{P \in \mathcal{P}_n \ : \ P(0) = 1; P(\mathbb{D}) \subset \Omega\}.$$

Then

$$P \in \mathcal{P}_n(\Omega) \Rightarrow P(\mathbb{D}) \subset (1+\mathbb{D})^n \qquad\qquad (\Leftrightarrow P \prec (1+z)^n).$$

In other words:

$$\Omega_n := \bigcup_{P \in \mathcal{P}_n(\Omega)} P(\mathbb{D}) \ = \ (1+\mathbb{D})^n.$$

Note: There is one single polynomial (namely $(1+z)^n$) which describes the possible ranges of all $P \in \mathcal{P}_n(\Omega)$ for this domain Ω.

2. **Less well known...:** Let $\Omega := RH$, and

$$\mathcal{P}_n(\Omega) := \{P \in \mathcal{P}_n \ : \ P(0) = 1; P(\mathbb{D}) \subset \Omega\}.$$

Then, for $P \in \mathcal{P}_n(\Omega)$, (L. Fejér 1916)

$$\operatorname{Re} P(z) < n + 1, \ z \in \mathbb{D}.$$

This is sharp for the Fejér polynomials

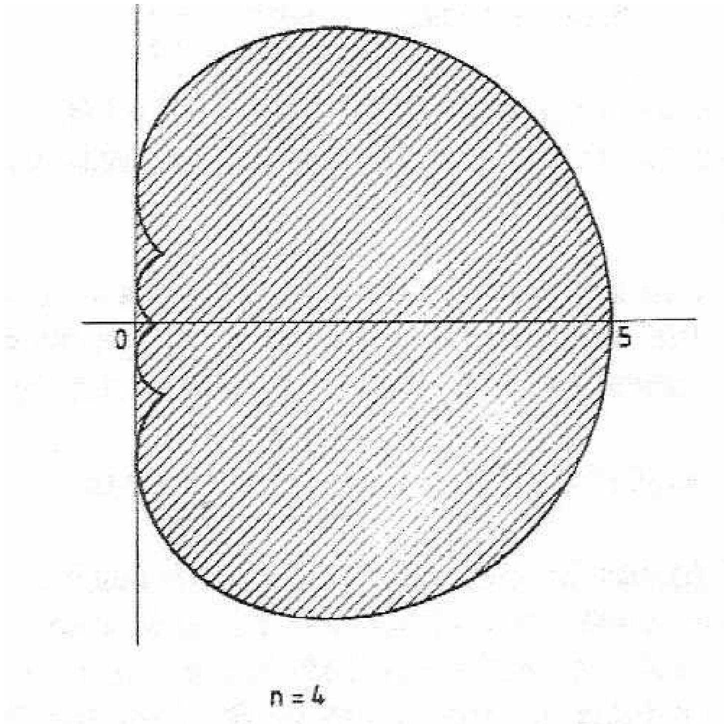
$$P(z) := F_n(z) = 1 + 2\sum_{k=1}^{n} \frac{n+1-k}{n+1} z^k \ ,$$

where $F_n(1) = n + 1$.

Theorem (A. Córdova, Ru. 1990):

$$\Omega_n := \bigcup_{P \in \mathcal{P}_n(\Omega)} P(\mathbb{D}) \ = \text{co}\, F_n(\mathbb{D}).$$

This is the case $n = 4$. The thick line is the border of $F_4(\mathbb{D})$, the hatched area is Ω_4.



Note that F_n is univalent in $\mathbb{D}$ with all zeros of its derivative on $\partial\mathbb{D}$.

Definition Let Ω be a domain in $\mathbb{C}$ with $1 \in \Omega$. Let

$$\mathcal{P}_n(\Omega) := \{P \in \mathcal{P}_n \ : \ P(0) = 1; P(\mathbb{D}) \subset \Omega\}.$$

Then

$$\Omega_n := \bigcup_{P \in \mathcal{P}_n(\Omega)} P(\mathbb{D})$$

is called the *Maximal Range* (of order n) of Ω.

Definition Let Ω be a domain in $\mathbb{C}$ with $1 \in \Omega$. Let

$$\mathcal{P}_n(\Omega) := \{P \in \mathcal{P}_n \ : \ P(0) = 1; P(\mathbb{D}) \subset \Omega\}.$$

Then

$$\Omega_n := \bigcup_{P \in \mathcal{P}_n(\Omega)} P(\mathbb{D})$$

is called the *Maximal Range* (of order n) of Ω.

Maximal Range problem: Describe the maximal range Ω_n for any given domain Ω and, in particular, describe the extremal polynomials for Ω_n.

Definition A polynomial $P \in \mathcal{P}_n(\Omega)$ is called *extremal polynomial* for Ω_n if

$$P(\partial\mathbb{D}) \cap (\partial\Omega_n \setminus \partial\Omega) \neq \emptyset.$$

Definition Let Ω be a domain in $\mathbb{C}$ with $1 \in \Omega$. Let

$$\mathcal{P}_n(\Omega) := \{P \in \mathcal{P}_n \ : \ P(0) = 1; P(\mathbb{D}) \subset \Omega\}.$$

Then

$$\Omega_n := \bigcup_{P \in \mathcal{P}_n(\Omega)} P(\mathbb{D})$$

is called the *Maximal Range* (of order n) of Ω.

Maximal Range problem: Describe the maximal range Ω_n for any given domain Ω and, in particular, describe the extremal polynomials for Ω_n.

Definition A polynomial $P \in \mathcal{P}_n(\Omega)$ is called *extremal polynomial* for Ω_n if

$$P(\partial\mathbb{D}) \cap (\partial\Omega_n \setminus \partial\Omega) \neq \emptyset.$$

Remark In our examples 1 and 2 the polynomials $(1 + xz)^n$ and $F_n(xz)$ (where $|x| = 1$) are the only extremal polynomials for the respective sets Ω_n.

Maximal ranges for disks with center at the origin

With T_{n+1} the Chebychev polynomials of degree $n+1$ we define the polynomials $Q_{\rho,n} \in \mathcal{P}_n$ given by:

$$Q_{\rho,n}(z^2) = \frac{-\rho\, z^{2n+3}}{n+1}\,\frac{d}{dz}\left\{ z^{-n-1}\, T_{n+1}\left(\rho^{-1/(n+1)}\frac{1+z^2}{2z}\right)\right\}.$$

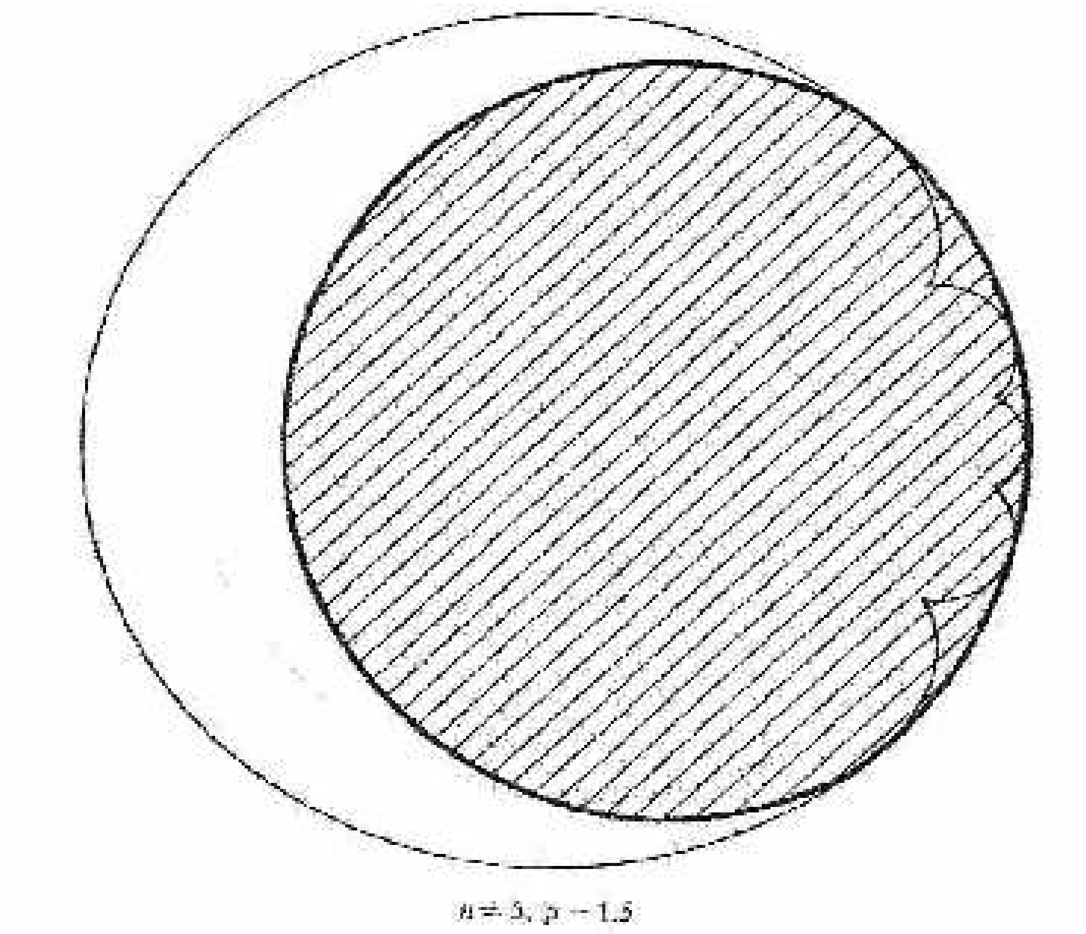
where

$$1 < \rho \le \rho_n := \left(\cos\frac{\pi}{n+1}\right)^{-n-1} = 1 + \frac{\pi^2}{2n} + \mathcal{O}(n^{-2}), \quad n \in \mathbb{N}.$$

For $\rho > 1$ we define $\Omega_\rho := \{z \,:\, |z| < \rho\}$ and $\Omega_{\rho,n}$ its maximal ranges.

It is known that the $Q_{\rho,n}$ belong to $\mathcal{P}_n(\Omega_\rho)$ and are univalent in $\mathbb{D}$ and have all zeros of their derivatives on the boundary of $\mathbb{D}$.

Note The $Q_{\rho,n}$ were introduced and studied by R. Varga and S.R. (1984), in a different context.



$Q_{\rho,n}$ for $\rho = 1.5,\ n = 5$

Theorem (A. Córdova, S.R. (1992)) *Let $n \in \mathbb{N}$ be fixed. Then the following statements hold.*

(i) For $1 < \rho \leq \rho_n$ the set $\Omega_{\rho,n}$ is the interior domain of the Jordan curve consisting of the two arcs

$$\begin{aligned} \mathcal{C}_1 : &= \left\{ Q_{\rho,n}(e^{i\tau}) \, : \, |\tau| \leq \tau_1 \right\}, \\ \mathcal{C}_2 : &= \left\{ \rho \, e^{i\tau} \, : \, |\tau| < \pi - \frac{n+1}{2}\tau_1 \right\}, \end{aligned}$$

where

$$\tau_1 := 2 \, \arccos \left(\rho^{1/(n+1)} \, \cos(\frac{\pi}{n+1}) \right).$$

(ii) For $\rho > \rho_n$ we have

$$\Omega_{\rho,n} \, = \, \Omega_\rho.$$

B. Solution of the $M_n(\alpha)$ Problem

Problem: *For $\alpha \geq 0$ and $n \in \mathbb{N}$ determine the numbers*

$$M_n(\alpha) := \inf_{P \in \mathcal{P}^*_{n-1}} (||zP(z) + \alpha|| - \alpha).$$

Let $0 < \alpha \leq \frac{1}{1+\rho_n} (< \frac{1}{2})$. Trivially we have

$$M_n(\alpha) \geq 1 - 2\alpha.$$

This is surprisingly best possible and sharp for the polynomials

$$P(z) = \frac{Q_{\rho_n,n}(xz) - 1}{(1 + \rho_n)z}, \quad |x| = 1,$$

which can be shown by an easy (?) calculation.

Theorem (M. Wołoszkiewicz, S.R. (2010)) *Let* $n \in \mathbb{N}$ *be fixed. Then*

(i) $M_n(\alpha)$ *is a differentiable, strictly decreasing and convex function of* α *in* $0 \leq \alpha < \infty$ *with* $M_n(0) = 1$ *and* $\lim_{\alpha \to \infty} M_n(\alpha) = \frac{1}{n}$.

(ii) Let $\alpha > \frac{1}{1+\rho n}$. *Then we have* $M_n(\alpha) = \alpha\,(s_n(\alpha) - 1)$, *where* $s = s_n(\alpha)$ *is the unique solution of the equation*

$$s\,T_{n+1}(s^{-1/(n+1)}) = 1 - 1/\alpha, \quad 1 < s < \rho_n\,,$$

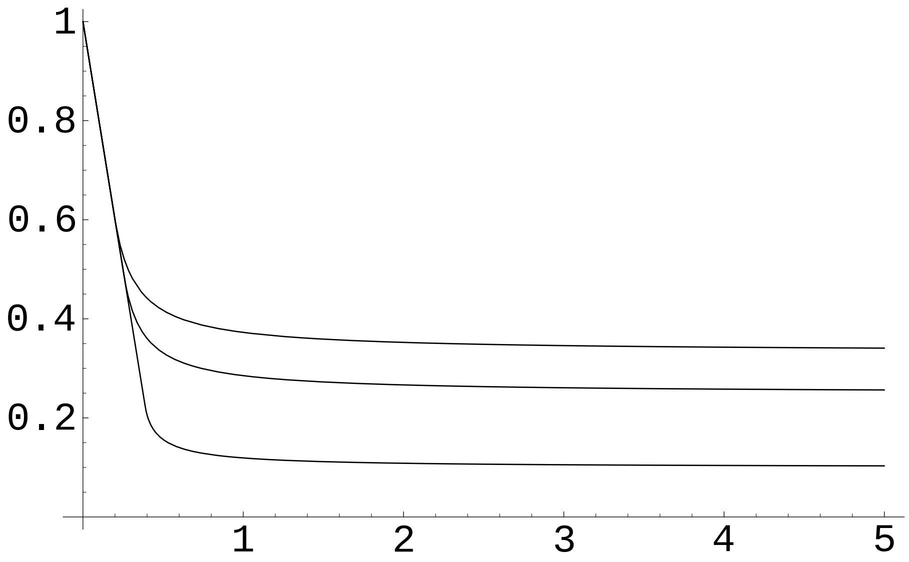
In this range of α *the only extremal polynomials* $P \in \mathcal{P}^*_{n-1}$ *with* $M_n(\alpha) = ||zP(z)+\alpha|| - \alpha$ *are*

$$P(z) = \alpha \frac{Q_{n,\rho}(xz) - 1}{z}\,, \quad |x| = 1,$$

where $\rho = \cos(s_n(\alpha))^{-n-1}$

(iii) For $0 \leq \alpha \leq \frac{1}{1+\rho n}$ *we have*

$$M_n(\alpha) = 1 - 2\alpha.$$



$M_n(\alpha)$ for $n = 3, 4, 10$

The proof is based on a proper reformulation of the "$M_n(\alpha)$ problem" into a "Maximal Range problem". Eventually one is left with the following technical inequality:

Lemma *For $n \in \mathbb{N}$ and $1 < \rho \le \rho_n$ we have*

$$|Q_{\rho,n}(z) - 1| \le |Q_{\rho,n}(1) - 1|, \quad |z| \le 1.$$

The proof is rather involved. There is numerical evidence for the following, even stronger statement:

Conjecture Let

$$Q_{\rho,n}(z) = 1 + \sum_{k=1}^{n} a_k(\rho, n) z^k.$$

Then, for $k = 1 \ldots, n$ and $1 < \rho \le \rho_n$

$$a_k(\rho, n) < 0.$$

Remark: Surprisingly, for a particular (non-trivial) value of α, namely $\alpha = 1$, we can evaluate M_n explicitly:

For $n \in \mathbb{N}$ we have

$$M_n(1) = \left(\cos\left(\frac{\pi}{2n+2}\right)\right)^{-n-1} - 1 = \frac{\pi^2}{8n} + \mathcal{O}\left(\frac{1}{n^2}\right).$$

C. Further results on Maximal Ranges.

Theorem (A. Córdova, C. Genthner, L. Salinas, S.R. (1990-2003)) *Let Ω be a simply connected domain with $1 \in \Omega$ and let $n \in \mathbb{N}$. Then every extremal polynomial $P \in \mathcal{P}_n(\Omega)$ is*

1) univalent in $\mathbb{D}$ with all zeros of its derivative on $\partial\mathbb{D}$.

2) On each arc of $\partial\mathbb{D}$ with endpoints in two such zeros x, y there exists a "point of contact", i.e. a $w \in \partial\mathbb{D}$ with $P(w) \in \partial\Omega$.

3) Let Ω be a convex domain and $\omega \in \partial\Omega_n \setminus \partial\Omega$. If $\omega = P(1)$ for an extremal polynomial from $\mathcal{P}_n(\Omega)$ then for each arc $\gamma_{(a,b)} := \{P(e^{it}) \,:\, t \in (a,b)\}$ with $0 \in (a,b)$ we have

$$\gamma_{(a,b)} \subset \Omega \Rightarrow \gamma_{(a,b)} \subset \partial\Omega_n.$$

In this case the extremal polynomial is uniquely determined.

Arc-Conjecture (A. Córdova, S.R.) Item 3) of the above Theorem holds for all simply connected domains.

The latest result concerning Maximal Ranges is about the following

Conjecture Let Ω be simply connected with $1 \in \Omega$. Then

$$(\Omega_n :=) \bigcup_{P \in \mathcal{P}_n(\Omega)} P(\mathbb{D}) = \bigcup_{P \in \mathcal{P}_n^u(\Omega)} P(\mathbb{D}),$$

where $\mathcal{P}_n^u(\Omega)$ stands for the set of all **univalent** polynomials in $\mathcal{P}_n(\Omega)$ (with all zeros of the derivative on $\partial\mathbb{D}$).

Theorem (V. Andrievskii, M. Wołoszkiewicz, S.R. (2011)) The conjecture holds for starlike domains Ω w.r.t. 1.

Finally:

There is some ongoing work with V. Andrievskii concerning the question of how fast a sequence of extremal polynomials for a given simply-connected Ω converges to a conformal map of $\mathbb{D}$ onto Ω. In principle, the extremal polynomials can be calculated numerically.

Finally:

There is some ongoing work with V. Andrievskii concerning the question of how fast a sequence of extremal polynomials for a given simply-connected Ω converges to a conformal map of $\mathbb{D}$ onto Ω. In principle, the extremal polynomials can be calculated numerically.

Thank you for your attention !!