

New Trends in Foundations of Mathematics

G. Mints

Stanford University
gmints@stanford.edu

Abstract: New emphasis on finitist methods and results in mathematics indicates a turn in foundations happening under the slogans of HARD ANALYSIS and PROOF MINING. While previously non-constructive or infinitistic methods were thought (by an influential minority) to be philosophically defective, the revival of interest is caused by mathematical needs. Some of the central results needed development of new tools that turned out to be instances of well-known constructions of proof theory.

The finitist trend is called “hard (quantitative) analysis” by T. Tao and contrasted with the ordinary or “soft” (qualitative) mathematical analysis. The latter works without any restrictions on the abstract notions and infinitistic methods. Proof mining initiated by G. Kreisel and developed by U. Kohlenbach applies proof theoretic tools to get essential strengthening of results proved in the mainstream mathematics. We present some examples.

The distinction between finitist and non-finitist methods crystallized in Hilbert’s foundational program. We discuss new developments in mathematics and its foundations emphasising finitist methods, and as a consequence of this, constructive methods. This is done under the slogans of **hard analysis** and **proof mining**.

Previously constructive methods were distinguished mainly on ideological grounds when non-constructive or infinitistic methods were thought to be defective from some philosophical considerations:

L. Brouwer, H. Weyl, A. Markov, E. Bishop, Dummett, E. Nelson, G. Stolzenberg.

H. Weyl: “So gebe Ich also jetzt meinen eigenen Versuch Preis und schliesse mich Brouwer an” (Weyl 1921, 56). [So I leave my own attempts aside and join Brouwer].

H. Weyl describes [in his obituary of Hilbert] how he restricted his research methods to intuitionistically acceptable ones.

Present revival of interest is caused by mathematical needs: obtaining estimates for some of the central results of modern number theory demanded development of new tools that turned out to be instances of well-known constructions of finitist approach. This new direction had been called “hard analysis” by T. Tao and contrasted with the ordinary or “soft” mathematical analysis. The latter works without any restrictions on the abstract notions and infinitistic methods. The idea that methods of logical proof theory can be useful in the mainstream mathematics independently of constructivist or finitist ideology had been stressed by G. Kreisel beginning with 1950s under the slogan of “unwinding” proofs. This approach led to several good but isolated results before it was jump-started by U. Kohlenbach at the beginning of 1990s and given the name “proof mining” (following suggestion by D. Scott).

The distinction between finitist and infinitistic methods goes back to D. Hilbert. It is based on the distinction between real (finitist, combinatorial) objects that can be completely encoded by natural numbers and imaginary or infinitistic objects that do not allow such coding.

For example real number defined by an infinite sequence of decimal approximations is an infinitistic object. Finitist methods work with finitist objects (=natural numbers) using computable functions on such objects (given in principle by computer programs).

In fact a sharpening of finitist restrictions to Kalmár-elementary (exponential) or even polynomial functions is often desirable: mathematicians accept faster growing functions only reluctantly.

Unnoticed by philosophers, these distinctions penetrated the working environment of mathematicians.

Here are some quotations from the blog by T. Tao of UCLA partially published in T. Tao, Structure and Randomness: pages from year one of a mathematical blog, American Mathematical Society, 2008.

“In the field of analysis, it is common to make a distinction between “hard”, “quantitative”, or “finitary” analysis on one hand, and “soft”, “qualitative”, or infinitary” analysis on the other. “Hard analysis” is mostly concerned with finite quantities (e.g. cardinality of finite sets, the measure of bounded sets, the value of convergent integrals, the norm of finite-dimensional vectors, etc.) and their *quantitative* properties (in particular upper and lower bounds). “Soft” analysis, on the other hand, tends to deal with more infinitary objects (e.g. sequences, measurable sets and functions, σ -algebras, Banach spaces, etc.) and their *qualitative* properties (convergence, boundedness, integrability, completeness, compactness, etc.).

To put it more symbolically, hard analysis is the mathematics of ϵ , N , $O()$ and \leq ; soft analysis is the mathematics of 0 , ∞ , ϵ and \rightarrow .

...Because of all these difficulties it is common for analysts to specialize in only one of the two types of analysis. For instance, as a general rule (and with notable exceptions), discrete mathematicians, computer scientists, real-variable harmonic analysts, and analytic number theorists tend to rely on “hard analysis” tools, whereas ‘functional analysts’, operator algebras, abstract harmonic analysts, and ergodic theorists tend to rely on “soft analysis” tools. ... There are examples of evolution of a field from soft analysis to hard (e.g. recent developments in extremal combinatorics, particularly in relation to the regularity lemma).

...In many cases qualitative analysis can be viewed as a convenient abstraction of quantitative analysis, in which the precise dependencies between various finite quantities has been effectively concealed from view by the use of infinitary notation. Conversely, quantitative analysis can often be viewed as a more precise and detailed refinement of qualitative analysis. Furthermore, a method from hard analysis often has some analogue in soft analysis and vice versa, though the language and notation may look completely different from that of the original. I therefore feel that it is often profitable for a practitioner of one type of analysis to learn about the other, as they both offer their own strengths, weaknesses and intuition, and knowledge of one gives more insight into the working of the other. I wish to illustrate this point here using a simple but not terribly well known result, which I shall call the “finite convergence principle”... Sometimes a careful analysis of a trivial result can be surprisingly revealing, as I hope to demonstrate here.

The infinite convergence principle is well known: every bounded monotone sequence x_n of real numbers is convergent.

Again: if $\epsilon > 0$ and

$$0 \leq x_1 \leq x_2 \leq \dots \leq 1,$$

there exists an N such that

$$|x_n - x_m| \leq \epsilon \text{ for all } n, m \geq N.$$

Randomness

Logic and especially proof theory accumulated a good supply of tools for conversion of ordinary mathematical proofs into proofs satisfying the needed restrictions. We consider different ways of obtaining bounds from existential proofs.

The simplest situation is direct constructivization. The original statement is existential

$$\exists xR(x)$$

and the original proof is non-constructive, since it proceeded by contradiction. An assumption to the contrary,

$$\forall x\neg R(x)$$

leads to an explicit contradiction. Constructivized proof provides a method of computing an object x satisfying $R(x)$.

Sometimes inessential modification of the original proof suffices.

Example. Infinity of primes. Euclid's proof (formally going by contradiction) already contains a bound for the next prime

$$p_{n+1} \leq p_1 \cdot \dots \cdot p_n + 1.$$

In other cases new constructions or even new ideas are needed.

Constructivization have been the core of several foundational schools, including intuitionism, Russian constructivism, Bishop analysis etc.; with some restrictions also explicit mathematics developed by S. Feferman can be mentioned here.

Proof mining has broader goals. It applies proof theoretic tools to get essential strengthening of results proved in the mainstream mathematics. New results are interesting to specialists in a given field (main example is non-linear analysis), and new proofs are stated completely in the framework of the field: no need for practitioners to learn any logic or constructive mathematics. The need to understand the underlying logical theory arises only when one tries to understand sources of the constructions used in the new proof and the reason exactly these constructions are used.

Examples follow

Important part of the picture is treatment of intermediate results. As pointed out before, the final result of interest is often an existential statement

$$\exists xA(x)$$

or more generally a statement

$$\forall x\exists yA(x, y)$$

with finitist (quantifier free) $A(x, y)$. We are interested in a function $Y(x)$ satisfying for every x

$$A(x, Y(x)).$$

However really deep mathematical proof of our goal statement in general contains lemmas of much more complicated form and uses deductive means which ordinarily are not expected to occur in proofs of such simple statements. There are several reductive tools that eliminate such "detours" from proofs of simple statements. Let's recall some of them having more obvious philosophical significance.

Absoluteness. Suppose an arithmetical statement is proved in ZFC using the axiom of choice AC. Then the relativization of this proof to constructible sets (slogan: $V=L$) does not use AC.

Functional interpretation. Gödel designed a transformation called *Dialectica interpretation* of arbitrary statement F of arithmetic or even analysis to a form

$$F' = \exists Y \forall x B(x, Y).$$

Here B is quantifier free, objects $x, Y(x)$ are in general of more complicated nature (of higher type) than natural numbers or numerical functions. This transformation preserves *constructive* proofs: if F_1, \dots, F_n is a proof, then F'_1, \dots, F'_n is a proof too (after a simple extension), moreover it is a quantifier free proof, finitistic in an extended sense.

As noticed by Kolmogorov, Gödel, Gentzen there is a simple translation of non-constructive proofs into constructive proofs of a **weaker statement**: just double-negate everything: $\neg\neg$.

This operation (*negative translation*) transforms

$$\exists x R(x) \text{ into } \neg\neg\exists x\neg\neg R(x)$$

when R is atomic. Dialectica interpretation provides (from a non-constructive proof of $\exists x R(x)$) some instance n for $\exists x$.

Modifications of Dialectica interpretation turned out to be suitable for analysis and strengthening of theorems of the mainstream mathematics (mainly non-linear analysis).

Example. Existence of the best approximation by polynomials in L^1 -metric.

U. Kohlenbach gave the first constructive proof and very specific estimates.

Friedman-Dragalin translation.

Example 1. There are irrational numbers x, y such that x^y is rational.

A non-constructive proof. If $\sqrt{2}^{\sqrt{2}}$ is rational, then we are done: $x := y := \sqrt{2}$.

Otherwise $x := \sqrt{2}^{\sqrt{2}}, y := \sqrt{2}$, since

$$\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{(\sqrt{2} \cdot \sqrt{2})} = \sqrt{2}^2 = 2.$$

This proof does not immediately provide x and y unless sophisticated proof (by A. Gelfond) that $\sqrt{2}^{\sqrt{2}}$ is irrational is used.

Constructivization by Dana Scott (cf. FOM newsgroup). Take

$$x := e, y := \ln 2; x^y = 2.$$

From the power series expansion of e^x it is easy to see that all integer powers of e^x are irrational. Now note

$$e^{m/n} = 2 \Rightarrow e^m = 2^n,$$

Another example from FOM.

Between any two irrationals $a < b$ there is another rational.

Proof (P. Halmos). Let $h := (b-a)/3$. Then one of $a+h, a+2h$ is irrational, since otherwise h is rational, hence a is rational too. But which one of $a+h, a+2h$?

Constructivization (D. Scott). Since $a < b$, there is an n such that

$$a < a + 1/n < b.$$

$a + 1/n$ is irrational, since otherwise $a = (a + 1/n) - 1/n$ is rational.

My contribution. Constructivization of the non-constructive proof of Herbrand's Theorem.

Reference

- T. Tao, Structure and Randomness: pages from year one of a mathematical blog, American Mathematical Society, 2008.