Nonlinear Waves in Disordered Media: Localization and Delocalization

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S. Flach, MPIPKS Dresden

Two lectures:

- Obtaining Anderson localization
- Destruction of Anderson localization

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Lecture I:

- Obtaining Anderson localization
- Destruction of Anderson localization

Glossary

Particles: zero/full transmission below/above barrier,
no interference, phase does not matter

Waves: partial transmission below/above barrier,
interference, phase matters

Quantum / classical waves:
Identical description for single qm particle / linear case

Quantum many body waves: linear equations in
VERY high-dimensional Hilbert (vector) space

Classical nonlinear waves: nonlinear equations, e.g.
from mean field approximation for MANY quantum particles

Nonlinearity: wave-wave (mode-mode) interactions

Localization: waves start to travel, but never get away

Philip W. Anderson
The Nobel Prize in Physics 1977

Nobel Lecture
Nobel Lecture, December 8, 1977

Local Moments and Localized States

I was cited for work both in the field of magnetism and in that of disordered systems, and I would like to describe here one development in each held which was specifically mentioned in that citation. The two theories I will discuss differed sharply in some ways. The theory of local moments in metals was, in a sense, easy: it was the condensation into a simple mathematical model of ideas which were very much in the air at the time, and it had rapid and permanent acceptance because of its timeliness and its relative simplicity. What mathematical difficulty it contained has been almost fully cleared up within the past few years.

Localization was a different matter: very few believed it at the time, and even fewer saw its importance; among those who failed to fully understand it at first was certainly its author. It has yet to receive adequate mathematical treatment, and one has to resort to the indignity of numerical simulations to settle even the simplest questions about it .

As usually, any new result has been obtained already before, And of course by others …

"Well, in my country," said Alice, still panting a little, "you would generally get to somewhere else, if you ran very fast for a long time, as we've been doing". "A slow sort of country!", said the queen. "Now *here*, it takes all the running you can do, to stay in the same place."

Experimental Evidence for Anderson Localization

waves in disordered media – Anderson localization for:
electrons, phonons, photons, BEC, …

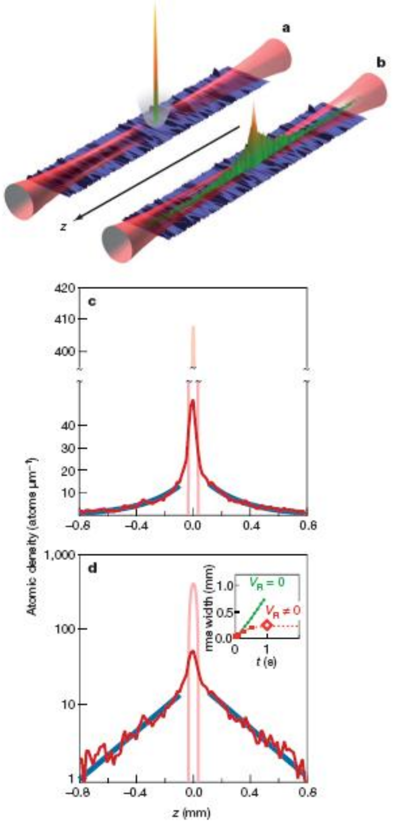


Figure 1 | Observation of exponential localization. a, A small BEC

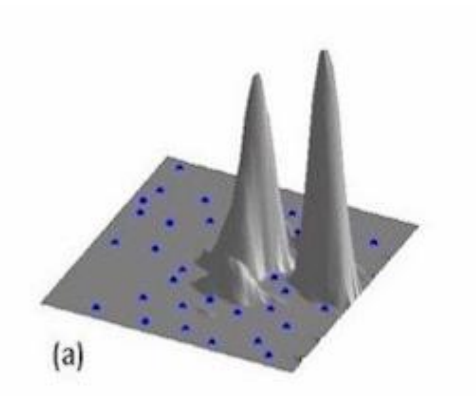
***Electrons:* in: Akkermans et al 2006**

***Ultrasound:* Weaver 1990**

***Microwaves:* Dalichaoush et al 1991, Chabanov/Pradhan/ et al 2000**

***Light:* Wiersma et al 1997, Scheffold et al 1999, Stoerzer et al 2006, Schwartz et al 2007, Lahini et al 2008**

***BEC:* Billy et al 2008, Roati et al 2008**



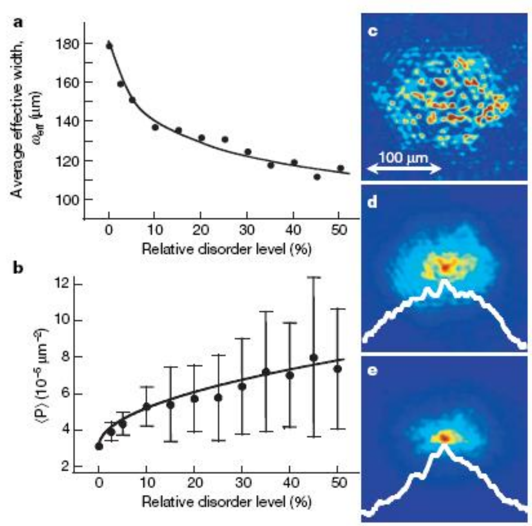
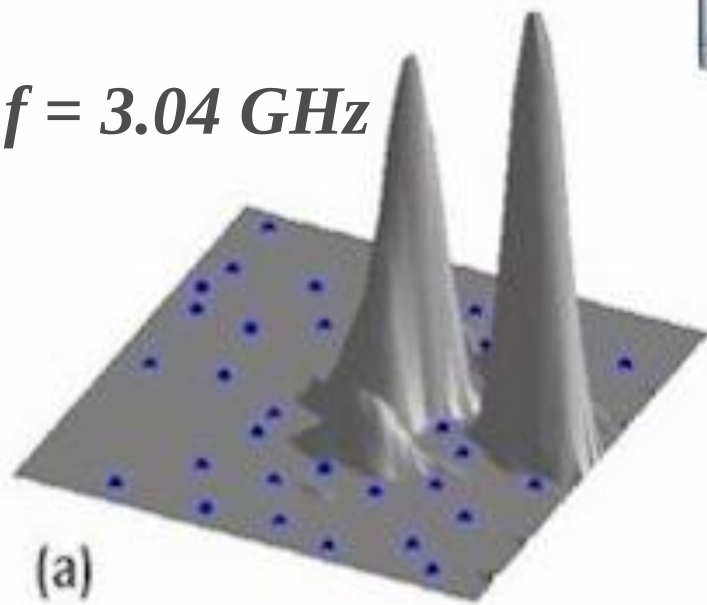


Figure 2 | Experimental results for propagation in disordered lattices.

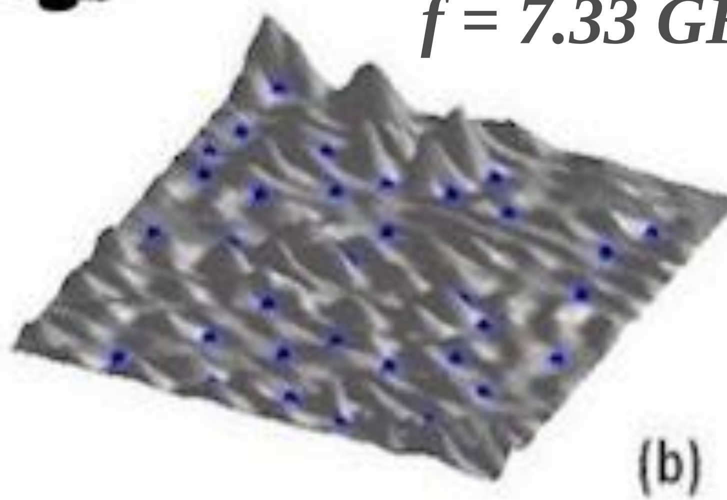
Correlations due to Localization in Quantum Eigenfunctions of Disordered Microwave Cavities

Prabhakar Pradhan and S. Sridhar
Department of Physics, Northeastern University, Boston, Massachusetts 02115
(Received 28 February 2000)

***f* = 3.04 GHz**



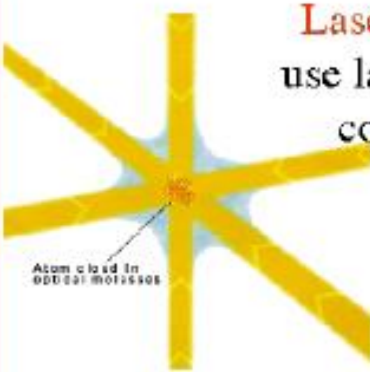
***f* = 7.33 GHz**



Localized State
Anderson Insulator

Extended State
Anderson Metal

1st stage: laser cooling

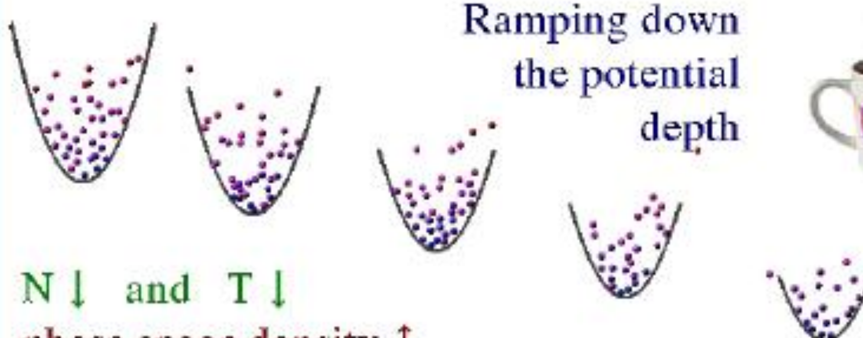


Laser cooling
use laser light to cool and stop atoms.

Trapping of atoms
combination of optical and magnetic traps

T~μ K

2nd stage: evaporative cooling



Ramping down the potential depth

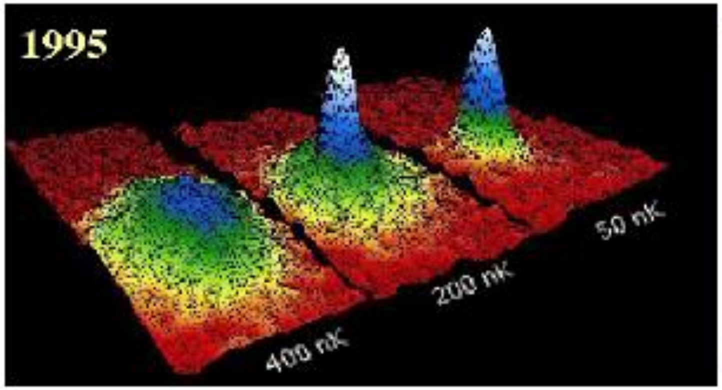
N ↓ and T ↓
phase space density ↑

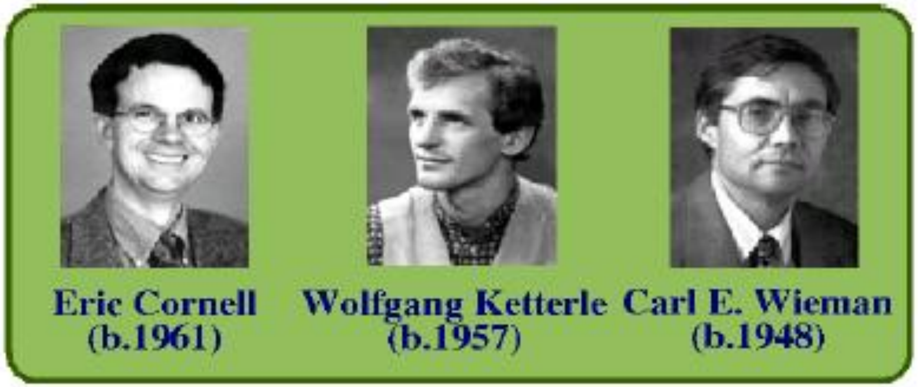
T~nK ⇒ Bose-Einstein condensation

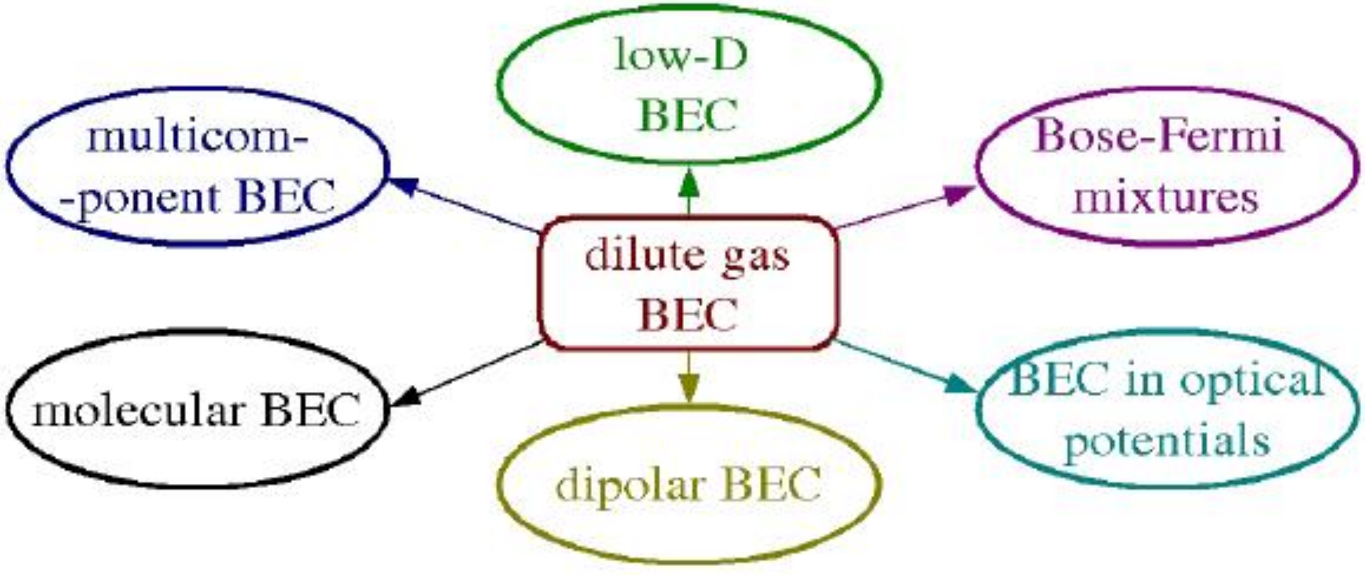
Steven Chu (b.1948)

Claude Cohen-Tannoudji (b.1933)

William D. Phillips (b.1948)


1995
400 nK
200 nK
50 nK


Eric Cornell
(b.1961)
Wolfgang Ketterle
(b.1957)
Carl E. Wieman
(b.1948)


low-D
BEC
multicom-
-ponent BEC
Bose-Fermi
mixtures
dilute gas
BEC
molecular BEC
BEC in optical
potentials
dipolar BEC

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Anderson Localization of Expanding Bose-Einstein Condensates in Random Potentials

L. Sanchez-Palencia,[1] D. Clément,[1] P. Lugan,[1] P. Bouyer,[1] G. V. Shlyapnikov,[2,3] and A. Aspect[1]
[1]Laboratoire Charles Fabry de l'Institut d'Optique, CNRS and Univ. Paris-Sud, Campus Polytechnique, RD 128, F-91127 Palaiseau cedex, France
[2]Laboratoire de Physique Théorique et Modèles Statistiques, Univ. Paris-Sud, F-91405 Orsay cedex, France
[3]Van der Waals-Zeeman Institute, Univ. Amsterdam, Valckenierstraat 65/67, 1018 XE Amsterdam, The Netherlands
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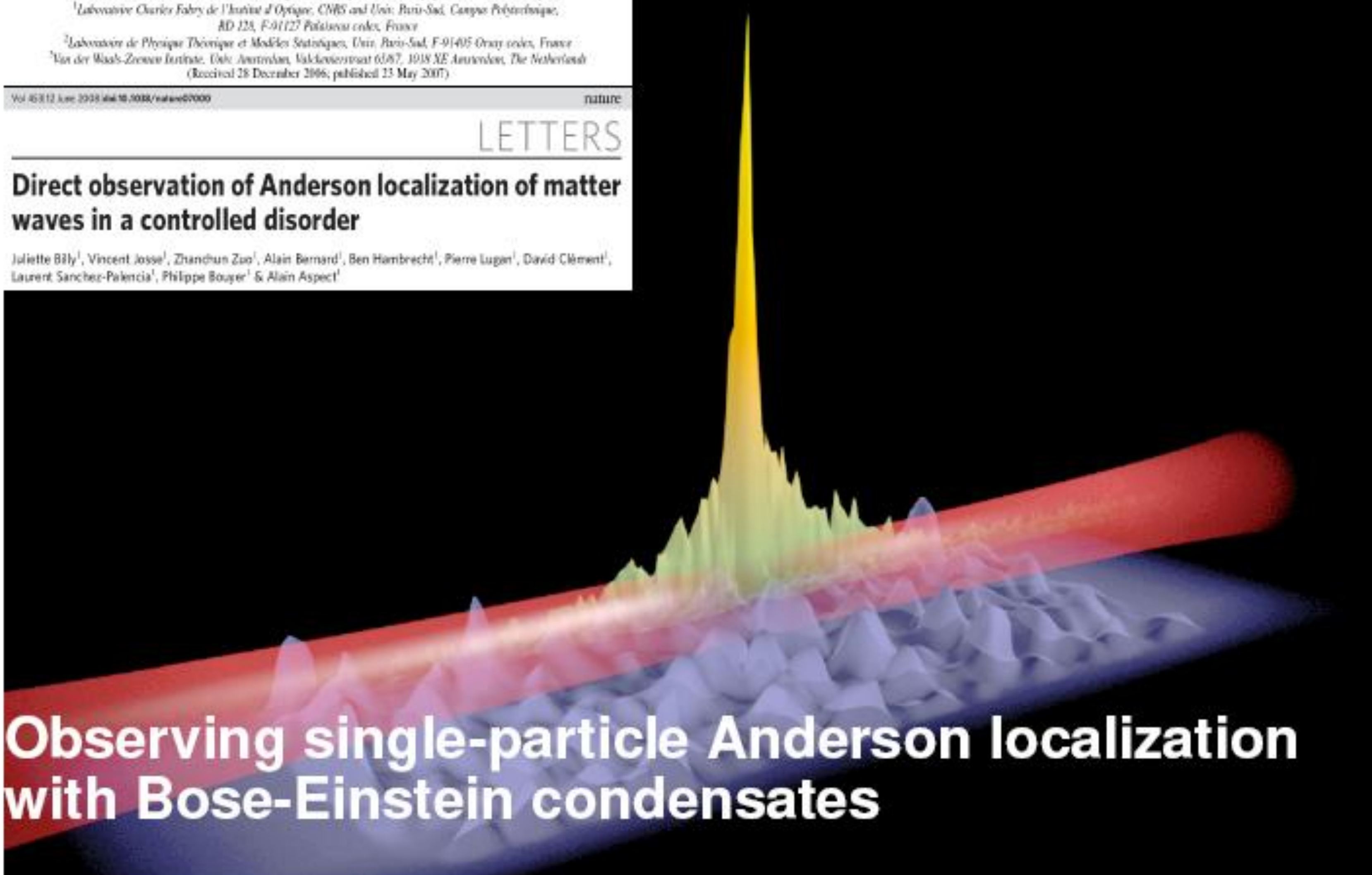
LETTERS

Direct observation of Anderson localization of matter waves in a controlled disorder

Juliette Billy[1], Vincent Josse[1], Zhanchun Zuo[1], Alain Bernard[1], Ben Hambrecht[1], Pierre Lugan[1], David Clément[1], Laurent Sanchez-Palencia[1], Philippe Bouyer[1] & Alain Aspect[1]



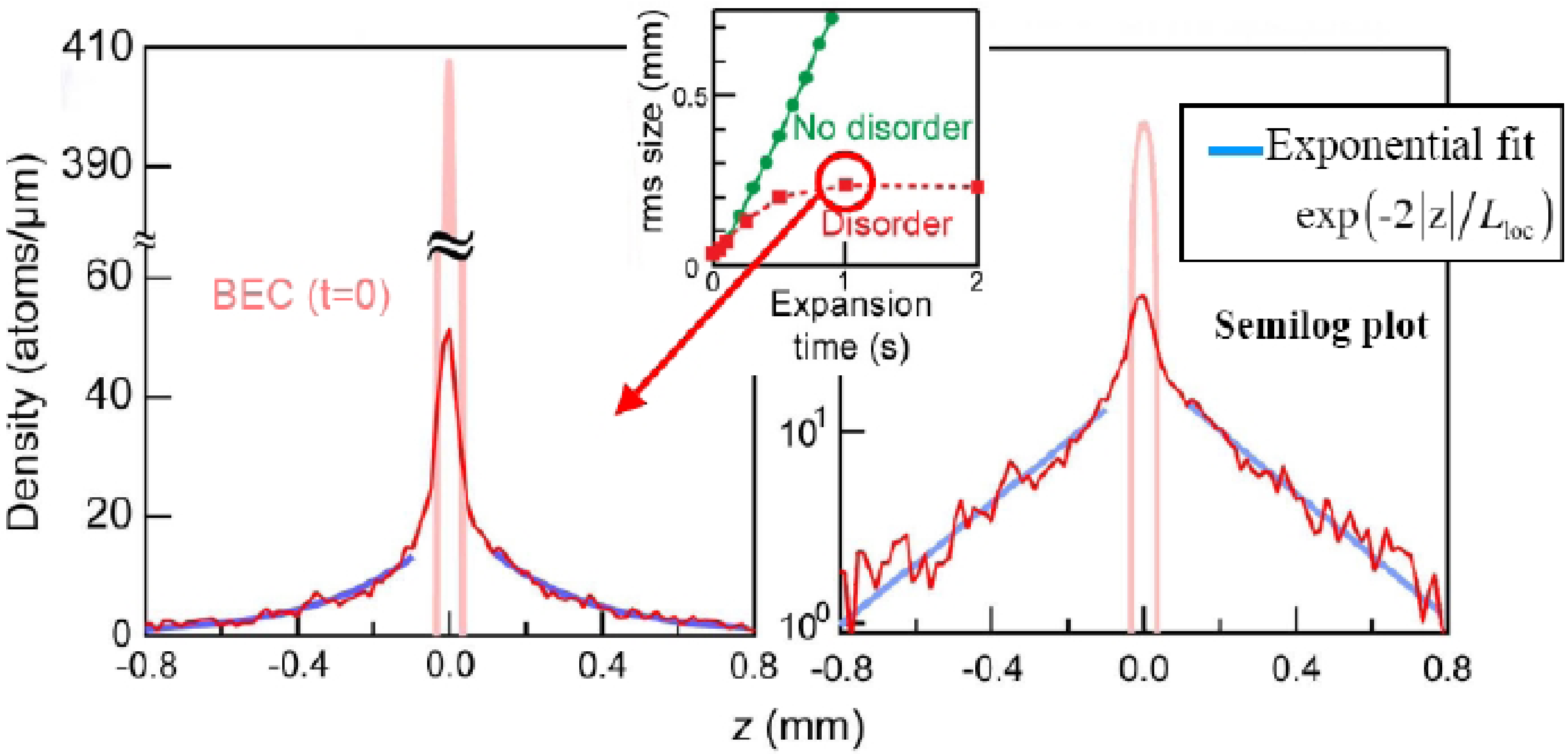
Observing single-particle Anderson localization with Bose-Einstein condensates

Observation of the signature of AL

BEC parameters : N=1.7 10^4 atoms, (μ_{in}=220Hz)

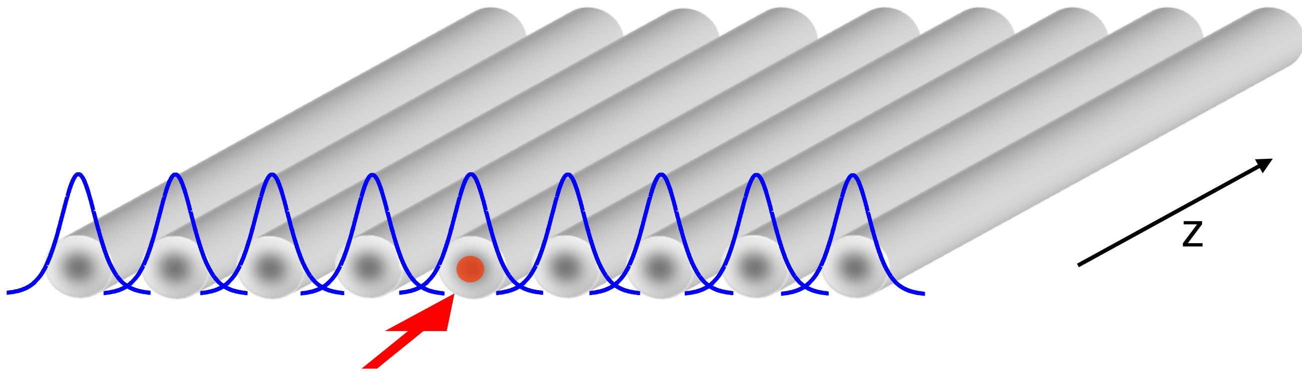
Weak disorder : V_R/μ_{in} =0.12 << 1

$k_{max}/\ k_c$ = 0.63 +/-0.09



⇒ Exponential decay of the density in the wings : L_{loc}=530 +/- 80 µm

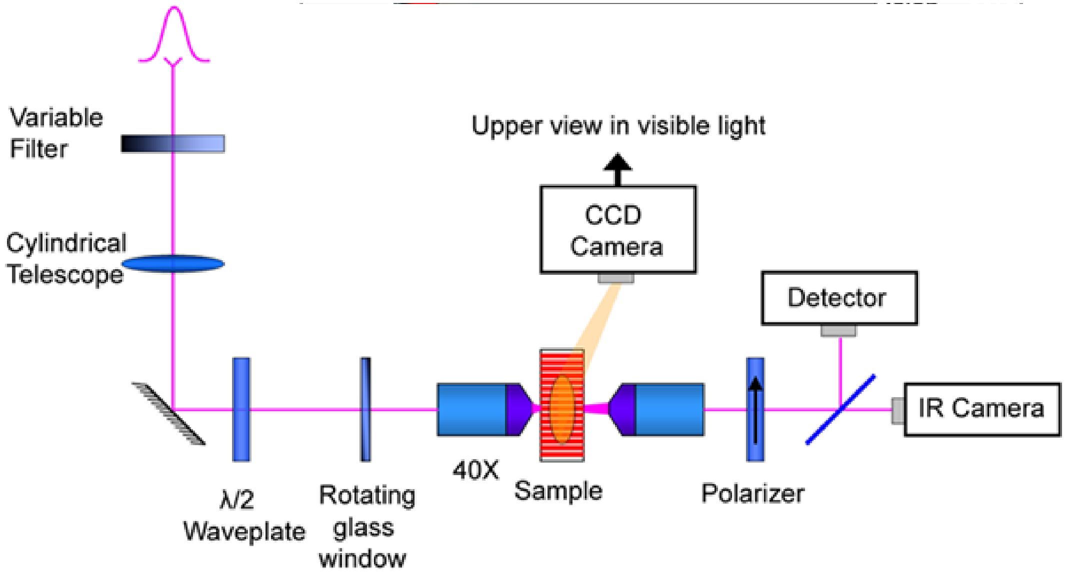
An optical one-dimensional waveguide lattice (Silberberg et al '08)



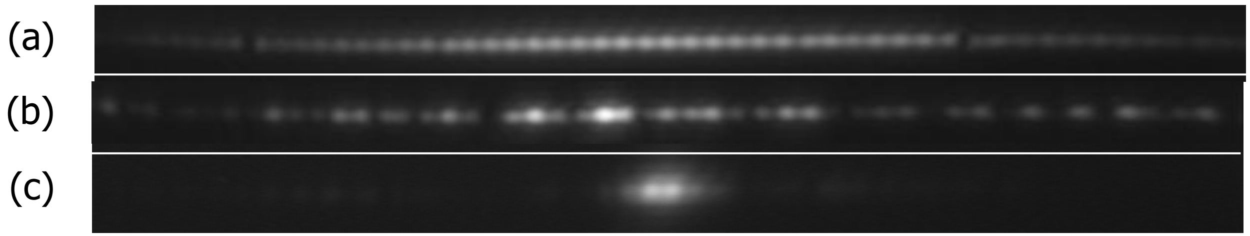
- Evanescent coupling between waveguides
- Light coherently tunnels between neighboring waveguides
- Dynamics is described by the Tight-Binding model

$$i\frac{\partial U_n}{\partial z} = \beta_n U_n + C_{n,n\pm1}\left[U_{n+1} + U_{n-1}\right]$$

β_n – waveguide's refraction index /width
$C_{n,n\pm1}$ – separation between waveguides



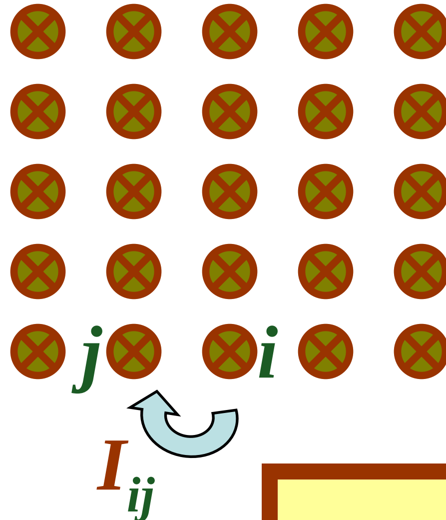
- Injecting a narrow beam (~3 sites) at different locations across the lattice



(a) Periodic array – *expansion*
(b) Disordered array - *expansion*
(c) Disordered array - *localization*

SOME FACTS, THOUGHTS AND IDEAS

Anderson Model



- *Lattice - tight binding model*
- *Onsite energies* ε_i *- random*
- *Hopping matrix elements* t_{ij}

$$-W/2 < \varepsilon_i < W/2$$

uniformly distributed

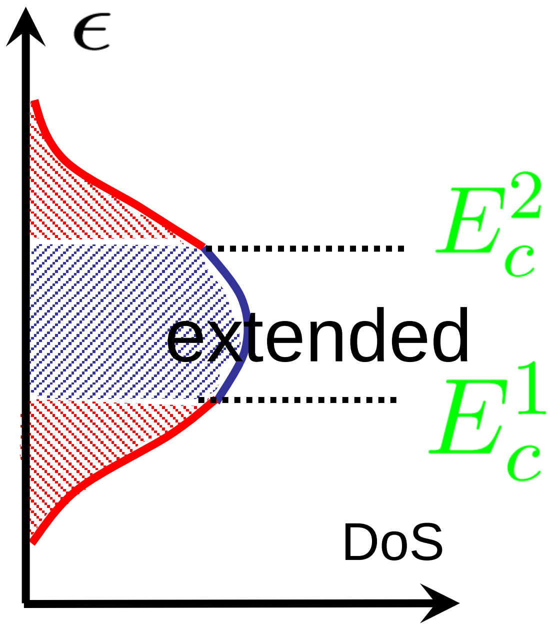
$$t_{ij} = \begin{cases} t & i \text{ and } j \text{ are nearest neighbors} \\ 0 & \text{otherwise} \end{cases}$$

Anderson Transition

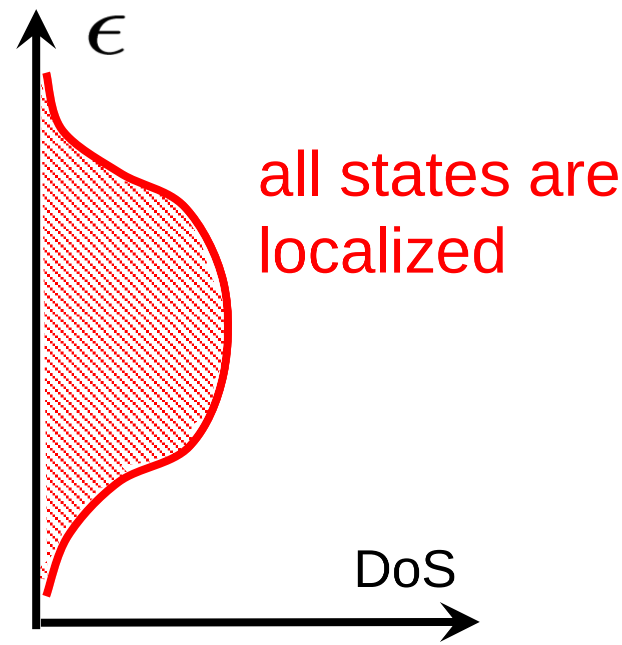
$t < t_c$	$t > t_c$
Insulator	**Metal**
All eigenstates are localized	There appear states extended all over the whole system

Anderson Transition

$t > t_c$



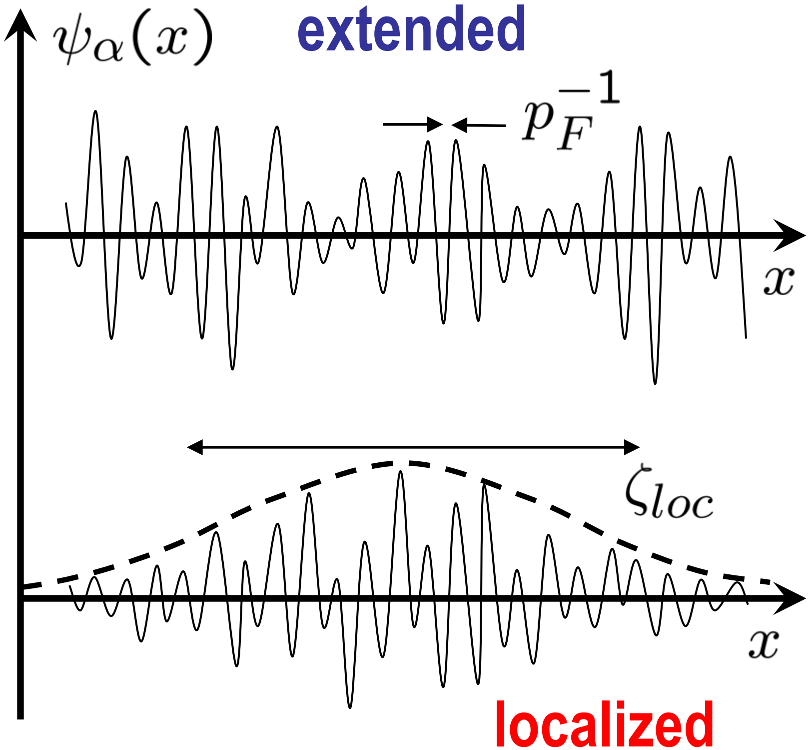
$t < t_c$



E_c - mobility edges (one particle)

Localization of single-particle wave-functions. Continuous limit:

$$\left[-\frac{\boldsymbol{\nabla}^2}{2m} + U(\boldsymbol{r}) - \epsilon_F\right]\psi_\alpha(\boldsymbol{r}) = \xi_\alpha \psi_\alpha(\boldsymbol{r})$$



d=1: **All states are localized**

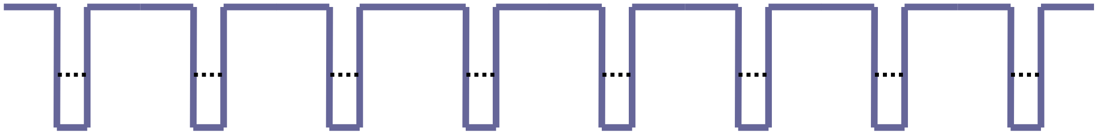
d=2: **All states are localized**

d >2: **Anderson transition**

The one-dimensional tight-binding model

- The periodic Lattice (Bloch, 1928)

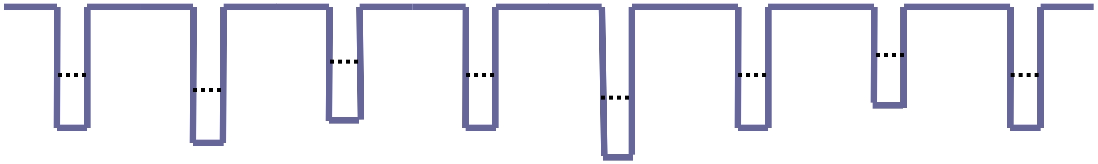
$$\boxed{-i\frac{\partial\psi_n}{\partial t} = E\,\psi_n + T\left[\psi_{n+1} + \psi_{n-1}\right]}$$



Eigenfunctions extend over entire lattice (Bloch functions)

- The disordered lattice (Anderson, 1958)

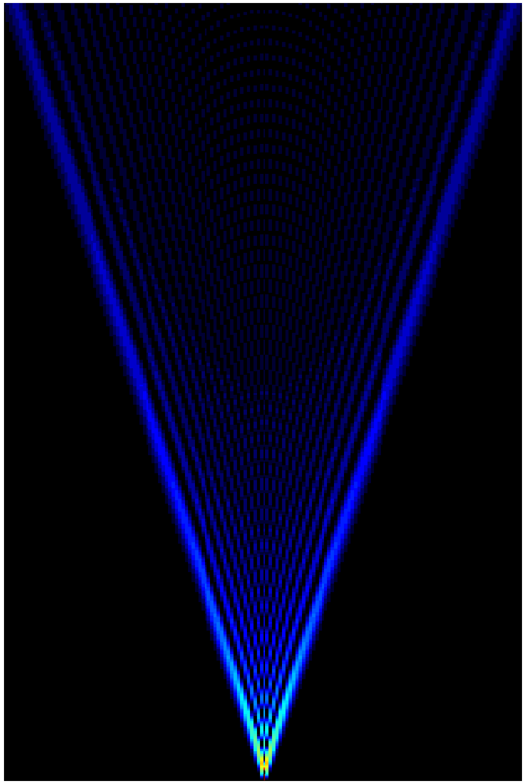
$$\boxed{-i\frac{\partial\psi_n}{\partial t} = E_n\psi_n + T_{n,n\pm1}\left[\psi_{n+1} + \psi_{n-1}\right]}$$

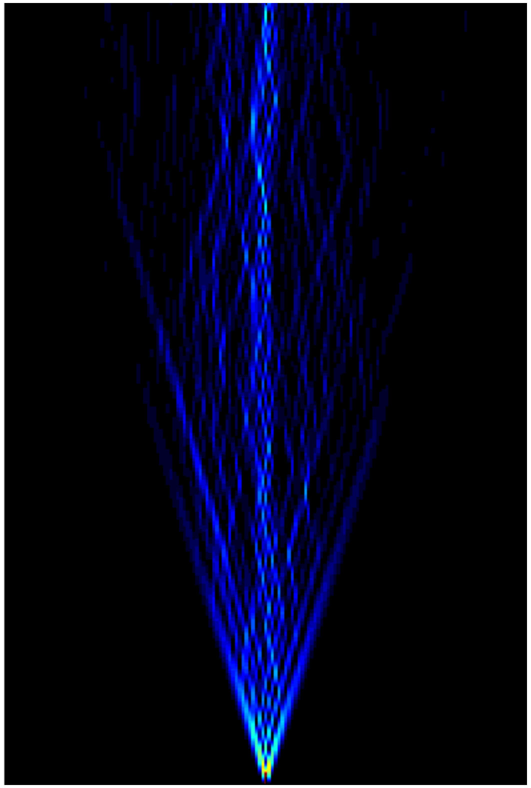


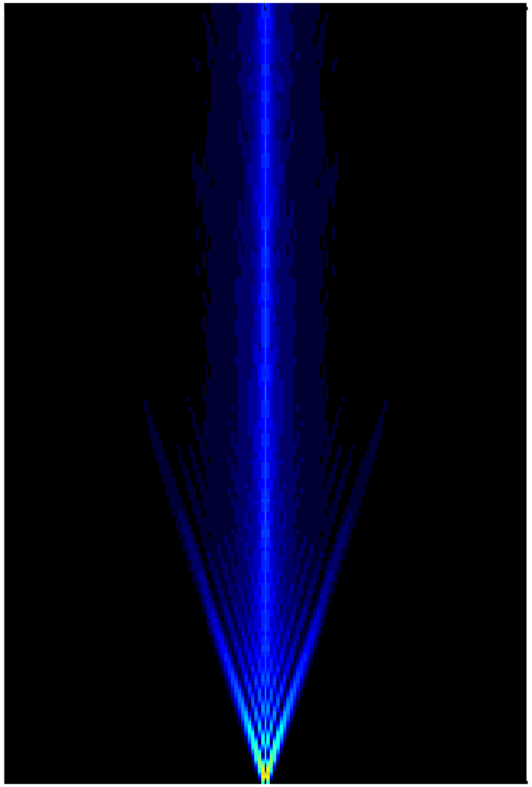
Wave packet evolution

- Exciting a *single* site as an initial condition

Ordered lattice **Disordered lattice** **Disordered lattice - averaged**







Properties of disordered states in the 1d Anderson model:

Stationary states: $\lambda A_l = \epsilon_l A_l - A_{l-1} - A_{l+1}$

Normal mode (NM) eigenvectors: $A_{\nu,l} \ \left(\sum_l A^2_{\nu,l} = 1\right)$

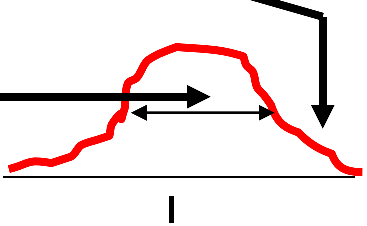
Eigenvalues: $\lambda_\nu \in \left[-2 - \frac{W}{2}, 2 + \frac{W}{2}\right]$

Width of EV spectrum: $\Delta_D = W + 4$

Asymptotic decay: $A_{\nu,l} \ \sim \ \mathrm{e}^{-l/\xi(\lambda_\nu)}$

Localization length: $\xi(\lambda_\nu) \ \leq \ \xi(0) \ \approx \ 100/W^2$

Localization volume of NM: V
V(W < 4) ≈ 3ξ V(W > 10) ≈ 1



What happens when we add nonlinear terms to the equations of motion?

- Eigenmodes of the linear equations can be continued as periodic orbits, however there are infinitely many resonances; still many modes stay localized, with frequencies inside or outside the spectrum of the lin. equations (S. Aubry, ’00, ’01)

- Finite sets of eigenmodes can be continued as quasiperiodic orbits as well, with similar properties as for periodic orbits (Wang/Bourgain ’08)

- All these statements are about manifolds of zero measure in phase space. What about the rest?

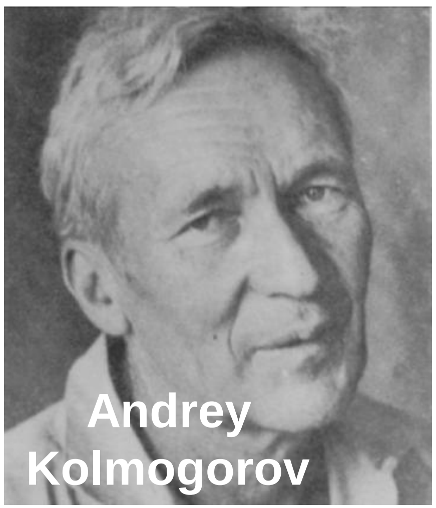
- Linear wave equations correspond to integrable dynamical systems

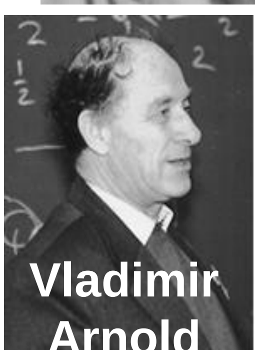
- Nonlinear terms will in general destroy integrability

- Will they also destroy localization?

Kolmogorov – Arnold – Moser (KAM) theory

A.N. Kolmogorov,
Dokl. Akad. Nauk SSSR, 1954.
Proc. 1954 Int. Congress of Mathematics, North-Holland, 1957



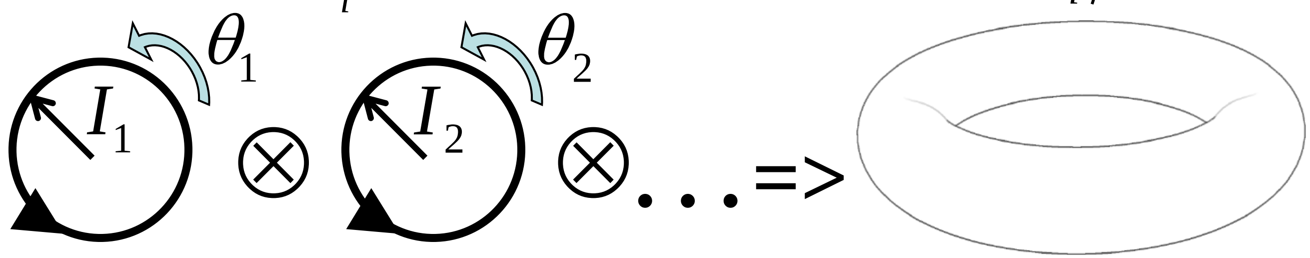


Integrable classical Hamiltonian $\hat{H}_0$, *d>1*:

Separation of variables: ***d*** sets of action-angle variables $I_1, \theta_1 = 2\pi\omega_1 t; .., I_2, \theta_2 = 2\pi\omega_2 t; ..$

Quasiperiodic motion: set of the frequencies, $\omega_1, \omega_2, .., \omega_d$ which are in general incommensurate

Actions I_i are integrals of motion $\partial I_i / \partial t = 0$



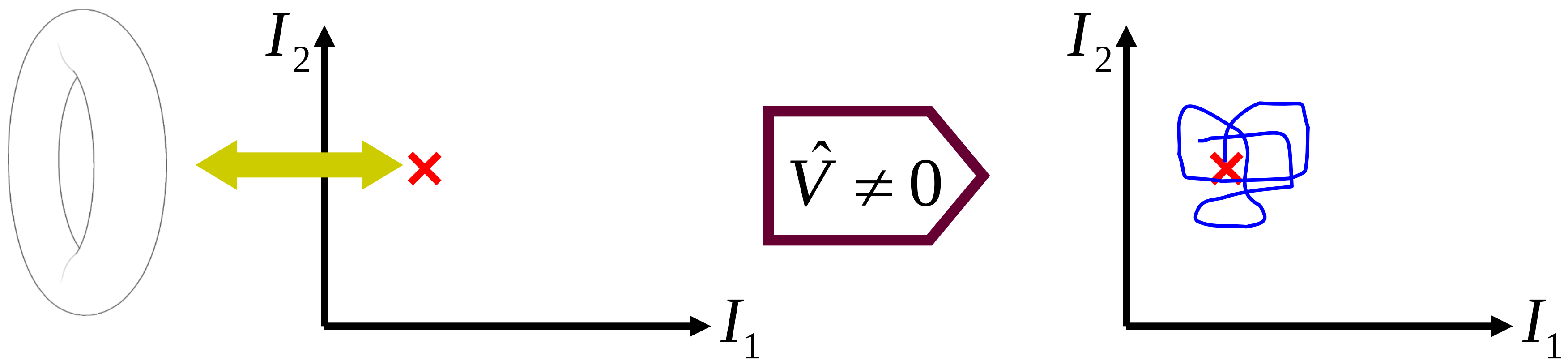
Q: Will an arbitrary weak perturbation $\hat{V}$ of the integrable Hamiltonian $\hat{H}_0$ destroy the tori and make the motion ergodic (when each point at the energy shell will be reached sooner or later)

A: **Most of the tori survive weak and smooth enough perturbations**

KAM theorem

KAM theorem:

Most of the tori survive weak and smooth enough perturbations



Each point in the space of the integrals of motion corresponds to a torus and vice versa

Finite motion. Localization in the space of the integrals of motion ?

- KAM applies to finite systems
- Does it apply to waves in infinite systems?
- How are KAM thresholds scaling with number of degrees of freedom?
- Will nonlinear waves observe KAM regime?
- If they do – then localization remains
- If they do not – waves can delocalize

•Some answers will be obvious from the next lecture …