Classical and Intuitionistic Geometric Logic

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Abstract: Geometric sequents "A implies C" where all axioms A and conclusion C are universal closures of implications of positive formulas play distinguished role in several areas including category theory and (recently) logical analysis of Kant's theory of cognition. They are known to form a Glivenko class: existence of a classical proof implies existence of an intuitionistic proof. Existing effective proofs of this fact involve superexponential blow-up, but it is not known whether such increase in size is necessary. We show that any classical proof of such a sequent can be polynomially transformed into an intuitionistic geometric proof of (classically equivalent but intuitionistically) weaker geometric sequent.

Keywords: geometric formulas, Glivenko classes, intuitionistic logic

Introduction

Geometric sequents (see definition below) play distinguished role in several areas including category theory (Goldblatt, 1984). This fragment of first order logic attracted new attention in the light of recent work by Theodora Achouriotti and Michiel van Lambalgen (Achourioti & van Lambalgen, 2011) who propose a translation of the philosophical language of Kant's theory of judgements into the language of elementary logic and provide a convincing justification of their view.

Geometric sequents are known to form a Glivenko class: existence of a classical proof of a geometric sequent S implies existence of an intuitionistic proof. Existing proofs of this fact involve superexponential blow-up, but we do not know whether such increase in size is necessary. We show that any classical proof of S can be polynomially transformed into an intuitionistic geometric proof of (classically equivalent but intuitionistically) slightly weaker geometric sequent.

We consider formulas of first order logic.

Definition 1. *Positive formulas* are constructed from atomic formulas and the constant $\perp$ by $\&, \vee, \exists$.

Geometric implications are positive formulas, implications of positive formulas and results of prefixing universal quantifiers to such implications.

Geometric sequents are expressions of the form

$$I_1, \ldots, I_n \Rightarrow I$$

where $I_1, \ldots, I_n, I$ are geometric implications.

A *geometric derivation* is a derivation consisting of geometric sequents.

The second proof of Theorem 1 given below is non-effective, but the first one allows one to derive some complexity bound. The proof begins with construction of a cut-free derivation, therefore the only obvious bound is the same as for cut-elimination, that is hyperexponential one. This contrasts with the most prominent Glivenko class, namely that of negative formulas. When a classical derivation of a negative formula is given, its intuitionistic derivation is constructed by "negativizing" all formulas in the derivation plus local changes to reinstate the inferences that were destroyed by this transformation. These transformations are polynomial.

We show here a weaker result for geometric sequents. Any classical proof (with cut) of a geometric sequent $\Gamma \Rightarrow I$ can be polynomially transformed into an intuitionistic geometric proof of a geometric sequent sequent $D, \Gamma \Rightarrow I$ where D is obtained by introducing abbreviations for some formulas. In fact $D, \Gamma \Rightarrow I$ is intuitionistically derivable iff $\Gamma \Rightarrow I$ is intutionistically derivable, but on the surface the definitions in D are only classical.

In section 1 we give two proofs of the Glivenko property of geometric sequents.

Section 2 describe depth-reducing transformations we need for our proofs. As far as I know, this use of formulas (17-19) especially to achieve that the whole proof is geometric is new. It is inspired by similar use of (18) by V. Orevkov (1968) in a different situation.

Section 3 contains the proof of the main result.

We use $\equiv$ for literal coincidence of syntactic objects and $\leftrightarrow$ for a logical equivalence connective.

LK,LJ are Gentzen's systems for classical and intuitionistic logic, both with cut.

$\vdash^c, \vdash^i$ denote derivability in classical or intuitionistic logic, that is in LK,LJ with cut.

A *formula translation* of a sequent $S \equiv A_1, \ldots, A_n \Rightarrow B_1, \ldots B_m$ is a formula $S^f \equiv (A_1 \& \ldots \& A_n \to B_1 \vee \ldots \vee B_m)$. Many notions defined for formulas are generalized to sequents via the formula translation. For example $S \leftrightarrow T$ for sequents S, T means $S^f \leftrightarrow T^f$.

c-models are ordinary models for the classical predicate logic, i-models are Kripke models.

1. Geometric Sequents Have Glivenko Property

The next theorem is well-known. The deductive proof given here is due to V. Orevkov (1968) and can be traced back to the work of H. Curry (1977).

Theorem 1. *A geometric sequent is derivable classically iff it is derivable intuitionistically.*

1. *A deductive proof.* Consider a cut-free proof of a geometric sequent

$$\Gamma \to I$$

in LK. Since the succedent rules for $\to, \forall$ are invertible in LK, we can analyze away initial universal quantifiers and implication in I, then assume that I is a positive formula. After that *the sequent* $\Gamma \Rightarrow I$ *contains only connective occurrences that give rise to rules*

$$\Rightarrow \&, \Rightarrow \vee, \Rightarrow \exists, \& \Rightarrow, \vee \Rightarrow, \exists \Rightarrow, \to\Rightarrow .$$

These rules are common for LK and LJm, hence our LK-derivation is already LJm-derivation, as required. $\vdash$

2. *A model-theoretic proof.* The idea here is rather similar, but I have not seen this proof in literature. Suppose a geometric sequent $\Gamma \Rightarrow I$ with positive formula I is underivable in LJm. Consider its proof search tree in LJm (see for example Mints, 2000). This tree is not a derivation, and hence has a non-closed branch generating a Kripke countermodel for $\Gamma \Rightarrow I$. The rules for analysis of the connectives $\forall, \to$ in succedent are not applied in this tree. But these are exactly the rules that add new worlds to a model. Therefore resulting model has just one world, and hence it is a classical model refuting our sequent. $\vdash$

2. Reducing Formula Depth

Familiar depth-reducing transformations by introduction of new predicate variables are modified here to preserve geometric sequents. There are subtle points noted below. Let's first recall well-known facts.

Let's define a relation between formulas (widely used in literature without a special name) which is weaker than provable equivalence but in some respects similar to it.

Write $F \trianglerighteq^s G$ where $s \in \{c, i\}$ if

$$G \equiv\ F' \to F \text{ and } \vdash^s F'[P_1/F_1, \ldots, P_n/F_n]$$

where P_i/F_i are substitutions (performed in this order) for predicate variables $P_1, \ldots P_n$ not occurring in F.

Lemma 1. *Assume $F \trianglerighteq^s G$. Then*

1. $\vdash^s F$ *iff* $\vdash^s G$,
2. *s-models for G are expansions (with respect to $P_1, \ldots, P_n$) of s-models for F.*

Proof. 1. $\vdash^s F \to G$ is obvious. If $\vdash^s G$ then since $G \equiv (F' \to F)$ the substitutions $P_1/F_1, \ldots, P_n/F_n$ and modus ponens yield $\vdash^s F$.

2. Similarly to 1.

$\vdash$

Notation $\mathbf{x}$ below stands for $x_1, \ldots, x_n$ with distinct variables $x_1, \ldots, x_n$.

Lemma 2. *If $\mathbf{x}$ contains all free variables of formulas $A(\mathbf{x}), B(\mathbf{x})$ then*

$$LJ \vdash \forall \mathbf{x}(A(\mathbf{x}) \leftrightarrow B(\mathbf{x})) \to\ (F(A) \leftrightarrow F(B)).$$

Proof. Induction on F. $\vdash$

Lemma 3. *If P is a fresh n-ary predicate symbol, $\mathbf{x}$ contains all free variables of the formula $A(\mathbf{x})$ then for $L \in \{LJ, LK\}$*

$$L \vdash\Rightarrow F(A) \textit{ iff } \qquad L \vdash \forall \mathbf{x}(A(\mathbf{x}) \leftrightarrow P(\mathbf{x})) \Rightarrow\ F(P)$$

Proof. If $L \vdash F(A)$, apply the previous Lemma.

If $L \vdash \forall \mathbf{x}(A(\mathbf{x}) \leftrightarrow P(\mathbf{x})) \Rightarrow\ F(P)$, substitute A for P. The antecedent of the sequent becomes $\forall \mathbf{x}(A(\mathbf{x}) \leftrightarrow A(\mathbf{x}))$. $\vdash$

For a given formula F assume that for every non-atomic subformula G of F a fresh predicate symbol P_G is chosen with the same arity as the number of free variables of G. In particular P_F has free variables of F as arguments. Atomic subformula $P(t_1, \ldots, t_n)$ is not changed.

Symbols P_G can be treated as pointers to subformulas of F. This informal observation can be formalized by assigning equivalences E_G to subformulas G in the following way:

If $G(\mathbf{x}) \equiv H(\mathbf{y}) \odot K(\mathbf{z})$ for $\odot \in \{\&, \vee, \to\}$ then

$$E_G \equiv\ \forall \mathbf{x}(P_G(\mathbf{x}) \leftrightarrow (P_H(\mathbf{y}) \odot P_K(\mathbf{z}))) \tag{1}$$

where $\mathbf{y}, \mathbf{z} \subseteq \mathbf{x}$.

If $G(\mathbf{x}) \equiv\ QyH(\mathbf{x}, y)$ for $Q \in \{\forall, \exists\}$ then

$$E_G \equiv\ \forall \mathbf{x}(P_G(\mathbf{x}) \leftrightarrow QyP_H(\mathbf{x}, y)). \tag{2}$$

Lemma 4. *Let $G, H \ldots, F$ be all non-atomic subformulas of F. Then for $L \in \{LJ, LK\}$*

$$L \vdash F \leftrightarrow\ L \vdash E_G, E_H, \ldots E_F \Rightarrow P_F.$$

Proof. Apply previous Lemma successively to subformulas, beginning with the innermost ones. ⊢

Let's rewrite equivalences (1),(2) as pairs or triples of implications, transforming these implications in LJ-equivalent way.

$$\forall\mathbf{x}(P_{G\&H}(\mathbf{x}) \rightarrow P_G(\mathbf{y})), \tag{3}$$
$$\forall\mathbf{x}(P_{G\&H}(\mathbf{x}) \rightarrow P_H(\mathbf{z})), \tag{4}$$
$$\forall\mathbf{x}(P_G(\mathbf{y})\&P_H(\mathbf{z}) \rightarrow P_{G\&H}(\mathbf{x}); \tag{5}$$
$$\forall\mathbf{x}(P_G(\mathbf{y}) \rightarrow P_{G\vee H}(\mathbf{x})), \tag{6}$$
$$\forall\mathbf{x}(P_H(\mathbf{z}) \rightarrow P_{G\vee H}(\mathbf{x})), \tag{7}$$
$$\forall\mathbf{x}(P_{G\vee H}(\mathbf{x}) \rightarrow (P_G(\mathbf{y}) \vee P_H(\mathbf{z})); \tag{8}$$
$$\forall\mathbf{x}(P_{\exists yP_G}(\mathbf{x}) \rightarrow \exists yP_G(\mathbf{x}, y)) \tag{9}$$
$$\forall\mathbf{x}\forall y(P_G(\mathbf{x}, y) \rightarrow P_{\exists yP_G}(\mathbf{x})) \tag{10}$$
$$\forall\mathbf{x}\forall y(P_{\forall yP_G}(\mathbf{x}) \rightarrow P_G(\mathbf{x}, y)); \tag{11}$$
$$* \forall\mathbf{x}(\forall yP_G(\mathbf{x}, y) \rightarrow P_{\forall yP_G}(\mathbf{x})) \tag{12}$$
$$* \forall\mathbf{x}(\neg P_G(\mathbf{x}) \rightarrow P_{\neg G}(\mathbf{x})) \tag{13}$$
$$\forall\mathbf{x}(P_G(\mathbf{x})\&P_{\neg G}(\mathbf{x}) \rightarrow \bot) \tag{14}$$
$$\forall\mathbf{x}(P_{G\rightarrow H}(\mathbf{x})\&P_G(\mathbf{y}) \rightarrow P_H(\mathbf{z})) \tag{15}$$
$$* \forall\mathbf{x}((P_G(\mathbf{y}) \rightarrow P_H(\mathbf{z})) \rightarrow P_{G\rightarrow H}(\mathbf{x})) \tag{16}$$

All these universally quantified implications are geometric except the three marked by a *. Let's replace them by classically equivalent geometric implications.

$$\forall\mathbf{x}\exists y(P_G(\mathbf{x}, y) \rightarrow P_{\forall yP_G}(\mathbf{x})) \tag{17}$$
$$\forall\mathbf{y}(P_G(\mathbf{y}) \vee P_{\neg G}(\mathbf{y})) \tag{18}$$
$$\forall\mathbf{x}((P_H(\mathbf{z}) \rightarrow P_{G\rightarrow H}(\mathbf{x})) \;\&\; (P_G(\mathbf{y}) \vee P_{G\rightarrow H}(\mathbf{x}))) \tag{19}$$

Denote the resulting set of geometric implications (3-11), (14,15) and (17,18,19) for subformulas of a set **F** of formulas by $\mathrm{DEF}_{\mathbf{F}}$.

3. Transformation of Classical Derivations

In this section we mean by intuitionistic predicate calculus a multiple-succedent formulation LJm (cf. Mints, 2000) which differs from LK only in the requirement that the list Δ is empty in the succedent rules for $\rightarrow, \neg, \forall$:

$$\frac{A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg A} \qquad \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \rightarrow B} \qquad \frac{\Gamma \Rightarrow \Delta, A(b)}{\Gamma \Rightarrow \Delta, \forall xA(x)}$$

Definition 2. Formulas $\neg A, A \rightarrow B, \forall xA$ introduced by these rules in an LK-derivation are called below *special* formulas when Δ is non-empty.

Let d be a derivation of a geometric sequent S in LK. Then $\mathbf{f}(d)$ denotes the set of all cut formulas in d and DEF_d denotes $\mathrm{DEF}_{\mathbf{f}(d)}$.

Theorem 2.

1. *Let d be a derivation of a geometric sequent $\Pi \Rightarrow \Phi$ in LK. Then it can be* polynomially *transformed into a geometric derivation in LJm of the sequent*

$$\mathrm{DEF}_d, \Pi \rightarrow \Phi$$

consisting of geometric sequents.

2. $\mathrm{DEF}_d, \Pi \Rightarrow \Phi \ \trianglerighteq^c \ \Pi \Rightarrow \Phi$.

3. $\vdash^c \mathrm{DEF}_d, \Pi \Rightarrow \Phi$ *iff* $\quad \vdash^i \mathrm{DEF}_d, \Pi \Rightarrow \Phi$ *iff* $\quad \vdash^i \Pi \Rightarrow \Phi$

Proof. We assume that all axioms $A, \Gamma \rightarrow \Delta, A$ have atomic A. Using if needed inversion transformations we assume that Φ consists of positive formulas. Then every special formula F is traceable to a cut formula. More precisely, $F \equiv F'(\mathbf{t})$ where $F'(\mathbf{x})$ is a subformula of some cut formula. Formula F' has a "representative" $P_{F'}(\mathbf{x})$ in DEF_d where $\mathbf{x}$ are free variables of F'. In this sense any occurrence of a formula F traceable to a cut formula has a representative which we write as $P_F(\mathbf{t})$.

Denote by d^+ the result of replacing every such occurrence of $F(\mathbf{t})$ as a separate formula in a sequent in d by $P_F(\mathbf{t})$.

This replacement destroys inferences having such $F(t)$ as principal formulas. Consider these inferences in turn to show they can be repaired using DEF_d.

Axioms are assumed to be atomic, therefore they are preserved. The cut inferences become cuts on atomic formulas.

Antecedent inferences are repaired using geometric implications in Def_d. For example $\rightarrow$-antecedent inference

$$\frac{\Gamma \Rightarrow \Delta, G(\mathbf{t}_1) \quad H(\mathbf{t}_2), \Gamma \Rightarrow \Delta}{G(\mathbf{t}_1) \rightarrow H(\mathbf{t}_2), \Gamma \Rightarrow \Delta}$$

goes into the figure

$$\frac{\Gamma \Rightarrow \Delta, P_G(\mathbf{t}_1) \quad P_H(\mathbf{t}_2), \Gamma \Rightarrow \Delta}{P_{G\rightarrow H}(\mathbf{t}), \Gamma \Rightarrow \Delta}$$

which is transformed using the formula $P_{G\rightarrow H}(\mathbf{t}) \& P_G(\mathbf{t}_1) \rightarrow P_H(\mathbf{t}_2)$ denoted below by I which is an instance of a formula (15) in DEF_d.

$$\dfrac{\dfrac{\dfrac{\overset{\text{axiom}}{P_{G\rightarrow H}(\mathbf{t}) \Rightarrow P_{G\rightarrow H}(\mathbf{t})} \quad \Gamma \Rightarrow \Delta, P_G(\mathbf{t}_1)}{P_{G\rightarrow H}(\mathbf{t}), \Gamma \Rightarrow \Delta, P_{G\rightarrow H}(\mathbf{t}) \& P_G(\mathbf{t}_1)} \quad P_H(\mathbf{t}_2), \Gamma \Rightarrow \Delta}{I, P_{G\rightarrow H}(\mathbf{t}), \Gamma \Rightarrow \Delta}}{\mathrm{DEF}_d, P_{G\rightarrow H}(\mathbf{t}), \Gamma \Rightarrow \Delta} \ \forall \Rightarrow$$

Other antecedent rules and succedent rules common to LK and LJm are treated similarly. Of the remaining rules consider $\neg, \rightarrow$ and $\forall$ in succedent. Given derivations are transformed as follows. The derivation

$$\frac{G, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg G}$$

goes to

$$\dfrac{\dfrac{P_G(\mathbf{t}), \Gamma \Rightarrow \Delta \quad P_{\neg G}(\mathbf{t}) \Rightarrow P_{\neg G}(\mathbf{t})}{P_G(\mathbf{t}) \vee P_{\neg G}(\mathbf{t}), \Gamma \Rightarrow \Delta, P_{\neg G}(\mathbf{t})} \ \vee \Rightarrow}{\mathrm{DEF}_d, \Gamma \Rightarrow \Delta, P_{\neg G}(\mathbf{t})}$$

The derivation

$$\frac{G, \Gamma \Rightarrow \Delta, H}{\Gamma \Rightarrow \Delta, G \rightarrow H}$$

goes to

$$\dfrac{\dfrac{P_G(\mathbf{t}_1), \Gamma \Rightarrow \Delta, P_H(\mathbf{t}_2) \quad \text{axioms}}{P_H(\mathbf{t}_2) \rightarrow P_{G\rightarrow H}(\mathbf{t}), P_G(\mathbf{t}_1) \vee P_{G\rightarrow H}(\mathbf{t}), \Gamma \Rightarrow \Delta, P_{G\rightarrow H}(\mathbf{t})} \ \vee \Rightarrow, \rightarrow\Rightarrow}{\mathrm{DEF}_d, \Gamma \Rightarrow \Delta, P_{G\rightarrow H}(\mathbf{t})}$$

The derivation

$$\frac{\Gamma \to \Delta, G(b)}{\Gamma \to \Delta, \forall y G(y)}$$

goes to

$$\dfrac{\dfrac{\Gamma \to \Delta, P_G(\mathbf{t}, b) \quad P_{\forall y G(y)}(\mathbf{t}) \to P_{\forall y G(y)}(\mathbf{t})}{P_G(\mathbf{t}, b) \to P_{\forall y G(y)}(\mathbf{t}), \Gamma \to \Delta, P_{\forall y G(y)}(\mathbf{t})} \to\Rightarrow}{\dfrac{\exists y (P_G(y, \mathbf{t}) \to P_{\forall y G(y)}(\mathbf{t})), \Gamma \to \Delta, P_{\forall y G(y)}(b, \mathbf{t})}{\mathrm{DEF}_d, \Gamma \to \Delta, P_{\forall y G(y)}(\mathbf{t})}} \exists \to$$

This completes the proof of the first part of the theorem.

The second part follows from *classical* derivability of the results of substitution P_G/G into formulas in DEF_d.

For the third part, if $\vdash^c \mathrm{DEF}_d, \Pi \to \Phi$ then substitution P_G/G for $G \in \mathbf{f}(d)$ yields $\vdash^c \Pi \Rightarrow \Phi$, then (by Theorem 1) $\vdash^i \Pi \Rightarrow \Phi$ and hence $\vdash^c \mathrm{DEF}_d, \Pi \Rightarrow \Phi$ completing the chain of equivalences. As pointed out in the Introduction, the transformation in Theorem 1 is not polynomial. $\vdash$

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