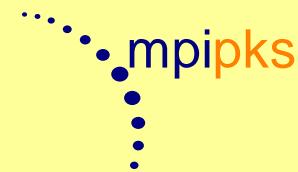


Nonlinear waves in localizing media

S. Flach, MPIPKS Dresden



- Localizing media: what is that?
- Properties of linear waves in localizing media
- Nonlinear waves in localizing media:
spreading, subdiffusion, delocalization

The Dresden Team:

P. Anghel-Vasilescu



Ch. Skokos



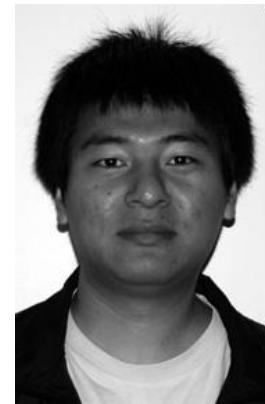
J. Bodyfelt



T. Lapteva



N. Li



G. Gligoric



Collaborations:

S. Aubry (Saclay)

M. Ivanchenko (N. Novgorod)

R. Khomeriki (Tbilissi)

S. Komineas (Heraklion)

D. Krimer (Tuebingen)

M. Tribelsky (Moscow)

R. Vicencio (Santiago)

Defining the problem

- a localizing medium
- linear waves: all eigenstates are localized
- add short range nonlinearity (interactions)
- follow the spreading of an initially localized wave packet

Will it delocalize?

Yes because of nonintegrability and ergodicity

No because of energy conservation –
spreading leads to small energy density,
nonlinearity can be neglected,
dynamics becomes integrable, and
localization is restored

Waves in localizing media

$$i\dot{\psi}_l = \epsilon_l \psi_l - \psi_{l+1} - \psi_{l-1}$$

$$\psi_l = A_l \exp(-i\lambda t) \quad \lambda A_l = \epsilon_l A_l - A_{l+1} - A_{l-1}$$

- uncorrelated random potential: Anderson localization
- quasiperiodic potential: Aubry-Andre (Harper) localization
- dc bias $\epsilon(\mathbf{l})=\mathbf{E}\cdot\mathbf{l}$: Wannier-Stark localization (Bloch oscillations)
- quantum kicked rotor: localization in momentum space,
loosely similar to quasiperiodic potential case

In all cases all (or almost all) eigenstates are spatially localized,
with finite upper bounds on the localization length / volume.

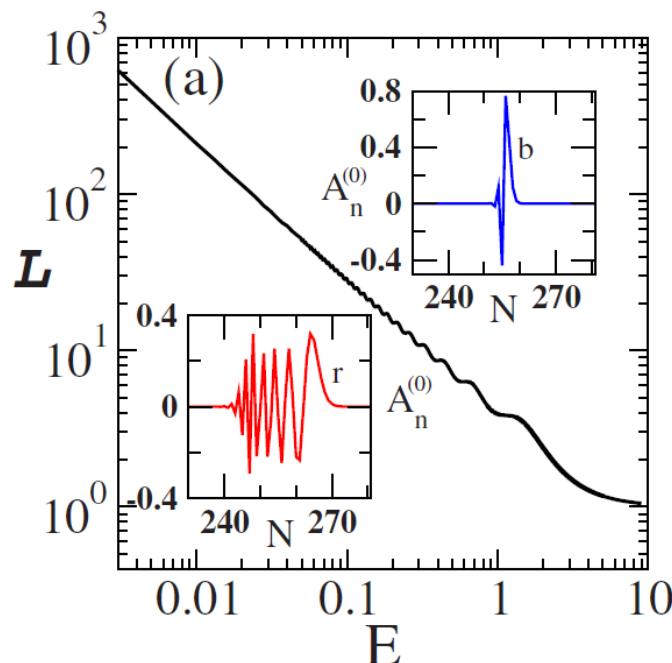
Wannier-Stark ladder

Wannier (1960)

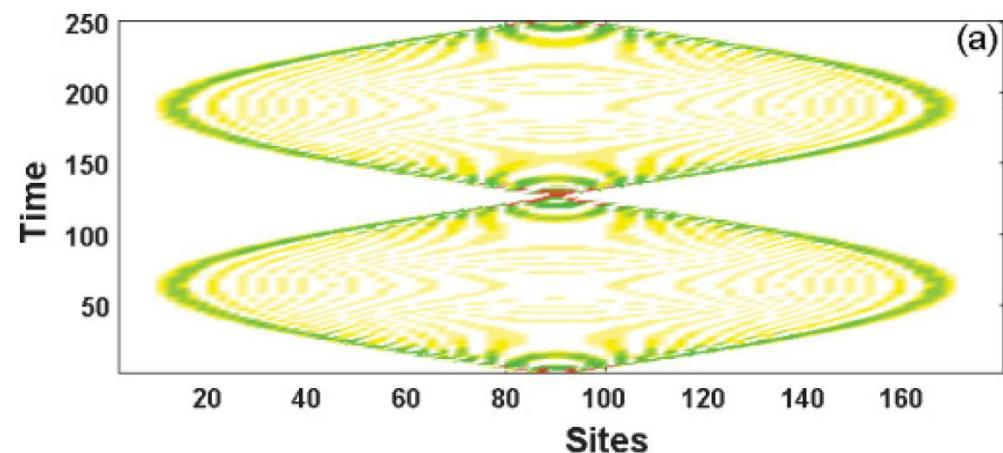
$$i\dot{\psi}_l = lE\psi_l - \psi_{l+1} - \psi_{l-1}$$

$$\lambda_\nu = E\nu \quad A_{l+\mu}^{\nu+\mu} = A_l^\nu \quad A_l^{(0)} = J_l(2/E)$$

Eigenfunctions Localization volume



Bloch oscillations for $E=0.05$



Krimer,Khomeriki,SF (2009)

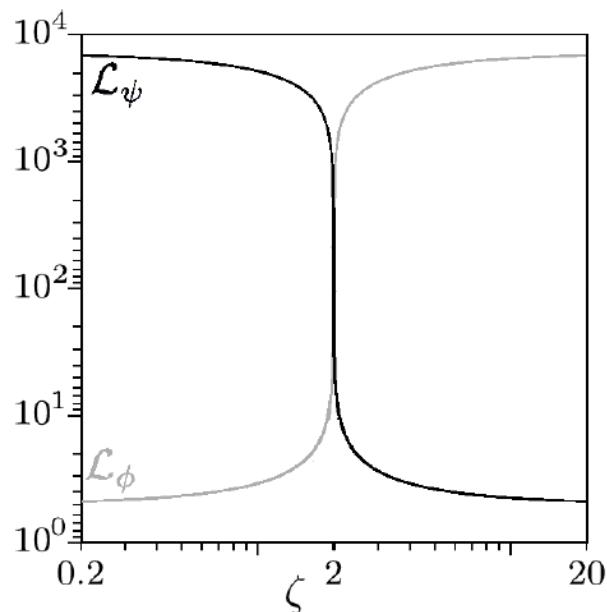
Aubry-Andre model

Aubry, Andre (1980)

$$i \frac{\partial \psi_l}{\partial t} = \zeta \cos(2\pi\alpha l) \cdot \psi_l - \psi_{l+1} - \psi_{l-1}$$

Self duality: $\psi_l = \sum_k e^{2\pi i \alpha k l} \phi_k$

$$i \frac{\partial \phi_k}{\partial t} = 2 \cos(2\pi\alpha k) \cdot \phi_k - \frac{\zeta}{2} \phi_{k+1} - \frac{\zeta}{2} \phi_{k-1}$$



adapted from Aulbach et al (2004)

Anderson localization

Anderson (1958)

$$i \frac{\partial \psi_l}{\partial t} = \epsilon_l \psi_l - \psi_{l+1} - \psi_{l-1} \quad \{ \epsilon_l \} \text{ in } [-W/2, W/2]$$

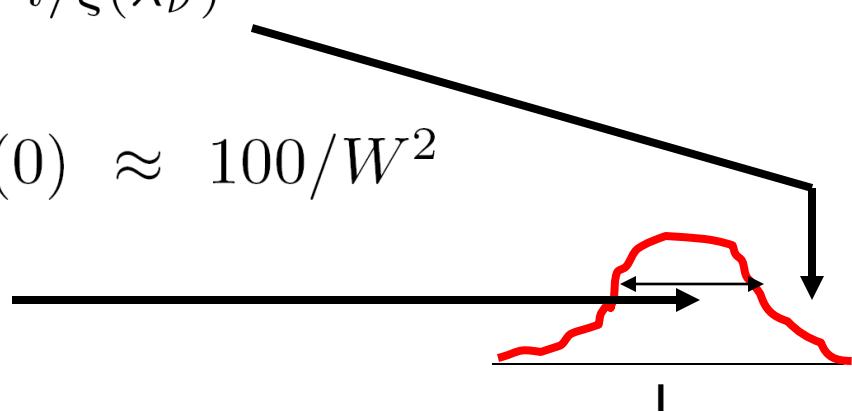
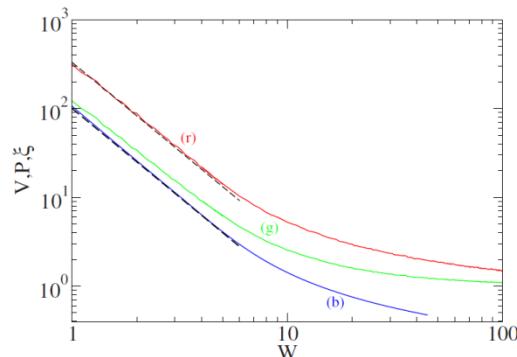
Eigenvalues: $\lambda_\nu \in \left[-2 - \frac{W}{2}, 2 + \frac{W}{2} \right]$

Width of EV spectrum: $\Delta = 4 + W$

Asymptotic decay: $A_{\nu,l} \sim e^{-l/\xi(\lambda_\nu)}$

Localization length: $\xi(\lambda_\nu) \leq \xi(0) \approx 100/W^2$

Localization volume of NM: L



Krimer,SF (2010)

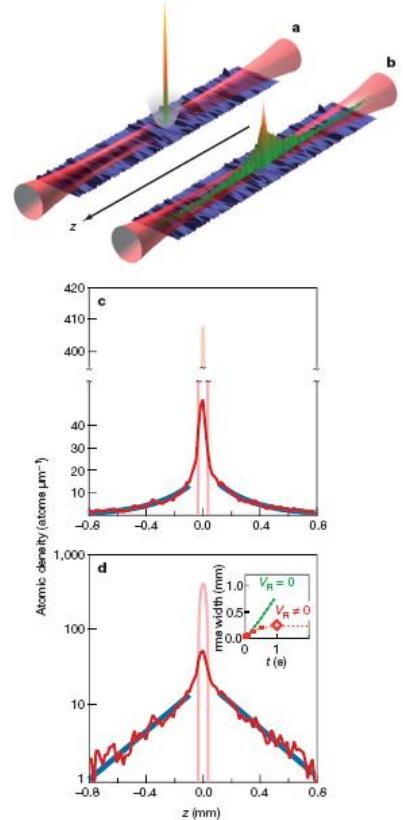
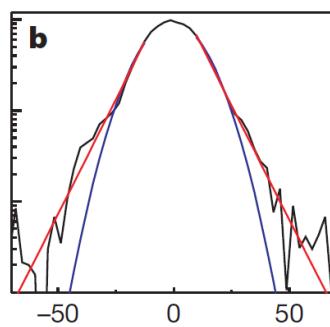
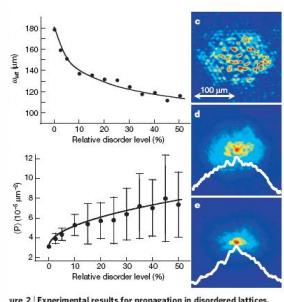
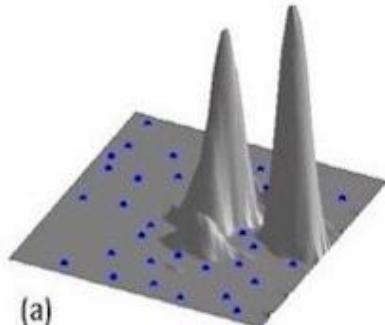
Experimental Evidence for Wave Localization

Ultrasound: Weaver 1990

Microwaves: Dalichaoush et al 1991, Chabanov et al 2000

Light: Wiersma et al 1997, Scheffold et al 1999, Pertsch et al (1999), Morandotti et al (1999), Stoerzer et al 2006, Schwartz et al 2007, Lahini et al 2008

BEC: Anderson et al (1988), Morsch et al (2001), Billy et al 2008, Roati et al 2008



The discrete nonlinear Schrödinger Equation

$$\mathcal{H}_D = \sum_l \epsilon_l |\psi_l|^2 + \frac{\beta}{2} |\psi_l|^4 - (\psi_{l+1} \psi_l^* + \psi_{l+1}^* \psi_l) \quad \dot{\psi}_l = \partial \mathcal{H}_D / \partial (i\psi_l^*)$$

$$i\dot{\psi}_l = \epsilon_l \psi_l + \beta |\psi_l|^2 \psi_l - \psi_{l+1} - \psi_{l-1}$$

Conserved quantities: energy and norm $S = \sum_l |\psi_l|^2$

Varying the norm is strictly equivalent to varying β

Equations model high intensity light propagation in structured media, and cold atom dynamics in optical potentials

Equations in normal mode space:

$$i\dot{\phi}_\nu = \lambda_\nu \phi_\nu + \beta \sum_{\nu_1, \nu_2, \nu_3} I_{\nu, \nu_1, \nu_2, \nu_3} \phi_{\nu_1}^* \phi_{\nu_2} \phi_{\nu_3}$$

$$I_{\nu, \nu_1, \nu_2, \nu_3} = \sum_l A_{\nu, l} A_{\nu_1, l} A_{\nu_2, l} A_{\nu_3, l}$$

NM ordering in real space: $X_\nu = \sum_l l A_{\nu, l}^2$

Characterization of wavepackets in normal mode space:

$$z_\nu \equiv |\phi_\nu|^2 / \sum_\mu |\phi_\mu|^2 \quad \bar{\nu} = \sum_\nu \nu z_\nu$$

Second moment: $m_2 = \sum_\nu (\nu - \bar{\nu})^2 z_\nu \longrightarrow$ location of tails

Participation number: $P = 1 / \sum_\nu z_\nu^2 \longrightarrow$ number of strongly excited modes

Compactness index: $\zeta = \frac{P^2}{m_2} \begin{cases} \nearrow & K \text{ adjacent sites equally excited: } \zeta = 12 \\ \searrow & K \text{ adjacent sites, every second empty or equipartition: } \zeta = 3 \end{cases}$

Scales, regimes

- Localization volume of eigenstate: L
- Frequency spectrum width
inside localization volume: Δ
- Average frequency spacing
inside localization volume: $d = \Delta/L$
- Nonlinearity induced frequency shift: $\delta_l = \beta |\psi_l|^2$

Three expected evolution regimes:

Weak chaos : $\delta < d$

Strong chaos : $d < \delta < \Delta$

Self trapping : $\Delta < \delta$

Disordered chains:

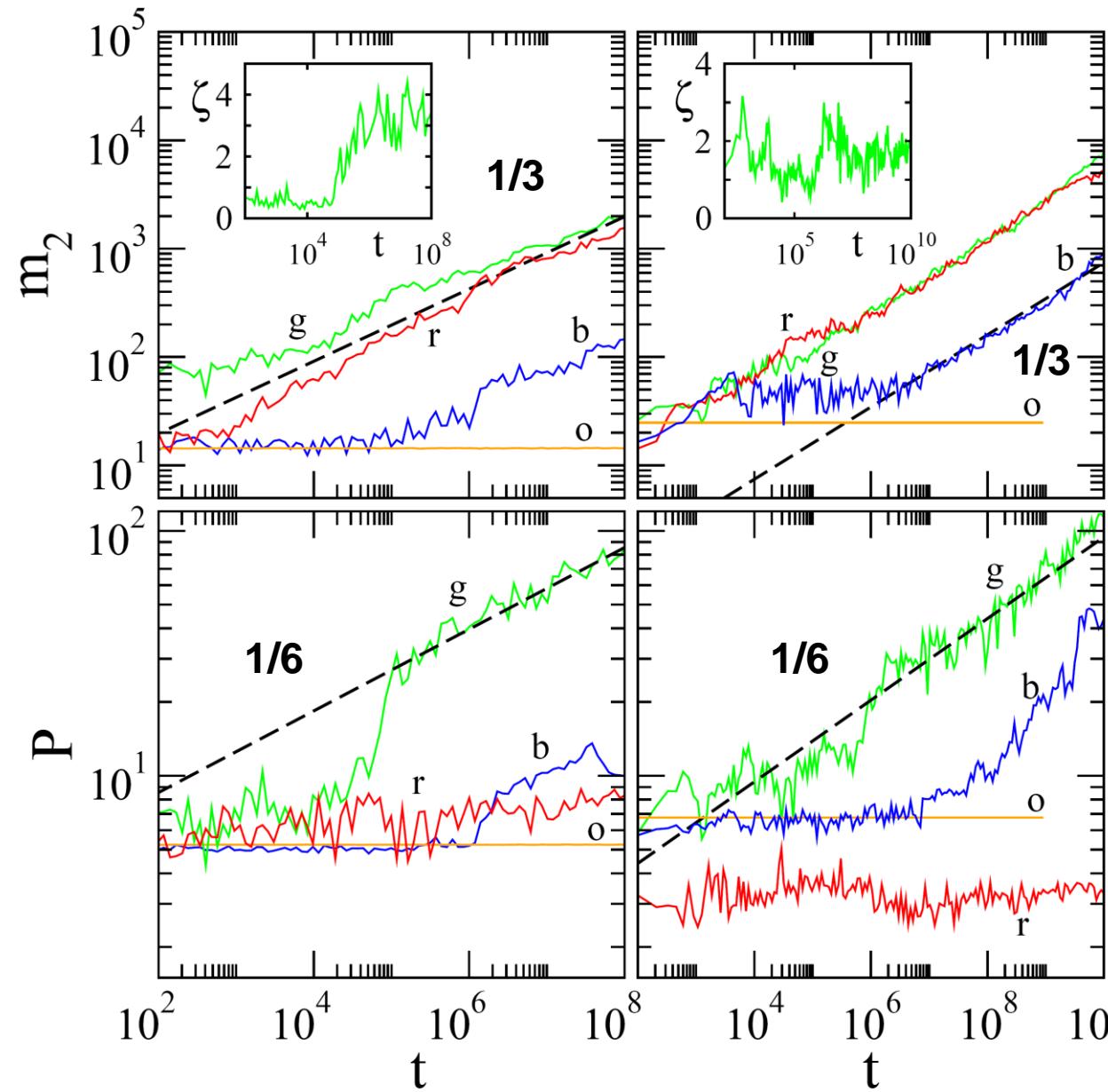
$$\psi_l = \delta_{l,l_0} \quad \epsilon_{l_0} = 0$$

DNLS $W=4$, $\beta = 0, 0.1, 1, 4.5$

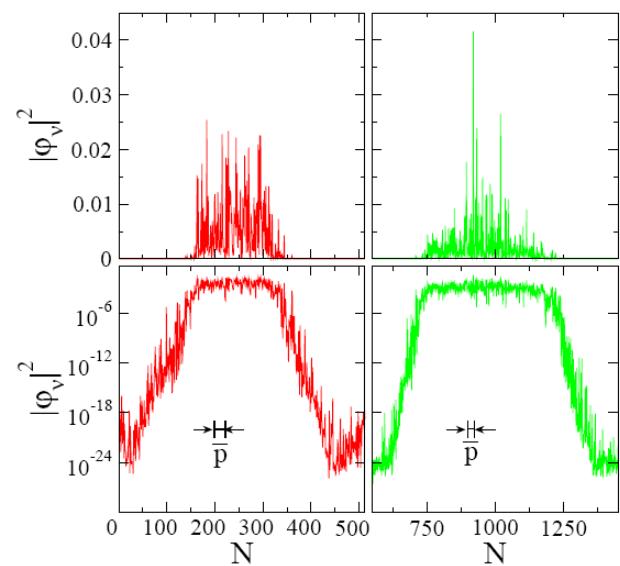
KG $W=4$, $E = 0, 0.05, 0.4, 1.5$

SF,Krimer,Skokos (2009)

Skokos,Krimer,Komineas,SF (2009)



Wavepacket spreads
way beyond localization
volume.
DNLS at $t = 10^8$



Disordered chains:

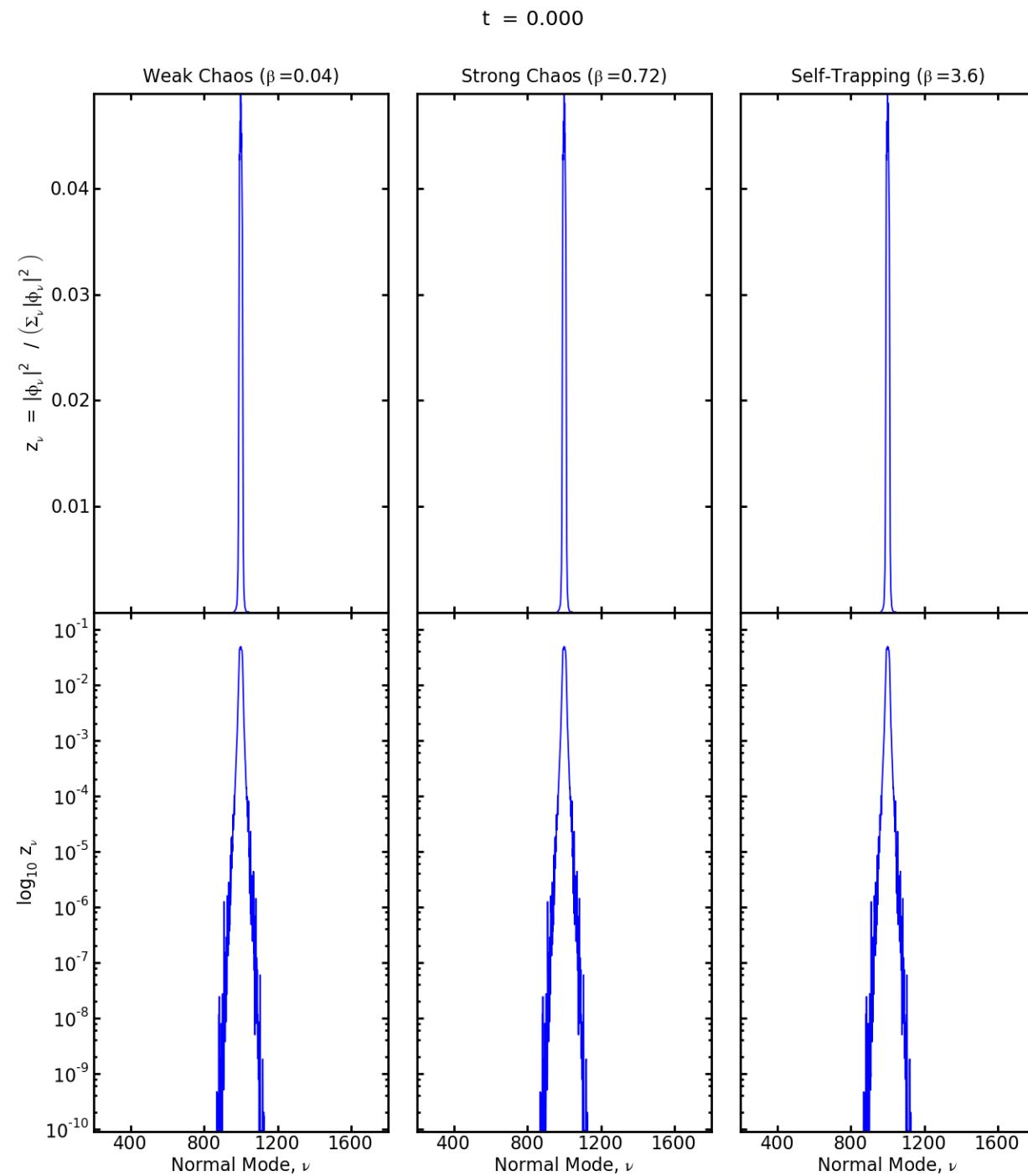
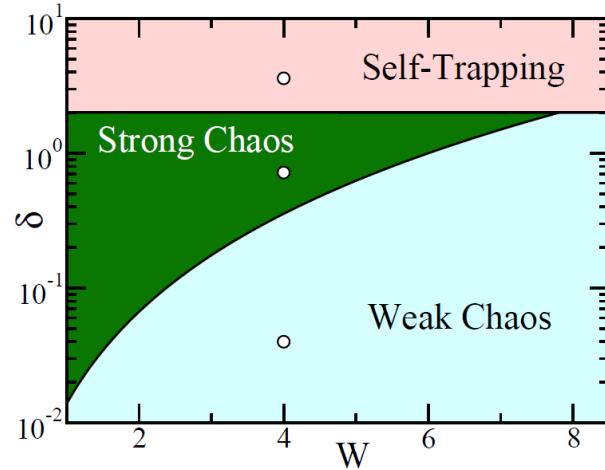
$W=4$

Wave packet, $L=V=20$ sites

Norm density $n=1$

Random initial phases

Averaging over
1000 realizations



Disordered chains:

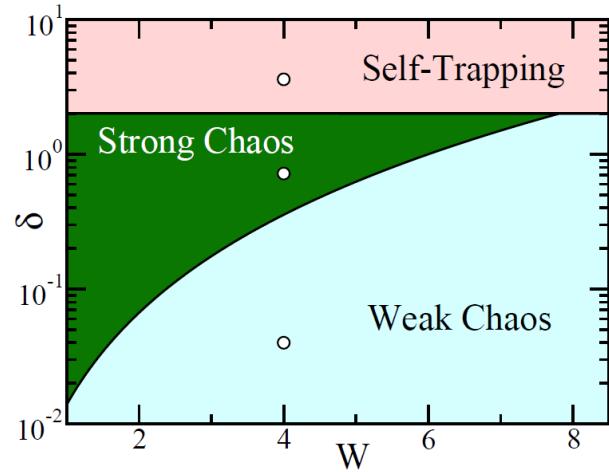
$W=4$

Wave packet, $L=V=20$ sites

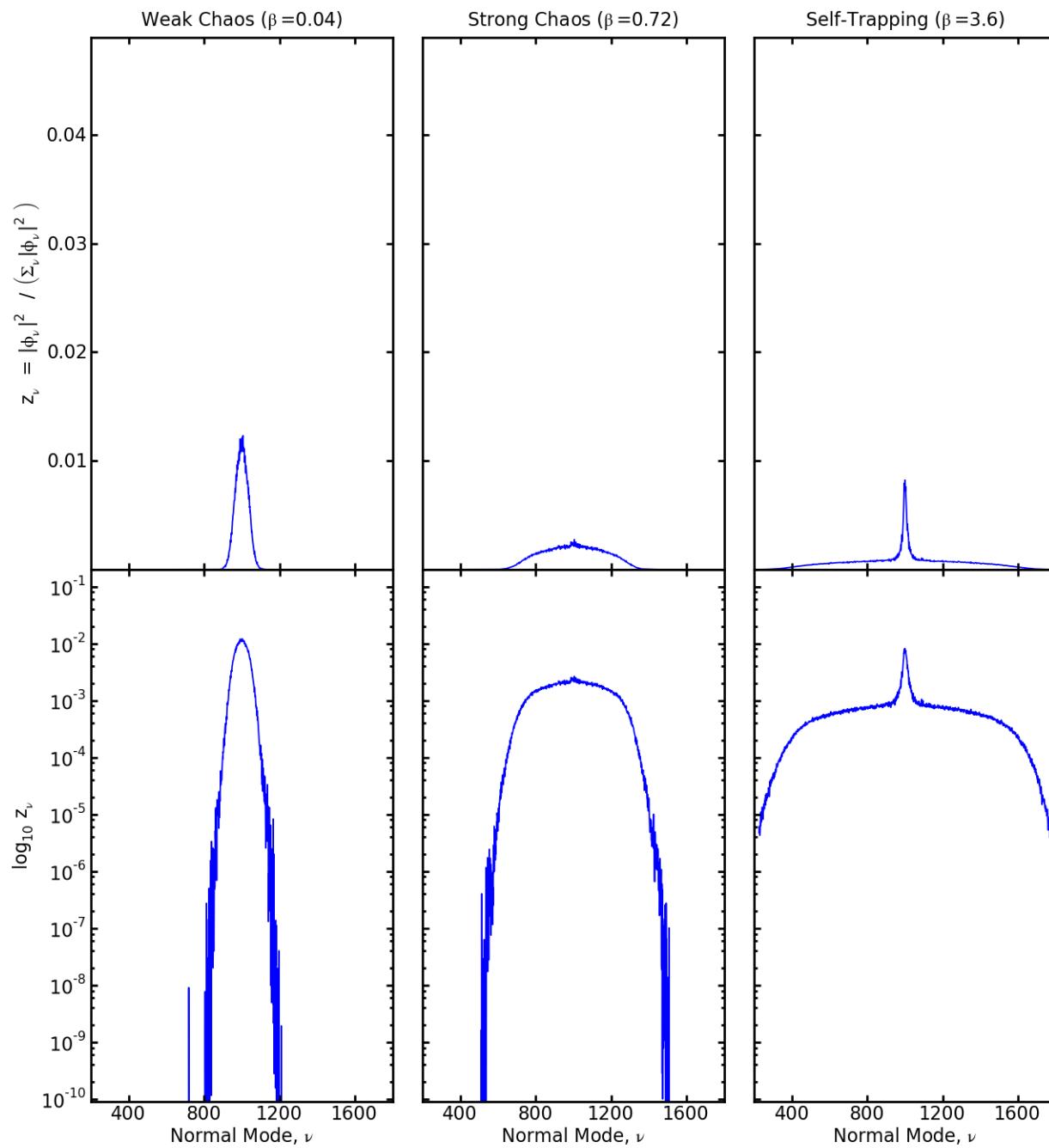
Norm density $n=1$

Random initial phases

Averaging over
1000 realizations

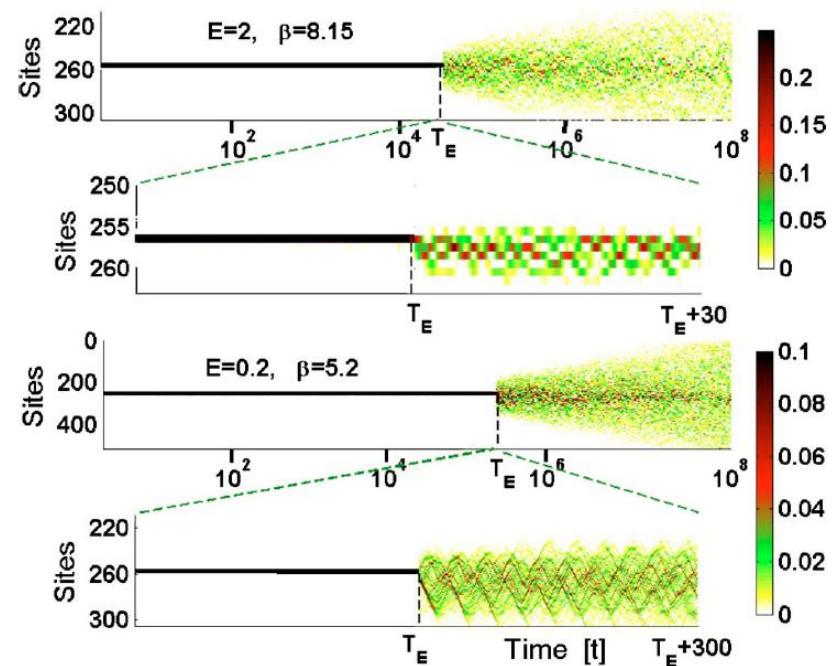
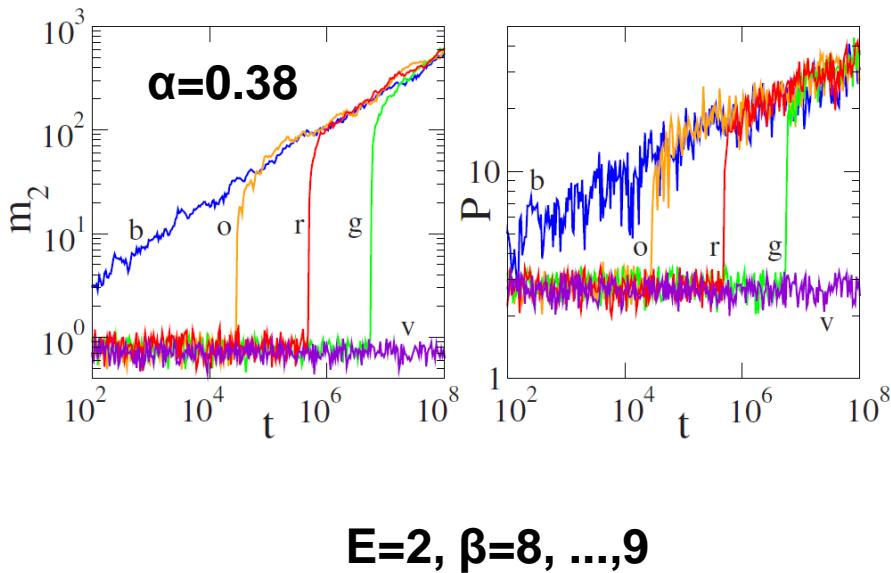


$\log_{10} t = 7.000$



Spreading in nonlinear Stark ladders

$$i\dot{\Psi}_n = -(\Psi_{n+1} + \Psi_{n-1}) + nE\Psi_n + \beta|\Psi_n|^2\Psi_n$$



Delocalization of a disordered bosonic system by repulsive interactions

B. Deissler¹*, M. Zaccanti¹, G. Roati¹, C. D'Errico¹, M. Fattori^{1,2}, M. Modugno¹, G. Modugno¹ and M. Inguscio¹

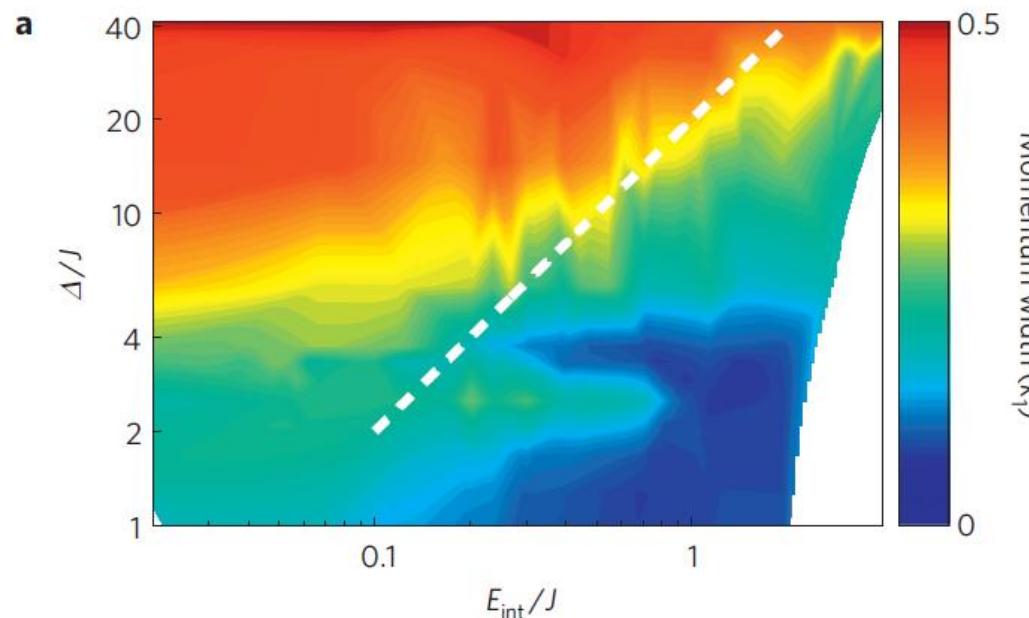
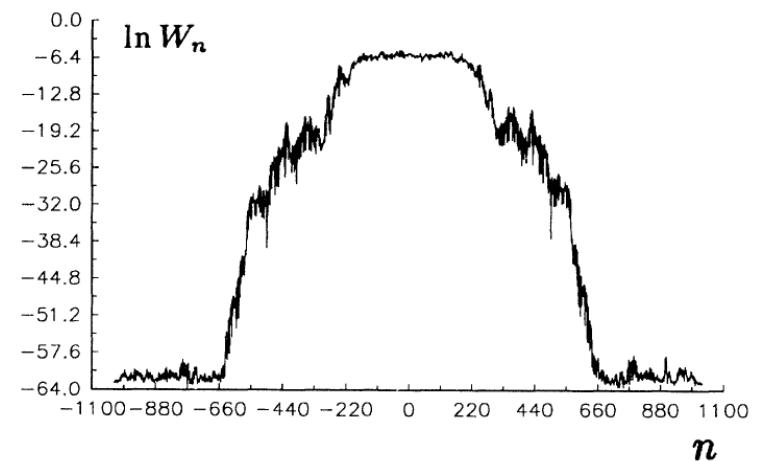
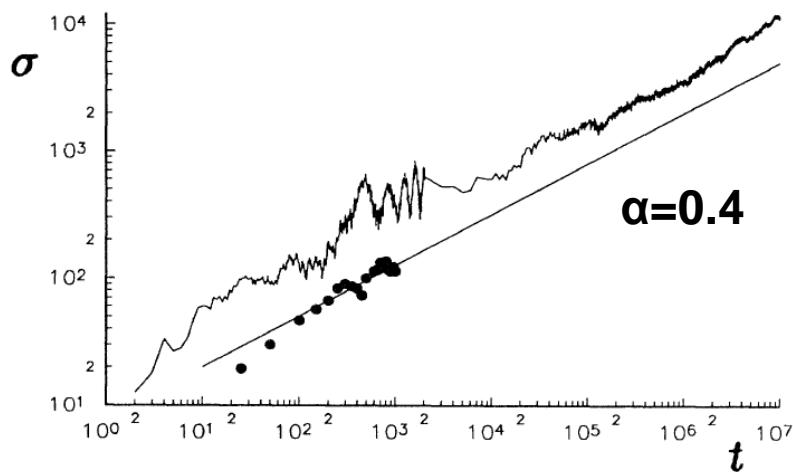


Figure 3 | Probing the interaction-induced delocalization. **a**, Root-mean-square

Spreading in kicked rotors with nonlinearity

$$i \frac{\partial \psi}{\partial t'} = \left[-\frac{1}{2} \frac{\partial^2}{\partial \vartheta^2} - u |\psi|^2 + k \cos(\vartheta) \sum_{t=0}^{+\infty} \delta(t' - t\tau) \right] \psi$$

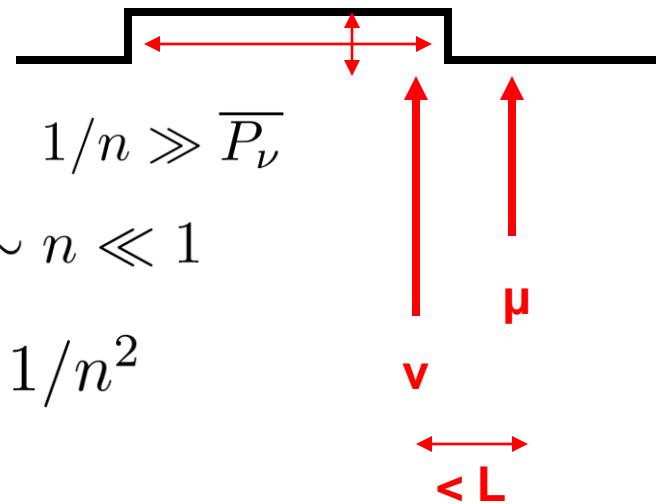


$k=5, \beta=1$

D. Shepelyansky (1993)

Explaining subdiffusion?

- at some time t packet contains $1/n$ modes: $1/n \gg \bar{P}_\nu$
- each mode on average has norm $|\phi_\nu|^2 \sim n \ll 1$
- the second moment amounts to $m_2 \sim 1/n^2$



Two mechanisms of exciting a cold exterior mode:

- heated up by the packet (nonresonant process)
- directly excited by a packet mode (resonant process)
- in both cases the relevant modes are in a layer of the width of the localization volume at the edge of the packet

Heating

Simplest assumption:

- some modes in packet interact resonantly and therefore evolve chaotic
- Probability of resonance: $P(\beta n)$
- the packet modes induce a stochastic force with amplitude proportional to $P(\beta n)$
- spreading = heating of cold exterior

exterior mode: $i\dot{\phi}_\mu \approx \lambda_\mu \phi_\mu + \beta n^{3/2} \mathcal{P}(\beta n) f(t)$

$$\langle f(t) f(t') \rangle = \delta(t - t')$$

$$|\phi_\mu|^2 \sim \beta^2 n^3 (\mathcal{P}(\beta n))^2 t$$

The momentary diffusion rate of packet equals the inverse time the exterior mode needs to heat up to the packet level:

$$D = 1/T \sim \beta^2 n^2 (\mathcal{P}(\beta n))^2$$

Disordered chains:

$C \neq 0!$

$$\mathcal{P} = \int_0^{\beta n} \mathcal{W}(x) dx$$

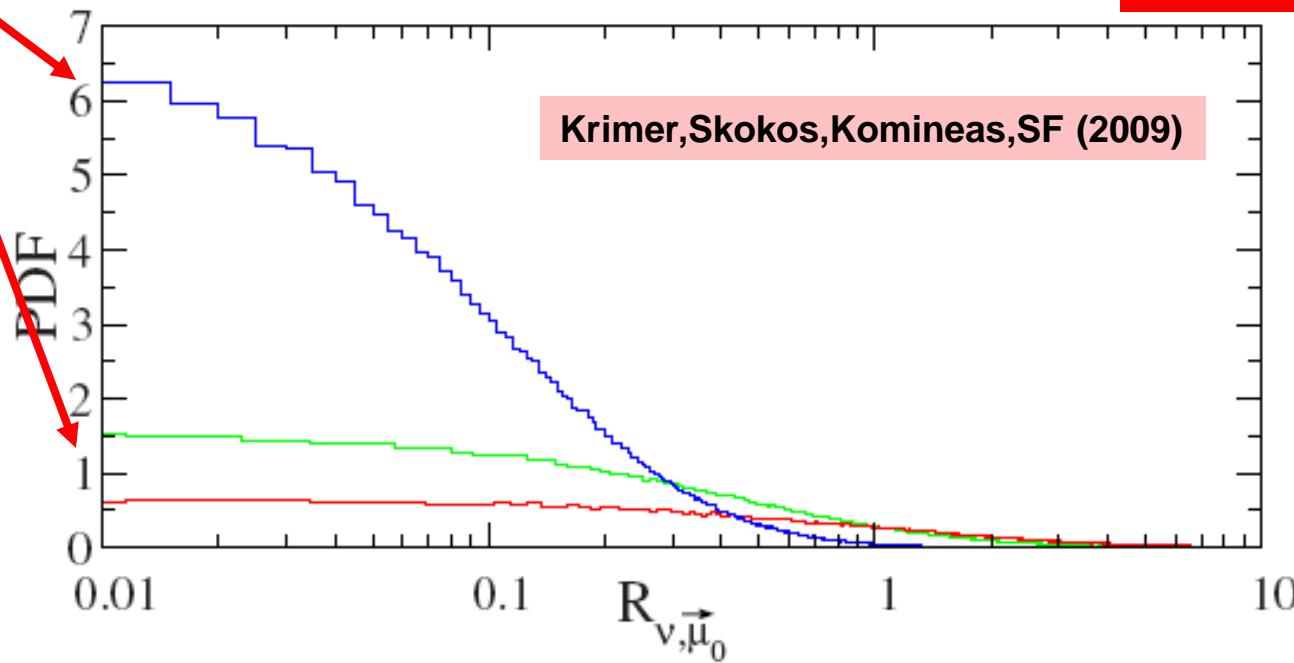


FIG. 9. (Color online) Statistical properties of NMs of the DNLS model. Probability densities $\mathcal{W}(R_{\nu, \vec{\mu}_0})$ of NMs being resonant (see Sec. IV B for details). Disorder strength $W=4, 7, 10$ (from top to bottom).

$$\mathcal{W}(R) \approx C e^{-CR}$$

$$\mathcal{P} = 1 - e^{-C\beta n}$$

$$\frac{1}{C} \approx d$$

SF (2010)

Disordered chains:

$$D = 1/T \sim \beta^2 n^2 (\mathcal{P}(\beta n))^2$$

With $m_2 \sim 1/n^2$ the diffusion equation $m_2 \sim Dt$ yields

$$\frac{1}{n^2} \sim \beta(1 - e^{-C\beta n})t^{1/2}.$$

$$m_2 \sim (\beta^2 t)^{1/2}, \text{ strong chaos , } C\beta n > 1$$

$$m_2 \sim C^{2/3} \beta^{4/3} t^{1/3}, \text{ weak chaos , } C\beta n < 1$$

Crossover from strong to weak chaos: $C\beta n_c \approx 1.86$

SF (2010)

Conclusions

- nonlinearity destroys integrability and generates chaos, this leads to decoherence and subdiffusion: **wavepacket delocalizes**
- nonlinearity may also generate resonant mode-mode interaction in general two decay channels: **coherent and incoherent**
- crossover from strong to asymptotic weak chaos, **no signature of stop!**
- spreading is universal due to nonintegrability

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