What is a Morse complex?
A survey from R. Thom to K. Fukaya,
via S. Smale and E. Witten

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Given a compact manifold M equipped with a real function $f : M \to \mathbb{R}$, whose critical points are quadratic non-degenerate and whose critical values are distinct, Marston Morse was able to deduce the so called *Morse inequalities* from a local study of the homology near the critical points only. In his 1949 Note, his first publication indeed, René Thom used the gradient of a Morse function for decomposing M into cells. But, the transversality condition of the stable and unstable manifolds was missing.

This condition on the gradient vector field, now called the *Morse-Smale condition*, was pointed out by Steve Smale in the early sixties. In high dimension and when M is 1-connected, Smale was able to cancel by pairs the critical points which do not contribute to the homology. On this occasion, Smale discovered the so-called *Morse complex*, an algebraic $\mathbb{Z}$-complex based on the critical points of f whose differentials count the gradient orbits which connects critical points of successive indices.

In 1982, Edward Witten rediscovered this complex by deforming the Laplacian by mean of the given Morse function. The de Rham-Witten complex, that is, the space $\Omega^*(M)$ of differential forms on M equipped with a co-boundary operator deformed *à la Witten*, has the Morse complex as semi-classical limit. This approach allowed analysts to take the case when M has a non-empty boundary into account. More recently, I found a direct topological approach to this, that is, a right notion of pseudo-gradient *adapted* to the boundary allowing me to build a complex which calculates the homology of M, or the relative homology $H_*(M, \partial M)$, according to the type of adapted pseudo-gradient which is used.

All these complexes are stable under small perturbations of the function f and its pseudo-gradient X. This is the starting point of Kenji Fukaya when M is closed. By *multi-intersecting* the stable/unstable manifolds of X with stable/unstable manifolds of perturbations of X, Fukaya equipped the Morse complex with a rich mutiplicative structure, an A_∞-structure indeed.

This also works in the case of non-empty boundary. For instance, consider the Borromean link L in S^3 endowed with the standard height function and define M as the complementary in S^3 of a small open tubular neighborhood of L. In this case, the third product in the infinite series yields a Morse approach to the Massey product.