Quasi-smooth Fano 3-fold hypersurfaces in the 95 families.

The 95 families of weighted Fano 3-fold hypersurfaces
Quasismooth anticanonically embedded weighted Fano 3-fold hypersurfaces with terminal singulari

$$X_d \subset \mathbb{P}(1, a_1, a_2, a_3, a_4)$$

where $d = a_1 + \cdots + a_4$ $\quad a_1 \leq a_2 \leq a_3 \leq a_4$.

1979 M. Reid codimension 1 weighted K3 surface
95 families of (K3 surfaces in 3-dim'l weighted projective spaces.)
"Canonical 3-folds"

1988 A.R. Iano-Fletcher
95 families of weighted Fano 3-fold hypersurfaces.
"Working with weighted complete intersections"

2001. J. Johnson and J. Kollár
Fano hypersurfaces in weighted projective 4-spaces. Exp. Math 10 (2001).

Examples. $X_4 \subset \mathbb{P}^4$.

$X_5 \subset \mathbb{P}(1,1,1,1,2)$ $\quad \frac{1}{2}(1,1,1)$

$X_6 \subset \mathbb{P}(1,1,1,1,3)$

$X_{66} \subset \mathbb{P}(1,5,6,22,33)$.

$\frac{1}{5}(1,2,3)$ $\quad \frac{1}{2}(1,1,1)$

$\frac{1}{3}(1,1,2)$ $\quad \frac{1}{11}(1,5,6)$

Many properties of ~~the~~ hypersurfaces in the 95
have been studied (since 2000).

- ✓ Non-rationality (birational rigidity).
- ✓ Groups of birational automorphisms
- ✓ Elliptic fibration structures
- ✓ Pencils of K3 surfaces.
- ✓ Global log canonical thresholds.
- ✓ Arithmentic properties (potential density).

2000. A Corti. A. Pukhlikov, M. Reid.
A <u>general</u> hypersurface in each of the 95 families of weighted Fano 3-fold hypersurfaces is ~~birational~~ birationally rigid.

<u>Def</u> A terminal $\mathbb{Q}$-factorial Fano variety V with $\mathrm{Pic}(V) \cong \mathbb{Z}$ is birationally rigid (super-rigid. resp.) if

(i) V cannot be birationally transformed into another Mori fibration except itself.
((ii) $\mathrm{Bir}(V) = \mathrm{Aut}(V)$. resp.)

Conjecture (CPR)
The theorem of CPR holds for eve
hypersurface in each family.

Classical result

$\mathbb{P}(1,1,1,1.$

◎ Every smooth quartic $X_4 \subset \mathbb{P}^4$ is
super-rigid. (Iskovskikh – Manin
○ Every smooth double cover of $\mathbb{P}^3$ r
a sextic surface is birationally s
$X_6 \subset \mathbb{P}(1,1,1,1,3)$ (Isk

<u>Thm</u> (Cheltsov, P–) the
Every quasi-smooth hypersurface in ea
95 families whose general members
super-rigid is birationally super-rig

<u>Rmk</u> ○ 48 families out of 95 famili

○ The conjecture has be checked f
about 87 families.

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- A quasi smooth hypersurface in families of the
95 families whose general members birationally
- rigid can be birationally super-rigid.

$$X_5 \subset \mathbb{P}(\underset{x}{1}, \underset{y}{1}, \underset{z}{1}, \underset{t}{1}, \underset{w}{2})$$

$$tw^2 + x^5 + y^5 + z^5 + t^5 = 0$$

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Find

- Fano Inequality

a terminal $\mathbb{Q}$-factorial Fano variety
with $\mathrm{Pic}(X) \cong \mathbb{Z}$.

very positive integer n and every mobile
l system $\mathcal{M} \subset |-nK_X|$

he pair $(X, \frac{1}{n}\mathcal{M})$ has canonical singulariti

is birationally super-rigid.

the theorem we use the arguments of CPR
ome modifications + α.

$X := X_d \subset \mathbb{P}(1, a_1, a_2, a_3, a_4)$ quasi-smooth
$\mathcal{M} \subset |-nK_X|$ a mobile linear system.
$(X, \frac{1}{n}\mathcal{M})$ is <u>not</u> <u>canonical</u>.

a <u>contradiction !!</u>

~~(Canonical threshold)~~

canonic

$c := \max\{ \lambda \mid (X, \lambda M) \text{ is canonical} \} < \frac{1}{n}.$

$\Rightarrow$ $\exists$ an <u>extremal extraction</u> $f : Y \to X$
(Y : terminal, E : irreducible, $-K_Y$: f-ample)

with exceptional divisor E s.t.

$$c = a_E(K_X) / \mathrm{mult}_E(M)$$

(i.e. $K_Y + c \cdot f_*^{-1}(M) = f^*(K_X + cM)$)

The extraction is called strong maximal singularity of (X, M)

$f(E) \subset X$ is the center of a strong maximal singularities.

The center can be $\begin{cases} \text{a smooth point of } X \\ \text{a singular point of } X \\ \text{an irreducible curve of } X. \end{cases}$

$(X, \frac{1}{n}M)$ is not canonical $\Rightarrow$ $\exists$ an center of a strong max singularities.
Exclude these three possibilities.

◎ Smooth point cannot be a center.

$p \in X$ smooth point.
the center of a strong max. sing of $(X, \frac{1}{n}$

$\Rightarrow \mathrm{mult}_p(M^2) > 4n^2$

s : positive integer

$H \subset |-sK_X|$ s.t.

(i) $p \in H$

(ii) it contains no 1-dimensional component of the base locus of M that contains p

~~$H \cdot M^2 \leq (-sn^2 K_X^3)$~~ $\Rightarrow$

$$\underset{\overset{\|}{-sn^2K_X^3}}{H\,M^2} \geq \mathrm{mult}_p H \ \mathrm{mult}_p M^2 > 4n^2$$

$\Rightarrow \quad -sK_X^3 > 4.$

~~Suppose~~ If we have such a surface H with $s \leq \frac{4}{-K_X^3}$

then we obtain an contradiction, so that a smooth point ~~can~~ could not be a center.

$\hat{a}_i = \text{l.c.m}\{a_1, a_2, a_3, a_4\} \setminus \{a_i\}$

X satisfies one of the following

⊙ X does not pass through O_w and $d \cdot \hat{a}_4 \leq 4a_1 \cdots a_4$

⊙ 〃 O_t $d \cdot \hat{a}_3$ 〃

⊙ 〃 O_z $d\hat{a}_2$ 〃

∴

$\pi_4 : X \to \mathbb{P}(1, a_1, a_2, a_3)$. projection from O_w.

$|\mathcal{O}_{\mathbb{P}(1,a_1,a_2,a_3)}(\hat{a}_4)|$ is base point free

Choose a general member in $|\mathcal{O}_{\mathbb{P}(1,a_1,a_2,a_3)}(\hat{a}_4)|$ that passes through the point $\pi_4(p)$.

ⓑ Irreducible curve cannot be a center.

$L \subset X$ an irreducible curve.

Suppose L is a center.

Suppose $L \subset$ smooth locus of X.

$\Rightarrow \text{mult}_L(\mathcal{M}) > n$.

$H_1, H_2 \in \mathcal{M}$ general member

$-K_X \cdot H_1 \cdot H_2 \geq (\text{mult}_L(\mathcal{M}))^2 \cdot (-K_X \cdot L) > -n^2$

$\overset{\shortparallel}{-n^2 K_X^3}$

$\Rightarrow 1 \leq -K_X \cdot L < -K_X^3$.

Our cases have $-K_X^3 < 1$ except quartic 3-fo sextic doubl solid.

∴ L is a center $\Rightarrow$ L must contain a singular point p.

$\xRightarrow[\text{Kawamata}]{}$ p is a center.

L is not a center.

@A Singular point cannot be a center.

$p \in X$ a center. type $\frac{1}{r}(1, a, r-a)$.

$f : (E \subset Y) \to (p \in X)$ extremal extraction.

= weighted blow up at p with weight $(1, a, r-a)$.

$$X := X_d \subset \mathbb{P}(\underset{\overset{\uparrow}{x}}{1}, a_1, a_2, a_3, a_4)$$

$A = f^*(-K_X)$.

$B = -K_Y$.

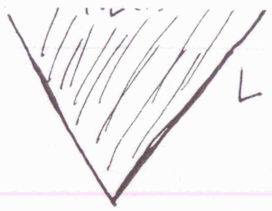
$S :=$ the proper transform of the surface cut by $x=0$.

$B^2 \in \mathrm{Int}(\overline{NE}Y) \subset N_1 Y$.

$m_E(M) > n\, a_E(K_X)$

$$B = \underset{\overset{\vee}{0}}{\underbrace{\frac{m-an}{m}}_{\alpha}} A + \underset{\overset{\vee}{0}}{\underbrace{\frac{a}{m}}_{\beta}} \cancel{M}' \Leftrightarrow K_Y + \frac{a}{m} M' = f^*(K_X + \frac{a}{m} M)$$

$$B^2 = \alpha^2 A^2 + 2\alpha\beta A \cdot M' + \beta^2 M'^2$$



(Case I) $B^3 \leq 0$. (except $\overset{N=82.}{X_{36} \subset \mathbb{P}(1,1,5,12,18)}$ $O_z = \frac{1}{5}(1,2,3$

$\exists$ a surface T in $|bB + cE|$ with $b>0$, $\frac{b}{r} \geq c \geq c$

s.t. (1) $\Gamma := S \cap T$ irred.

(2) $T \cdot \Gamma \leq 0$.

Conditions $\Rightarrow$ $Q = [\Gamma]$.

$\Gamma = B \cdot (bB + cE)$ $\qquad B = A - \frac{1}{r} E$

$= bB^2 + cL$

$B^2 = \frac{1}{b}\Gamma - \frac{c}{b} L \,.\, \notin \mathrm{Int}(\overline{NE(Y)})$

(~~General cases ER~~ With generality condition, this covers all)

$\Gamma := S \cap T$ consists of irreducible curves that are numerically propotional each other.

T is defined by a simple functions so that we could see the irr. components of Γ.

$\exists$ an index i s.t.

(i) $\exists$ a surface T in $|a_i A - \frac{m}{r} E|$ s.t. $a_i \geq m > 0$

(ii) $\Gamma := S \cap T$ consists of - - - - -

(iii) $T \cdot \Gamma \leq 0 \iff ra(r-a)a_i^2 A^3 \leq m^2$

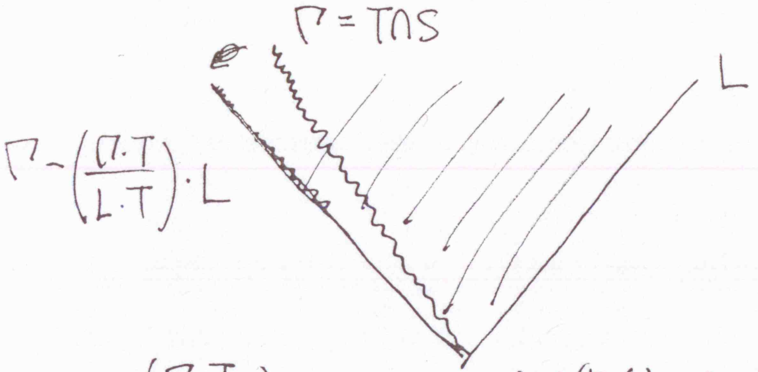
Extra cases.

$\exists$ an index i s.t.

(1) $\exists$ a divisor T in $|a_i A - \frac{m}{r} E|$ s.t. $m > 0$.

(2) T is nef.

(3) $ra(r-a)a_i A^3 \leq m$.



$$\Gamma - \left(\frac{\Gamma \cdot T}{L \cdot T}\right) L = \Gamma - \left(\frac{a(r-a)}{m} a_i^2 A^3 - \frac{m}{r}\right) L$$

$$= B\left(a_i B + \left(\frac{a_i}{r} - \frac{m}{r}\right) E\right) - \quad ''$$

$$= a_i B^2 + \underline{\left(\frac{a_i}{r} - \frac{a(r-a)}{m} a_i^2 A^3\right)} L.$$

If $\qquad \geq 0$, then $B^2 \notin \text{Int}$.

T is the proper transform of a surface cut by a coordinate function.

Case II) $B^3 > 0$.

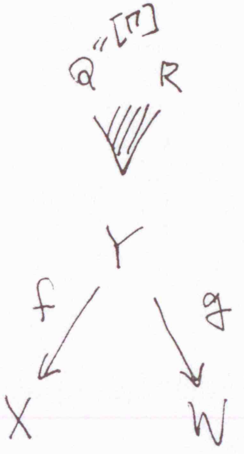
$\exists$ an index i s.t.

(1) $\exists$ an irreducible surface T in $|a_iA - \frac{m}{r}E|$ such that $m > a_i$

(2) $\Gamma = S \cap T$. irreducible curves that are numerically propotional each other.

(3) $B\cdot\Gamma \leq 0$.

$B\cdot\Gamma = 0$.



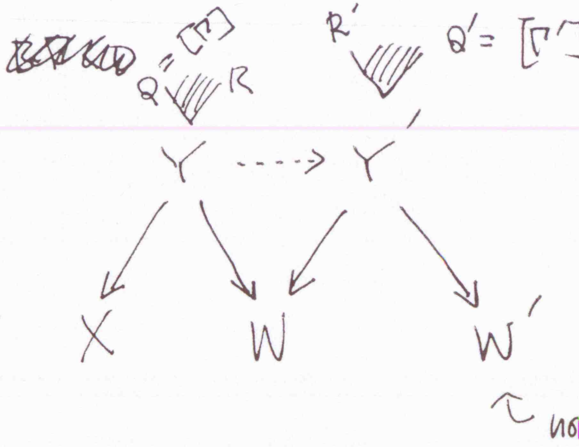
Since $B\cdot\Gamma = 0$, g cannot be a flopping contraction.

g is either a divisorial cont. or a fibre space.

$a_T(K_W) = 0 \Rightarrow W$ has a canonical but nonterminal sing.

$K_Y\cdot\Gamma = 0 \Rightarrow \varphi$ is not a Mori fibre space.

$D\cdot 1 < 0$.



One can show that ~~even~~ fo

$-C.K_Y = 0$.

$\Rightarrow$ $K_Y + f^{-1}(\frac{1}{n}M) = f^*(K_X + \frac{1}{n}$

$K_Y.C + f^{-1}(\frac{1}{n}M).C = -$

$f^{-1}(\frac{1}{n}M).C$

$\Rightarrow f(C) \subset M$.

$\Rightarrow f(T) \subset M$ $*$.

In this way, we can also exclu

terminal

every curve C on T

$M) - c\bar{E}$

$E.C$

< 0.

de singular points.