Stanley-Reisner Degenerations of Mukai Varieties

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General Question: Given some variety X, find "nice" toric varieties Z to which X degenerates.

Motivation Mirror Symmetry [Prz]

- [AB] "Toric degenerations of spherical varieties"
  X spherical variety; $\underline{w_0}$ decomposition of longest word in Weyl group. $\rightsquigarrow$ String polytopes $\Delta_{\underline{w_0}}$
  X degenerates to $\mathbb{P}(\Delta_{\underline{w_0}})$.
- [Anderson]: X normal variety; $F_\bullet$ full flag in X, D ample Divisor
  $\rightsquigarrow \Delta_{F_\cdot, D}$ | If $\Delta_{F_\cdot, D}$ polyhedral, then
  X degenerates to Z, where $\tilde{Z} = \mathbb{P}(\Delta_{F_\cdot, D})$

Main Result: Explicit degenerations for special Fano varieties

I. Mukai Varieties
II. Stanley-Reisner Schemes
III. Stanley-Reisner Degenerations of Mukai
IV. Toric Degenerations
V. Hilbert scheme for degree 12 Fano 3-folds.

I. Mukai Varieties

[Isk] There are 17 deformation families of rk 1 Fano 3-folds

[Muk] rk 1 index 1 Fano threefolds can be embedden in WPS or homogeneous spaces:

		genus	
$M_3 =$	V_4'	3	$(4) \subset \mathbb{P}^4$
$M_4 =$	V_6	4	$(2,3) \subset \mathbb{P}^5$
$M_5 =$	V_8	5	$(2,2,2) \subset \mathbb{P}^6$
	V_{10}	6	$(1)^2 \subset G(2,5) \cap Q = M_6$
	V_{12}	7	$(1)^7 \subset SO(5,10) = M_7$
	V_{14}	8	$(1)^5 \subset G(2,6) = M_8$
	V_{16}	9	$(1)^3 \subset LG(3,6) = M_9$
	V_{18}	10	$(1)^2 \subset G_2 = M_{10}$

Main Result: Toric Degenations of linear sections of Mukai Varieties.

II Stanley - Reisner Schemes

$[n] = \{0, \dots, n\}$ $\quad \Delta_n = 2^{[n]}$

Def A simplicial complex K is a subset of Δ_n s.t. $f \in K$, if $g \subset f \Rightarrow g \in K$.

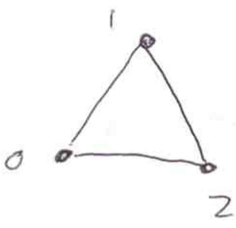
The elements of K are <u>faces</u>.

A face f has dimension $= \# f - 1$.

Edges $\Longleftrightarrow$ 1-dim faces.

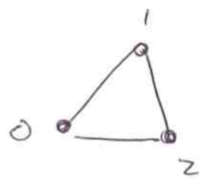
Vertices $\Longleftrightarrow$ 0-dim faces.

<u>Ex</u> $n=2$ $K = \{ \{0\}, \{1\}, \{2\}, \{ij\}, \emptyset \ i \neq j, \ i,j \in [2] \}$

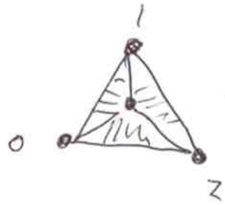


K, K' simplicial complexes, there <u>join</u>

$$K * K' := \{ f \vee g \mid f \in K, \ g \in K' \}$$

<u>Ex</u>: $K =$ 

$K' =$ 

$K * K'$ 

K is a simplicial complex with $n+1$ vert.

$S = \mathbb{C}[x_0, \ldots x_n]$

$I_K \subset S$ $\quad I_K = \{ x_p \mid p \in \Delta_n \setminus K \}$

$$x_p := \prod_{i \in p} x_i$$

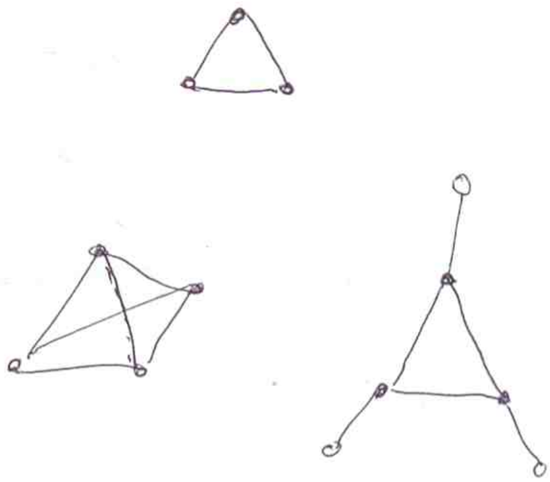
Correspondence between Square-free Mon. Ideals $\Longleftrightarrow$ Simpl. Complexes

$$\rightsquigarrow \quad \mathbb{P}(K) = \mathrm{Proj}\left(S/I_K \right)$$

$\uparrow$

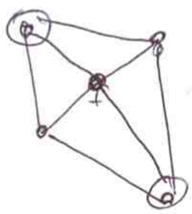
Stanley-Reisner scheme.

Triangulations of S^2

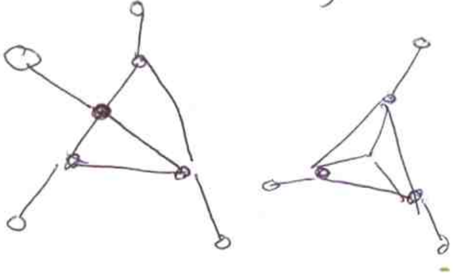


4 vertices: T_4

Given T_{n-1} construct T_n by finding an edge of T_{n-1} with opposing vertices of valency ≤ 4; subdivide edge



$T_4 \to T_5$



$T_4, \ldots T_{10}$

T_{11} from T_{10} by ~~sub~~subdividing an edge w/ opposing valencies 4 and 5.

$4 \leq n \leq 10$, T_n is the boundary complex of a convex deltahedron!

Consider any triangulation T of S^2

- $X = \mathbb{P}(T * \Delta_k)$ $k \geq 0$

$\rightsquigarrow$ $\omega_X = \mathcal{O}_X(-k-1)$ $\rightsquigarrow$ X Fano

- $4 \leq n \leq 10$ $X = \mathbb{P}(T_n * \Delta_m)$ $m \geq 0$

$\rightsquigarrow$ $T^2_{X/\mathbb{P}^{n+m+1}} = 0$ [Ishida, Oda, Altmann, Christophersen]

III. SR Degenerations

Thm $3 \leq g \leq 10$. Then $\mathcal{M}_g$ degenerates to $\mathbb{P}(T_{g+1} * \Delta_{i_g})$

$i_g = \mathrm{ind}(\mathcal{M}_g) - 1$ i.e. $i_6 = i_{10} = 2$,

$i_7 = 7$

$i_8 = 5$

$i_9 = 3$.

6

[Sturmfels]: Let T be a regular unimodular triangulation of Δ. Then $\mathbb{P}(\Delta)$ degenerates to $\mathbb{P}(T)$.

<u>Cor</u>: $-1 \le K \le i_g$ $\quad 3 \le g \le 9$. $\quad V = (1)^{i_g - k} \cap M_g$.

If $\exists\ \nabla \subset P$ with unimodular regular triang. of form $T_{g+1} * \Delta_K$, then $\mathbb{P}(\nabla)$ and V lie on same component of Hilbert scheme.

<u>Proof</u>: $\mathbb{P}(\nabla) \rightsquigarrow \mathbb{P}(T_n * \Delta_k)$

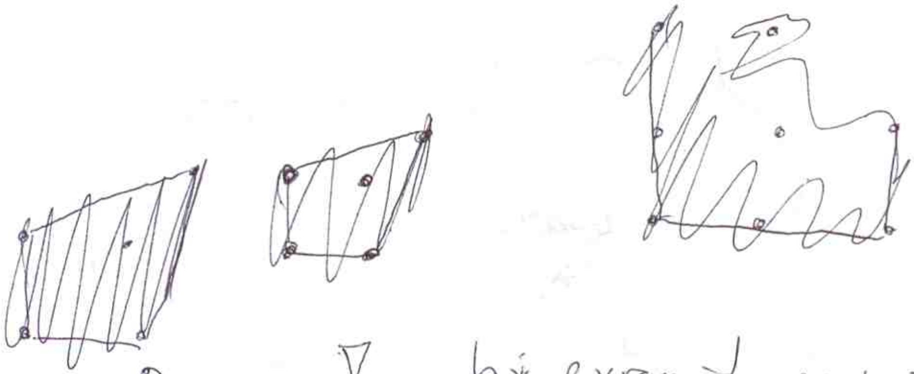
$V \rightsquigarrow \mathbb{P}(T_n * \Delta_k)$

lies on one-comp. of Hilbert scheme

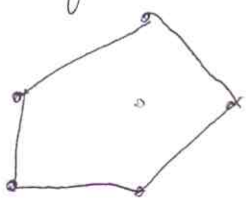
Thus $\mathbb{P}(\nabla)$ and V must also lie on this component.

<u>Remark</u>: In particular, this gives toric degenerations of $V_4, V_6 \dots V_{16}$

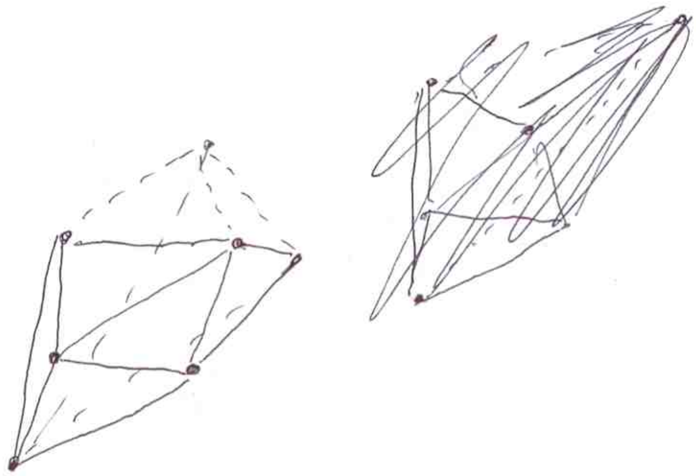
Ex



∇ by pyramid over this ?



∇



∇ has unimodular triangulation

to $T_7 * A_0$

$\rightsquigarrow$ degeneration of V_{10} to $\mathbb{P}(\nabla)$

V. An analysis of Hilbert scheme for deg 12 Fano 3-folds. $\longrightarrow$ embedded in $\mathbb{P}^8$

[MM]

Name	Index	$H^0(N)$
V_{12}	1	98
$V_{12,2,6}$	2	96
$V_{12,2,9}$	2	99
$V_{12,3}$	3	97

$B_{98}, - B_{99}$

[Ka] : $tor_{12} = \{$ Gorenstein toric Fano 3-folds with at most canonical singularities $\}$

$\# tor_{12} = 135$

For any variety $X \subset \mathbb{P}^8$ with $hilb(X) = hilb(V_{12})$, let P_X denote corresponding point in the Hilbert scheme.

<u>Q</u>: Given $X \in tor_{12}$, on which components B_i does P_X lie?

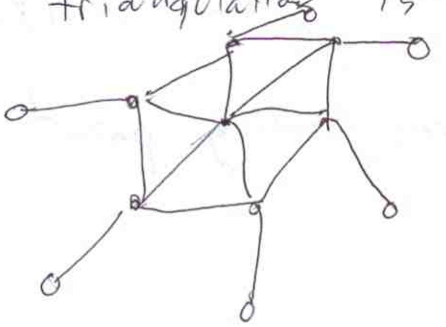
- Exactly 3 elements of tor_{12} with P_X a smooth point ~~point~~ on <u>some</u> 96-dim comp.
- 2 P_X smooth point on B_{97}
- 46 P_X smooth point on B_{98}
- Are 3 (at least) elements of tor_{12} with P_X smooth on B_{99}

- 75 elements of tor_{12} with P_X lying solely on B_{97}, B_{98}, and/or B_{99}.

- 6 remaining elements

Following triangulation is important

$\hat{T}$



Compute versal space for $\mathbb{P}(\hat{T} * \Delta_0)$

$\rightsquigarrow$ has 3 components: B_{97}, B_{98}, B_{99}.