Kähler–Einstein metrics, slope stability and Fano bundles

Weekly seminar   08. July 2011.   17:00
Laboratory of algebraic geometry and its applications.
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~~X: a complex Kähler mfd.~~

~~(X, ω)~~ (X,

A compact complex manifold is said to be Kähler–Einstein if it possesses a Kähler form $\omega = \frac{\sqrt{-1}}{2\pi} \sum \omega_{i\bar{j}} dz_i \wedge d\bar{z}_j$ satisfying Einstein condition

$$\mathrm{Ricci}(\omega) = \lambda \omega$$

for some real constant λ.

$$\mathrm{Ricci}(\omega) = -\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log \det(\omega_{i\bar{j}}).$$

<u>Question</u>

If a complex Kähler mfd has a negative, zero or positive 1st Chern class, does it have a K-E metric (in the same class as its Kähler metric, unique up to rescaling)
a constant scalar curvature Kähler form.

c_1 $-$ (general type)

Th. Aubin & S.T. Yau in 1976 in $c_1(K_X)$

c_1 0 (Calabi–Yau)

S.T. Yau. arbitrary class.

c_1 + (Fano mfd).
The answer is negative.
Blow up 1 & 2 points of $\mathbb{P}^2$.

Question X : a Fano manifold.

~~$(X, -K_X)$~~ Does X allow a K-E metric in $c_1(-K_X)$?

dim $X = 1$, $X = \mathbb{P}^1$ Fubini-Study metric

$|w|^2 = \sum |w_i|^2$ $\omega = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log |w|^2$

dim $X = 2$.

deg of X		
9	$\mathbb{P}^2$ Yes	
8	$\mathbb{P}^1 \times \mathbb{P}^1$ Yes,	$\mathbb{F}_1$ No
7	No	
6	Yes	
5	"	
4	"	
3	"	
2	"	
1	"	

dim $X \geq 3$.

How do
allows

Late 1990
K-E

Sufficient

α-invari

$\alpha(X)$

Example

① del Pezzo
q

$\Rightarrow$ S

② General
in $\mathbb{P}^{n+1}$

$\Rightarrow$ V

we determine whether a given Fano mfd
a K-E metric or not. ?

s Yau suggests.

$\Longleftrightarrow$ "algebro-geometric stability"

ondition. α-invariant.

ant is introduced by Tian & Yau.

$> \frac{n}{n+1} \Rightarrow$ X admits a K-E.

surface S of deg 1 $\alpha(S) \geq 5/6$

deg 2 $\alpha(S) \geq 3/4$.

admits a K-E

ypersurface. V of deg $n+1$ $\alpha(V) > \frac{n}{n+1}$.

, $n \geq 2$

admits a K-E

- Smooth double cover W of $\mathbb{P}^n$, $n \geq 2$ ramified along a hypersurface of degree $2n$

$$\alpha(W) \geq \frac{2n-1}{2n}.$$

$\Rightarrow$ W admits a K-E.

~~devis~~ α-invariant is not a necessary ~~cond~~ condition for K-E.

$\mathbb{P}^n$ has a K-E metric

$$\alpha(\mathbb{P}^n) = \frac{1}{n+1}.$$

A cubic surface with an Eckardt point

Necessary condition for K-E.

Late 1980's Yau suggests.

Existence of K-E $\leftrightsquigarrow$ "algebro-geometric stability"

<u>Conjecture</u>

(X, L) polarized mfd with ample line bundle L

$\exists$ a constant scalar curvature Kähler form in $c_1(L)$

$\Longleftrightarrow$ K-polystability of (X, L)

$(\Rightarrow)$ has been proved by Donaldson, Stoppa.

K-polystability is hard to check.

K-polystable $\Rightarrow$ K-stable $\Rightarrow$ Slope stable

$\not\Leftarrow$ (Ross & Thomas)

$\mathbb{F}_1$: not slope stable

not K-stable.

Two points blow-up of $\mathbb{P}^2$: slope stable, not K-stable.

Slope stability is relatively ~~easi~~ easy to check.

(X,L) : n-dim'l polarized mfd with an ample line bundle L.

$Z \subset X$: a proper closed subscheme of X.

The <u>Seshadri constant</u> of Z w.r.t. L on X

$$\varepsilon(Z) = \varepsilon(X, L : Z) := \max\{ c \mid \sigma^* L - cE \text{ is nef} \}$$

where $\sigma : \hat{X} \longrightarrow X$ is the blowup along Z with the exceptional divisor E.

Consider the Hilbert polynomial

$$\chi(\mathcal{O}_X(kL)) = a_0 k^n + a_1 k^{n-1} + O(k^{n-2}) \quad k \gg 0.$$

<u>The slope of $(X.L)$</u>

$$\mu(X) = \mu(X, L) := \frac{a_1}{a_0} = -\frac{nK_X . L^{n-1}}{2L^n}$$

In particular, $L = -K_X \Rightarrow \mu(X) = n/2$

$x \in \mathbb{Q}_{>0}$.

$$\chi(\mathcal{O}_{\hat{X}}(\sigma^*(kL) - xkE))$$
$$= a_0(x) k^n + a_1(x) k^{n-1} + O(k^{n-2})$$

$k \gg 0 \quad xk \in \mathbb{N}$

$a_i(x)$ can be extended to all $x \in \mathbb{R}$.

Set $\hat{a}_i(x) := a_i - a_i(x)$.

<u>The slope of Z</u> w.r.t. L and $\lambda \in (0, \varepsilon(Z)]$

$$\mu_\lambda(Z) = \frac{\int_0^\lambda \left(\hat{a}_1(x) + \frac{\hat{a}_0'(x)}{2}\right) dx}{\int_0^\lambda \hat{a}_0(x)\, dx}.$$

<u>Def</u> $(X.L)$ is <u>slope stable</u> (resp. slope semistab <u>w.r.t. Z</u> if

$$\mu_\lambda(Z) > \mu(X) \quad \text{for all } \lambda \in (0. \varepsilon(Z)].$$
(resp. $\geq$)

Consider $(X, -K_X)$ $\quad Z \subset X$ a smooth closed subvar. of codim r.

- <u>Simple observation by Ross</u>

$$\boxed{\varepsilon(Z) \leq r \Rightarrow (X, -K_X) \text{ is slope stable w.r.t. } Z.}$$

- <u>Another simple observation by P-</u>

$$\boxed{\varepsilon(Z) > r \Rightarrow Z \text{ and the exceptional divisor } E \text{ over } Z \text{ are Fano.}}$$

$$-(K_{\hat{X}} + E) = \sigma^*(-K_X) - rE.$$

$\varepsilon(Z) > r \Rightarrow -(K_{\hat{X}} + E)$ is ample

$\Rightarrow$ $-K_E$ is ample.

$\Rightarrow$ E is a Fano and hence Z is a Fano.

$(\sigma|_E : E \to Z$ a surjective smooth morphism$)$.

<u>Cor</u> $(X, -K_X)$ is slope stable w.r.t. every smooth non-rational curve.

<u>Further observation</u>

$V = \mathbb{P}(\bigoplus_{i=1}^{n-1} \mathcal{O}_{\mathbb{P}^1}(a_i))$ $\quad a_1 = 0, \ a_i \in \mathbb{Z}_{\geq 0}$.

is Fano

$\Rightarrow$ The bundle is trivial or $\mathcal{O}_{\mathbb{P}^1}^{n-2} \oplus \mathcal{O}_{\mathbb{P}^1}(1)$.

$(X, -K_X)$ is not slope stable w.r.t. a smooth rational curve Z.

$\Rightarrow \varepsilon(Z) > n-1. = \operatorname{codim}_X Z$

$\Rightarrow E \overset{= \mathbb{P}(N^*_{Z/X})}{} $ is a Fano

$\Rightarrow N^*_{Z/X} \cong (\mathcal{O}_{\mathbb{P}^1}(a))^{\oplus n-1}$

or

$(\mathcal{O}_{\mathbb{P}^1}(a))^{\oplus n-2} \oplus \mathcal{O}_{\mathbb{P}^1}(a+1)$

where $a \in \mathbb{Z}$

$\Rightarrow N_{Z/X} \cong (\mathcal{O}_{\mathbb{P}^1}(-a))^{\oplus n-1}$

or

$(\mathcal{O}_{\mathbb{P}^1}(-a))^{\oplus n-2} \oplus \mathcal{O}_{\mathbb{P}^1}(-a-1)$

The degree of $N_{Z/X}$.

Z : smooth rational curve. $\deg(N_{Z/X}) = -2 - K_X.Z$

⊙ $-K_X.Z = 1$.

$(X, -K_X)$ is slope (semi) stable w.r.t. Z

$\Longleftrightarrow$ $\varepsilon(Z) \underset{(<)}{\leq} \sqrt{n^2-1}$

⊙ $-K_X.Z = 2$

" " "

$\Longleftrightarrow$ $\varepsilon(Z) \underset{(<)}{\leq} n$

⊙ $-K_X.Z \geq 3$ & $\varepsilon(Z) \leq -K_X.Z$

$(X, -K_X)$ is slope semistable w.r.t. Z

$(X, -K_X)$ is not slope stable w.r.t. Z

$\Longleftrightarrow$ $\varepsilon(Z) = -K_X.Z = n+1$.

$0 \leq (\sigma^*(-K_X) - \varepsilon(Z)E).C \leq -K_X.Z - \varepsilon(Z)$.

$\Rightarrow$

$-K_X.Z \geq 3$. $\varepsilon(Z) = -K_X.Z = n+1$ happen only when $X = \mathbb{P}^n$, Z is a line.

⊙' $-K_X.Z \geq 3$.

$(X, -K_X)$ is slope stable w.r.t. Z

except when $X \cong \mathbb{P}^n$ and Z is a line.

(*)

Z has trivial normal bundle.

$\Rightarrow$ $\exists$ a rational curve C with $C \neq Z$, $C \cap Z \neq$ and $-K_X.C \leq n$.

$\Rightarrow$ $\varepsilon(Z) \leq -K_X.C \leq n$.

<u>Deformation of rational curves</u>

$-K_X.Z \geq 3 \Rightarrow \varepsilon(Z) \leq -K_X.Z$

$\because$ One can find a curve $C\ (\neq Z)$ s.t.

$C \equiv Z$ and $C \cap Z \neq \emptyset$

algebraically equiv.

Thm (Hwang, Kim, Lee, P–)

$(X, -K_X)$ is not slope stable w.r.t. Z

$\Rightarrow$

① Z has trivial normal bundle

② $N_{Z/X} = (\mathcal{O}_{\mathbb{P}^1})^{n-2} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$

③ $X \cong \mathbb{P}^3$ and Z is a line.

By (*).

$(X, -K_X)$ is slope semistable w.r.t. Z in case ①.

" " " in case ③

Suppose ③ is not the case.

$\Rightarrow -K_X.Z = 1$ or 2

$$\deg(N_{Z/X}) = -2 - K_X.Z = 0 \quad \text{if } -K_X.Z = 2$$
$$= -1 \quad \text{if } -K_X.Z = 1.$$

$$N_{Z/X} = (\mathcal{O}_{\mathbb{P}^1}(-a))^{n-1} \Rightarrow a = 0$$

$$\text{or } (\mathcal{O}_{\mathbb{P}^1}(-a))^{n-2} \oplus \mathcal{O}_{\mathbb{P}^1}(-a-1) \Rightarrow a = 0$$

$$-(n-1)\cdot a = 0 \text{ or } -1$$
$$-(n-1)a - 1 = 0 \text{ or } -1 \qquad \Rightarrow a = 0$$

Kento Fujita "Fano manifolds which are not slope stable along curves".

~~Today~~ ~~Also~~ ~~Again~~

X : Fano n-fold with $n \geq 3$

Z : a smooth curve.

$(X, -K_X)$ is not slope stable w.r.t. Z

$\Longleftrightarrow$ $(X, Z) \cong (\mathbb{P}^n, \text{line})$

$(\mathbb{P}^1 \times \mathbb{P}^{n-1}, \mathbb{P}^1 \times pt)$

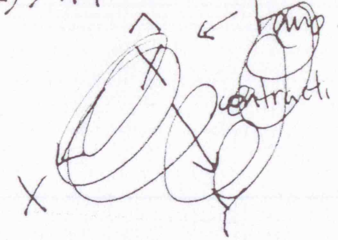
$(Bl_{\mathbb{P}^{n-2}} \mathbb{P}^n, \text{excp. line})$

$(X, -K_X)$ is not slope semistable w.r.t Z

$\Longleftrightarrow$ $(X, Z) \cong (Bl_{\mathbb{P}^{n-2}} \mathbb{P}^n, \text{excp. line})$

$n \geq 3$

$\varepsilon(Z) > n-1$



$K_{\hat{X}}$ - negative extreamal rays R with $(E \cdot R) > 0$ have of fiber type

or $\exists$ a prime divisor $D \subset X$ with $D \cong \mathbb{P}^{n-1}$ and $N_{D/X} \cong \mathcal{O}_{\mathbb{P}^{n-1}}$ s.t. D contains Z as a line

$\wedge$ Fano

an extremal ray $R \subset \overline{NE}(\hat{X})$ with a minimal rational curve $[C] \in R$ s.t. $E \cdot C > 0$.

Contraction of R

$$\begin{array}{ccc} \hat{X} & \longrightarrow & Y \\ \downarrow & & \\ X & & \end{array}$$

$\dim Y \geq n-2$.

$\dim Y = n-2$ $(\mathbb{P}^n, \text{line})$ $(\mathbb{Q}^n, \text{conic})$

$\dim Y = n-1$ $(\mathbb{Q}^n, \text{line})$, $(\mathbb{P}^1 \times \mathbb{P}^{n-1}, \mathbb{P}^1 \times pt)$

$(Bl_{\mathbb{P}^{n-2}} \mathbb{P}^n, \text{line disj. from } \mathbb{P}^{n-2})$

$(Bl_{\mathbb{P}^{n-2}} \mathbb{P}^n, \text{excep. line})$.