Smooth linear congruences of lines

A congruence of lines in PN is an (N-1)-dimensional subvariety of the Grassmann variety Gr(1,N). A congruence is called linear if it is cut in Gr(1,N) by a linear subspace of the ambient Plücker space. The order of a congruence is the number of its lines passing through a general point of PN. It is clear that the order of a linear congruence can be either 1 or 0. The points of PN through which there pass infinitely many lines of the family form the fundamental locus of the congruence; the fundamental locus has a natural scheme structure. We are particularly interested in smooth linear congruences of order 1 whose fundamental locus is smooth and connected.

We shall recall the notion of Severi variety. Severi varieties were introduced and classified by Fyodor Zak. To each smoothly projected Severi variety, one can associate the corresponding congruence of 3-secant lines. It is well known that this is a linear congruence of order 1 whose fundamental locus is the projected Severi variety.

In the same context, we introduce another interesting family; Palatini varieties . The 4-secant lines to a (smooth) Palatini variety form a linear congruence of order 1 whose fundamental locus coincides with the variety itself. Palatini varieties need to be classified (we only know examples in dimensions 3 and 6).

In the talk we will discuss the known examples of linear congruences of order 1 with smooth fundamental locus and the structure of their fundamental loci. Using a recent joint work with Laurent Gruson (the k-secant lemma which I will recall), we introduce an interesting invariant for linear congruences, the secant index. We observe that we do not know linear congruences of order 1 with smooth fundamental locus and secant index >4.

We close this talk with two conjectures related to the work of F. Zak.