The Heat Kernel on the AdS(2) Cone and Logarithmic Corrections to Extremal Black Hole Entropy

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Introduction

- Black Holes in a quantum theory of gravitation are expected to have entropy.

$$\boxed{S_{BH} = \frac{A_H}{4G_N}}$$

- This formula is obtained in two approximations:
  - Low energy,
  - Semi-classical.
- A complete theory of quantum gravity will encode corrections to this formula.
- They arise by weakening the two approximations:
  - Higher-derivative corrections to GR,
  - Quantum Corrections.

Introduction

- Higher-derivative corrections can be computed by means of the Wald formula.
- What about quantum corrections?
- It would be helpful to seek simpler settings where the problem can be solved explicitly.
- This might teach us useful lessons which can be extrapolated to the more general case.
- Extremal black holes are an ideal laboratory:
  - Their classical entropy is already very simple,
  - The quantum answer is explicitly known from string theory.
- Using AdS/CFT one can compute quantum entropy.

Extremal Black Holes

- Black holes generically have two event horizons.
- Consider a limit where the horizons coincide.
- This is an extremal black hole.
- The near horizon geometry is always $AdS_2 \otimes M$.
- We can use AdS/CFT to compute quantum entropy.
- Prescription: Calculate the string theory path integral in the black hole near horizon geometry. [Sen]
- This is the degeneracy associated to the event horizon.
- Reproduces S_{BH} in the classical limit.
- Can we compute more extensively? Quantum Effects?

Introduction

- Consider black holes for which the full quantum answer is known from string theory.
- This takes the form of a degeneracy $d\left(Q, P\right)$

$$d\left(Q, P\right) \sim e^{\frac{A_H(Q,P)}{4}}$$

- $\left(Q, P\right)$ are black hole electric and magnetic charges.
- Taking the log of both sides, we recover S_{BH}.
- If we zoom in closer on the degeneracy

$$\boxed{d\left(Q, P\right) \simeq A_H^m e^{\frac{A_H}{4}} + \sum_N A_H^p e^{\frac{A_H}{4N}}}$$

- m and p are numbers which have been computed.
- In principle, p can depend on N.

Introduction

Can we match this answer from the string path integral?

$$\mathcal{Z}_{str.} \overset{?}{\simeq} A_H^m e^{\frac{A_H}{4}} + \sum_N A_H^p e^{\frac{A_H}{4N}}$$

Useful to recall the origin of $e^{\frac{A_H}{4}}$:

- We will use the saddle-point approximation.
- Black holes with horizon geometry $AdS_2 \otimes S^2$.
- A saddle-point of $\mathcal{Z}_{str.}$ is the near horizon geometry itself.

$$ds^2 = a^2\left(d\eta^2 + \sinh^2\eta d\theta^2\right) + a^2\left(d\psi^2 + \sin^2\psi d\phi^2\right)$$

- $a^2 \simeq A_H$ upto constants. We will work in terms of a.
- The value of $\mathcal{Z}_{str.}$ at the saddle point is $e^{\frac{A_H}{4}}$.
- Quantum fluctuations about the saddle point give A_H^m.

Introduction

Question: Do the other terms have a similar origin?

- Consider the $\mathbb{Z}_N$ orbifold of the near-horizon geometry

$$(\theta, \phi) \mapsto \left(\theta + \frac{2\pi}{N}, \phi - \frac{2\pi}{N}\right)$$

- This is an admissible saddle-point of $\mathcal{Z}_{str.}$.
- At the saddle-point $\mathcal{Z}_{str.} = e^{\frac{A_H}{4N}}$.
- Reproducing A_H^p is the subject of this talk.

Terminology:

- A_H^m and A_H^p are called 'log terms'.
- String Path Integral $\Rightarrow$ Quantum Entropy Function.

Introduction

Question: Just like A^m_H, can we reproduce A^p_H from quantum fluctuations about the alternate saddle points?

- Why is this important?
  - Requires us to go beyond the classical limit and study quantum corrections to black hole entropy.
  - We can push the analysis to cases where the string theory answer is not available. New predictions!
- Why is this doable? Sen, 1205.0971
  - The log terms are determined purely from one-loop fluctuations of massless fields around the saddle-point.
  - Knowledge of two-derivative supergravity is enough to compute this contribution to the string path integral!
  - Remarkable simplification as string theory has an infinite number of massive fields of arbitrary spin.

Bonus: find new maths results while solving the problem!

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Gaussian Integrals

All the techniques we use can be explicitly demonstrated here.

- Consider an integral $Z = \int \prod_{i=1}^{n} dx_i \, e^{-x_i M_{ij} x_j}$.
- Then $Z = \det^{-\frac{1}{2}} M$.

We will now 'define' the determinant of M.

- Let M have eigenvalues κ_m with degeneracy d_m.
- Since determinant = product of eigenvalues

$$\ln \det M = \sum d_m \ln \kappa_m,$$

$$\sum d_m \ln \kappa_m = \sum_m \int_0^\infty \frac{dt}{t} d_m e^{-t\kappa_m}.$$

We will evaluate $\det(M)$ by explicitly enumerating κ_m and d_m.

Gaussian Integrals (Pitfalls!)

Let's look closer at $Z = \det(M)$. Suppose $M_{ij} = \kappa_i \delta_{ij}$.

$$Z = \int \left(\prod_{i=1}^{n} dx_i e^{-\kappa_i x_i^2} \right) = \sqrt{\frac{1}{\prod_{i=1}^{n} \kappa_i}} = \det{}^{-\frac{1}{2}} M.$$

- This is true only if $\kappa_i > 0 \, \forall \, i$.
- What if say $\kappa_n = 0$? i.e. M has a zero mode?

In that case

$$Z = \int \left(\prod_{i=1}^{n-1} dx_i e^{-\kappa_i x_i^2} \right) \int dx_n = \left(\det{}' M \right)^{-\frac{1}{2}} \int dx_n.$$

- We get a determinant over non-zero modes,
- The zero mode contribution has to be analyzed separately.

The One-Loop Determinant

- Consider a path integral for a field $\phi(x)$

$$\mathcal{Z}[\Phi] = \int \mathcal{D}\Phi e^{-\frac{i}{\hbar}S[\Phi]}.$$

- As $\hbar \to 0$, this is dominated by classical configurations Φ_{cl}

$$\frac{\delta}{\delta\Phi} S[\Phi]|_{\Phi=\Phi_{cl}} = 0$$

- In an expansion about Φ_{cl}, we have

$$S[\Phi_{cl} + \phi] \simeq S[\Phi_{cl}] + \int d^n x \sqrt{g}\phi(x) D\phi(x).$$

- We can easily evaluate the one-loop path integral.

$$\boxed{\mathcal{Z}_{1-\ell}[\phi] = \det^{-\frac{1}{2}}(D).}$$

The Degeneracy

- We will define $\det D$ exactly as we defined $\det M$.

$$\ln \det D = \sum_m \int \frac{dt}{t} d_m e^{-t\kappa_m}$$

- A prescription for the degeneracy d_m:
  - Let $\psi_{n,m}$ denote a complete set of orthonormal eigenfunctions of D with eigenvalue κ_m.
  - Then d_m is given by

$$\boxed{d_m = \sum_n \int dx \sqrt{g} \psi^*_{n,m}(x)\, \psi_{n,m}(x)\,.}$$

- This is perfectly well defined on S^2.
- AdS/CFT will make it well defined on $AdS_2 \otimes S^2$.

$\Rightarrow$ new maths!

Extracting The Log Term

- Consider $\mathcal{Z}_{str.} \sim A^p e^{\frac{A}{4N}}$. We need the term A^p.

$$\mathcal{Z}_{str.} \sim A_H^p e^{\frac{A_H}{4N}} \Rightarrow \ln \mathcal{Z}_{str.} = p \ln A_H + \cdots .$$
$$\ln \mathcal{Z}_{str.} = \ln \det D + \cdots \Rightarrow \ln \det D = p \ln A_H + \cdots .$$

- Only the one-loop determinant contributes to $\ln A_H$.
- Only massless fields contribute to $\ln A_H$.
- Further simplification: define the heat kernel

$$K(t) = \sum_m d_m e^{-t\kappa_m}.$$

- Only the t^0 term in K(t) contributes.

$$\boxed{p = \frac{1}{2} K(0; t).}$$

- This limit should be taken carefully.

Extracting The Log Term

- In principle, the zero mode integral can also contribute.
- Suppose a field ϕ has one zero mode of D.

$$\mathcal{Z}^{zero}_{str.} = A_H^{\frac{\beta_\phi}{2}} \mathcal{Z}_0.$$

- β_ϕ is a number which is known. $\mathcal{Z}_0$ does not scale with A_H.
- If ϕ has n^0_ϕ zero modes, then

$$\mathcal{Z}^{zero}_{str.} = A_H^{\frac{\beta_\phi}{2} n^0_\phi} \mathcal{Z}_0.$$

- Recall $AdS_2 \otimes S^2$ radius a: $A_H \sim a^2$.
- A–dependence of $\mathcal{Z}^{zero}_{str.}$ arises from the length scale a that the saddle-point has.

Extracting The Log Term

- We trade in the area A_H for the length scale a as $A_H \sim a^2$.
- Also $K(t)$ computes $\ln \det D$. We need $\ln \det' D$.
- Consider a path integral over a field ϕ.
- Suppose for some m_0, $\kappa_{m_0} = 0$. Then $d_{m_0} = n^0_\phi$.
- Define $K'(t) = K(t) - d_{m_0}$.
- We then have

$$\ln \mathcal{Z} = \left(K'(0;t) + \beta_\phi n^0_\phi\right) \ln a + \cdots .$$

- If we have multiple fields

$$\boxed{\ln \mathcal{Z} = \left(K(0;t) + \sum_\phi \left(\beta_\phi - 1\right) n^0_\phi\right) \ln a + \cdots .}$$

A toy model: The scalar on S^2

- Let's put this to work for the Laplacian!
- Consider a scalar field on a sphere of radius a.
- Eigenvalues: $E = \ell(\ell+1)$ degeneracies: $(2\ell+1)$.
- Then the heat kernel is

$$K(t) = \sum_{\ell=0}^{\infty} (2\ell+1)\, e^{-\frac{t}{a^2}\ell(\ell+1)}$$

- The small-t expansion is

$$K(t) = \frac{a^2}{t} + \frac{1}{3} + \frac{t}{15a^2} + \cdots .$$

- The log term is $\frac{1}{3}\ln a$.

Caveat: $\ell = 0$ is a zero mode, but $\beta = 1$ for the scalar.

A toy model: The scalar on AdS_2

- Another simple setting to understand the overall strategy.
- The metric on AdS_2 is

$$ds^2 = a^2 \left(d\eta^2 + \sinh^2 \eta d\phi^2\right).$$

- We need the spectrum of the scalar Laplacian, i.e. eigenvalues and degeneracies.
- eigenvalues are $E_\lambda = \frac{1}{a^2}\left(\lambda^2 + \frac{1}{4}\right)$.
- degeneracies are problematic! Eigenfunctions are

$$\Psi_{\lambda,m} = e^{im\theta} F_{\lambda,m}(\eta), \quad m \in \mathbb{Z}.$$

- 'degeneracy' $\sim$ 'number of eigenfunctions' of given E_λ.
- There are an infinite number of them!

A toy model: The scalar on AdS_2

We will use the following definition of degeneracy

$$d_\lambda = \sum_{m\in\mathbb{Z}} \int_{AdS_2} \Psi^*_{\lambda,m}(x)\, \Psi_{\lambda,m}(x)\,,$$

since AdS_2 is a homogeneous space,

$$d_\lambda = \left(\sum_{m\in\mathbb{Z}} |\Psi_{\lambda,m}(0)|^2 \right) (Vol_{AdS_2}).$$

- $|\Psi_{\lambda,m}(0)|^2 = 0$ unless $m = 0$.
- $|\Psi_{\lambda,m}(0)|^2 = \frac{1}{2\pi a^2}\lambda \tanh \pi\lambda$ if $m = 0$.

Then

$$d_\lambda = \frac{(Vol_{AdS_2})}{2\pi a^2}\lambda \tanh \pi\lambda.$$

Regulating the AdS Volume

The divergence in d_λ hides in the infinite volume of AdS_2.

$$Vol_{AdS_2} = \int_0^\infty d\eta \int_0^{2\pi} a^2 \sinh\eta .$$

We regulate it by cutting off the AdS_2 radius at a large η_0.

$$Vol_{AdS_2} = \int_0^{\eta_0} d\eta \int_0^{2\pi} a^2 \sinh\eta = 2\pi a^2 \left(\cosh\eta_0 - 1\right)$$

$$\Rightarrow Vol._{AdS_2} = 2\pi a^2 \left(e^{\eta_0} - 1\right) + \mathcal{O}\left(e^{-\eta_0}\right).$$

The regularised volume is the order-1 term. $\Leftarrow$ AdS/CFT.

$$Vol_{AdS_2} = -2\pi a^2 .$$

The regularised degeneracy is then

$$\boxed{d_\lambda = -\lambda \tanh \pi\lambda} \quad \Rightarrow \text{Plancherel Measure!}$$

A toy model: The scalar on AdS_2

Then the regulated heat kernel is

$$K(t) = \int_0^\infty d\lambda\, d_\lambda e^{-tE_\lambda} = -\int_0^\infty d\lambda\, \lambda \tanh \pi\lambda\, e^{-\frac{t}{a^2}\left(\lambda^2+\frac{1}{4}\right)}.$$

The short-time expansion of $K(t)$ is

$$K(t) = \left(-\frac{a^2}{2t} + \frac{1}{6} - \frac{t}{30a^2}\right) + \cdots .$$

- The log term is $\frac{1}{6} \ln a$.
- There are no zero modes.

$\therefore$ we have a well-defined heat kernel and log term.

The Analytic Continuation

The scalar heat kernel on S^2 is

$$K(t) = \left(\frac{a^2}{t} + \frac{1}{3} + \frac{t}{15a^2} \right) + \cdots .$$

The scalar heat kernel on AdS_2 is

$$K(t) = \frac{1}{2} \left(-\frac{a^2}{t} + \frac{1}{3} - \frac{t}{15a^2} \right) + \cdots .$$

- The bracketed terms are related by $a \mapsto ia$.
- The overall half is an artefact of the analytic continuation.
- Origin: $\mathrm{Vol}_{S^2} = 4\pi a^2 \mapsto -4\pi a^2$ under $a \mapsto ia$.
- But $\mathrm{Vol}_{AdS_2} = -2\pi a^2$. This is the reason for the $\frac{1}{2}$.

This will be useful for us because we can't use homogeneity to evaluate the heat kernel on the quotient space.

Summary: I

- Our overall goal is to extract log terms from $\mathcal{Z}_{str}$.
- We have seen how the heat kernel will help us do that.
- The methods we presented can compute the log term about the leading saddle point. 1005.3044, 1106.0080
- We want the log term about $\mathbb{Z}_N$ orbifolds of $\mathrm{AdS}_2 \otimes \mathrm{S}^2$.
- These heat kernel computations rely on homogeneity of spacetime and break down here.
- We will first extend the heat kernel techniques.

  $\Rightarrow$ generalise the Plancherel formula!
- We will then apply them to saddle-points of $\mathcal{Z}_{str}$.

The $\mathbb{Z}_N$ Orbifold

The $AdS_2 \otimes S^2$ spacetime is described by the metric

$$ds^2 = a^2 \left(d\eta^2 + \sinh^2\eta d\theta^2\right) + a^2 \left(d\psi^2 + \sin^2\psi d\phi^2\right)$$

We impose the following $\mathbb{Z}_N$ orbifold

$$(\theta, \phi) \mapsto \left(\theta + \frac{2\pi}{N}, \phi - \frac{2\pi}{N}\right).$$

This has fixed-points

- $(\eta = 0, \psi = 0)$ and
- $(\eta = 0, \psi = \pi)$

Near the fixed points the metric has the form

$$ds^2 = a^2 \left(d\eta^2 + \eta^2 d\theta^2\right) + a^2 \left(d\psi^2 + \psi^2 d\phi^2\right)$$

$\Rightarrow$ conical singularities, break translational invariance!

Analytic Continuation

Metric on $S^2 \otimes S^2$

$$ds^2 = a_1^2 \left(d\chi^2 + \sin^2\chi d\theta^2\right) + a_2^2 \left(d\psi^2 + \sin^2\psi d\phi^2\right)$$

We impose the following $\mathbb{Z}_N$ orbifold

$$(\theta, \phi) \mapsto \left(\theta + \frac{2\pi}{N}, \phi - \frac{2\pi}{N}\right).$$

The only difference: number of fixed points is doubled

- $(\chi = 0, \psi = 0)$
- $(\chi = 0, \psi = \pi)$
- $(\chi = \pi, \psi = 0)$
- $(\chi = \pi, \psi = \pi)$

Structurally, the fixed points are the same

$$ds^2 = a_1^2 \left(d\chi^2 + \chi^2 d\theta^2\right) + a_2^2 \left(d\psi^2 + \psi^2 d\phi^2\right)$$

$\Rightarrow$ the same conical singularities.

Analytic Continuation

Consider now the orbifold space $\left(S^2 \otimes S^2\right)/\mathbb{Z}_N$.

$$ds^2 = a_1^2\left(d\chi^2 + \sin^2\chi d\theta^2\right) + a_2^2\left(d\psi^2 + \sin^2\psi d\phi^2\right)$$

Analytically Continue: $(a_1, a_2) \mapsto (ia, a)$, $\chi \mapsto \eta$.

$$\Rightarrow ds^2 = a^2\left(d\eta^2 + \sin^2\eta d\theta^2\right) + a^2\left(d\psi^2 + \sin^2\psi d\phi^2\right)$$

which is $AdS_2 \otimes S^2$. The $\mathbb{Z}_N$ orbifold is the same

$$(\theta, \phi) \mapsto \left(\theta + \frac{2\pi}{N}, \phi - \frac{2\pi}{N}\right).$$

$$\therefore \left(S^2 \otimes S^2\right)/\mathbb{Z}_N \overset{\text{analytic continuation}}{\longleftrightarrow} \left(AdS_2 \otimes S^2\right)/\mathbb{Z}_N$$

This will give us the heat kernel on $\left(AdS_2 \otimes S^2\right)/\mathbb{Z}_N$.

Scalar on $S^2/\mathbb{Z}_N$

Strategy

- Enumerate eigenvalues and degeneracies on the sphere.
- Compute the heat kernel.
- Analytically continue to AdS.

We do this for the scalar on $S^2/\mathbb{Z}_N$ first.

- $ds^2 = a^2\left(d\psi^2 + \sin^2\psi d\phi^2\right)$
- $\mathbb{Z}_N$: $\phi \mapsto \phi + \frac{2\pi}{N}$.

We now compute the heat kernel on this quotient space.

Scalar on $S^2/\mathbb{Z}_N$

The spectrum of the scalar Laplacian on S^2:

- Eigenvalues: $E_\ell = \ell(\ell+1)$
- Eigenfunctions: $Y_{\ell,m}(\psi,\phi) = P_\ell^m e^{im\phi}, \quad -\ell \le m \le \ell.$

The heat kernel is

$$K(t) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} 1 \cdot e^{-\frac{t}{a^2}\ell(\ell+1)}$$

The $\mathbb{Z}_N$ orbifold:

- No change in eigenvalues
- Modes restricted to $m = Np, \quad p \in \mathbb{Z}, \quad -\ell \le m \le \ell,$

The degeneracy changes:

$$\boxed{d_\ell = \sum_{m=-\ell}^{\ell} \delta_{m,Np}.}$$

Scalar on $S^2/\mathbb{Z}_N$

We will use the following representation for δ

$$\delta_{m,Np} = \frac{1}{N}\sum_{s=0}^{N-1} e^{i\frac{2\pi ms}{N}}$$

Then the heat kernel on $S^2/\mathbb{Z}_N$ is

$$K(t) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(\frac{1}{N}\sum_{s=0}^{N-1} e^{i\frac{2\pi s}{N}m}\right)\cdot e^{-\frac{t}{a^2}\ell(\ell+1)}$$

Doing the sum over m

$$K(t) = \frac{1}{N}\sum_{\ell=0}^{\infty}\sum_{s=0}^{N-1}\frac{\sin\frac{(2\ell+1)\pi s}{N}}{\sin\frac{\pi s}{N}} e^{-\frac{t}{a^2}\ell(\ell+1)}$$

Scalar on $S^2/\mathbb{Z}_N$

Degeneracy of E_ℓ on $S^2/\mathbb{Z}_N$:

$$d_\ell = \frac{2\ell+1}{N} + \frac{1}{N}\sum_{s=1}^{N-1}\chi_\ell\left(\frac{\pi s}{N}\right)$$

χ_ℓ is the Weyl character of SU(2).

The heat kernel on $S^2/\mathbb{Z}_N$ is given by

$$K_{S^2/\mathbb{Z}_N}(t) = \frac{1}{N}K_{S^2} + \frac{N^2-1}{6N} + \mathcal{O}(t).$$

The structure of the answer:

- The first term is from the smooth part of $S^2/\mathbb{Z}_N$.
- The second term is from the fixed points.
- Note: No $\frac{1}{t}$ from the second term!

Scalar on $AdS_2/\mathbb{Z}_N$

A natural analytic continuation suggests itself.

- $\frac{1}{N}K_{S^2} \mapsto \frac{1}{N}K_{AdS_2}$
- $a \mapsto ia$ in second term (trivial, but not for $\mathcal{O}(t)$).
- multiply second term by $\frac{1}{2}$ ($\because$ # fixed points is halved).

We then obtain

$$K_{AdS_2/\mathbb{Z}_N}(t) = \frac{1}{N}K_{AdS_2} + \frac{1}{2}\cdot\frac{N^2-1}{6N} + \mathcal{O}(t)\,.$$

Log Terms:

$$K_{S^2/\mathbb{Z}_N}(0;t) = \frac{1}{3N} + \frac{N^2-1}{6N}$$
$$K_{AdS_2/\mathbb{Z}_N}(0;t) = \frac{1}{6N} + \frac{N^2-1}{12N}$$

Group Theory & Analytic Continuation

$$d_\ell = \frac{2\ell+1}{N} + \frac{1}{N}\sum_{s=1}^{N-1}\chi_\ell\left(\frac{\pi s}{N}\right)$$

- Weyl character of SU(2): $S^2 \equiv SU(2)/U(1)$.
- Now $AdS_2 \equiv \mathfrak{sl}(2,R)/U(1)$.

Question: Weyl Character $\mapsto$ Harish-Chandra Character?

It has worked in the past! 0911.5085, 1103.3627

Proposal:

$$\chi_\ell\left(\frac{\pi s}{N}\right) \mapsto \chi_\lambda\left(\frac{\pi s}{N}\right) = \frac{\cosh\left(\pi - \frac{2\pi s}{N}\right)\lambda}{\cosh\pi\lambda\sin\left(\frac{\pi s}{N}\right)}.$$

Then the degeneracy on $AdS_2/\mathbb{Z}_N$ is

$$d_\lambda = -\frac{\lambda\tanh\pi\lambda}{N} + \frac{1}{2N}\sum_{s=1}^{N-1}\chi_\lambda\left(\frac{\pi s}{N}\right)$$

Scalar on $\left(\mathrm{AdS}_2 \otimes \mathrm{S}^2\right)/\mathbb{Z}_N$

We follow the same procedure, so just the final results:

- The scalar is moded by $e^{im\theta}\, e^{in\phi}$.
- The $\mathbb{Z}_N$ orbifold is $(\theta, \phi) \mapsto \left(\theta + \frac{2\pi}{N}, \phi - \frac{2\pi}{N}\right)$.
- $\mathbb{Z}_N$ projects onto (m, n) : $m - n = Np, \quad p \in \mathbb{Z}$.

The heat kernel on $\left(\mathrm{S}^2 \otimes \mathrm{S}^2\right)/\mathbb{Z}_N$ is

$$K_{\mathbb{Z}_N} = \frac{1}{N}K + \frac{1}{N}\sum_{s=1}^{N}\sum_{\ell,\ell'=0}^{\infty}\chi_\ell\left(\frac{\pi s}{N}\right)\chi_{\ell'}\left(\frac{\pi s}{N}\right)e^{-tE_{\ell,\ell'}}$$

As $t \mapsto 0$ evaluate the second term

$$K_{\mathbb{Z}_N} = \frac{1}{N}K + \frac{N^4 + 10N^2 - 11}{180N} + \mathcal{O}(t)$$

Again conical terms are finite as $t \mapsto 0$.

Scalar on $\left(\mathrm{AdS}_2 \otimes \mathrm{S}^2\right)/\mathbb{Z}_N$

The analytic continuation to $\left(\mathrm{AdS}_2 \otimes \mathrm{S}^2\right)/\mathbb{Z}_N$ is carried out by

$$\chi_\ell\left(\frac{\pi s}{N}\right)\chi_{\ell'}\left(\frac{\pi s}{N}\right) \mapsto \chi_\lambda\left(\frac{\pi s}{N}\right)\chi_{\ell'}\left(\frac{\pi s}{N}\right)$$

The heat kernel is then given by

$$K_{\mathbb{Z}_N} = \frac{1}{N}K + \frac{1}{2N}\sum_{s=1}^{N}\sum_{\ell=0}^{\infty}\int_0^\infty d\lambda\,\chi_{\lambda,\ell}\left(\frac{\pi s}{N}\right)e^{-tE_{\lambda\ell}}$$

The degeneracy $d_{\lambda\ell}$ of the eigenvalue $E_{\lambda\ell}$ is

$$\boxed{d_{\lambda\ell} = -\frac{\lambda\tanh\pi\lambda\,(2\ell+1)}{N} + \frac{1}{2N}\sum_{s=1}^{N}\chi_{\lambda,\ell}\left(\frac{\pi s}{N}\right)}$$

The Graviphoton Background

- We have computed the heat kernel of the Laplacian on $\left(\mathrm{AdS}_2 \otimes \mathrm{S}^2\right)/\mathbb{Z}_N$.
- This defines the determinant of the Laplacian and yields the log term.
- However this is not the full story. We need to compute the determinant of the full kinetic operator.
- Fields couple to each other through the background electromagnetic fields.
- This shifts eigenvalues but not degeneracies.
- We now explain this in the context of fields on S^2.
- This will illustrate the last ingredients that go into the computation.

The Graviphoton Background

Consider a subset of the full heat kernel computation.
$\Rightarrow$ S^2 with the magnetic field

$$F_{\psi\phi} = \frac{p}{4\pi} \sin\psi.$$

A transverse vector and a scalar couple to each other.

$$\boxed{\mathcal{L}_{kin} = \begin{pmatrix} \Phi & \mathcal{A}_\alpha \end{pmatrix} \begin{pmatrix} -\Box - \frac{2}{a^2} & \frac{2i}{a}\varepsilon^{\gamma\beta} D_\gamma \\ \frac{2i}{a}\varepsilon^{\alpha\gamma} D_\gamma & -g^{\alpha\beta}\Box + \ldots \end{pmatrix} \begin{pmatrix} \Phi \\ \mathcal{A}_\beta \end{pmatrix}}$$

We have to compute the heat kernel of this operator.
Our approach generalises to the whole calculation.

The Graviphoton Background

$\mathcal{A}_\alpha$ is a transverse field on S^2

$$\mathcal{A}_\alpha = \epsilon_{\alpha\beta}\nabla^\beta \tilde{\Phi}$$

So modes of $\mathcal{A}$ are in 1-1 correspondence with scalar modes.

- Modes labelled with quantum numbers ℓ, m
- Eigenvalues labelled with ℓ
- Degeneracy $d_\ell = 2\ell + 1 \quad \Leftarrow e^{im\phi}$ moding.

Key Simplification: Not all modes mix!

- The only modes of $\mathcal{A}$ and Φ that mix with each other share the same ℓ and the same m.
- We can analyse the mixing just on this subset of modes.

The Graviphoton Background

We will just focus on the general structure.

- Suppose we turn off the flux for a moment. Fix an ℓ.
- We have $(2\ell + 1)$ modes $Y_{\ell m}$ from Φ with eigenvalue E^s_ℓ.
- We have $(2\ell + 1)$ modes $A_{\ell m}$ from $\mathcal{A}$ with eigenvalue E^v_ℓ.
- Fix m to $\tilde{m}$. Now one mode $Y_{\ell\tilde{m}}$ and one mode $A_{\ell\tilde{m}}$.
- Now turn the flux back on. $Y_{\ell\tilde{m}}$ and $A_{\ell\tilde{m}}$ interact.
- The interactions change the eigenvalues to E^a_ℓ and E^b_ℓ.
- But there is still one mode each for E^a_ℓ and E^b_ℓ.

Thus we arrive at the new spectrum:

- Eigenvalue E^a_ℓ, degeneracy $2\ell + 1$.
- Eigenvalue E^b_ℓ, degeneracy $2\ell + 1$.

This happens for all fluctuations we are computing over.

The Graviphoton Background

Now consider all fluctuations in $\left(\mathrm{AdS}_2 \otimes \mathrm{S}^2\right) / \mathbb{Z}_N$.

- Bosonic fields are scalars, vectors and the graviton.
- Vectors and gravitons $\Rightarrow$ derivatives of scalars.
- Quantum numbers (λ, ℓ, m, n) label all modes.
- Suppose we have n fields in $\mathrm{AdS}_2 \otimes \mathrm{S}^2$. Turn off the flux.
- Eigenvalues are $E^{(i)}_{\lambda\ell}$. Degeneracy of each is $d_{\lambda\ell}$.

We have computed $d_{\lambda\ell}$ above.

- Turn the flux back on.
- Eigenvalues are $\tilde{E}^{(i)}_{\lambda\ell}$. Degeneracy of each is $d_{\lambda\ell}$.
- Same heat kernel formula. New eigenvalues.

The same thing happens for fermions as well.

The Graviphoton Background

So all we have to do is to diagonalize one block and compute the new eigenvalues.

- For bosons the largest block is 12×12.
- For fermions the largest block is 40×40.

While very hard, its not impossible. 1106.0080

However, we find more simplifications.

- We just want the t^0 term of the heat kernel.
- The conical terms are finite.

$$\lim_{t\mapsto 0}\int\sum \chi_{\lambda\ell}e^{-tE_{\lambda\ell}} = \int\sum\chi_{\lambda\ell} = \text{'finite'}.$$

- The 'global' contribution is already known. 1106.0080
- So we don't have to diagonalise a 40×40 matrix. We have to multiply a number by 40, and add it to other numbers.

Counting Zero Modes

- The final piece of the puzzle is the zero mode contribution.
- It is determined by the number of zero modes.
- We compute the number of zero modes of the vector field on $\left(\mathrm{AdS}_2 \otimes \mathrm{S}^2\right)/\mathbb{Z}_N$.
- This is also a chance for us to explicitly evaluate the degeneracy without relying on analytic continuations.

The zero modes on are given by

$$\mathcal{A}_\eta = \partial_\eta \Phi,\ \mathcal{A}_\theta = \partial_\theta \Phi,\ \mathcal{A}_\psi = \mathcal{A}_\phi = 0,$$

where

$$\boxed{\Phi = \left(\frac{\sinh\eta}{1+\cosh\eta}\right)^{|m|} e^{im\theta}, \quad |m| = N, 2N, \cdots .}$$

Counting Zero Modes

The number of zero modes is

$$\boxed{n_0 = \sum_m \int_0^{\eta_0} d\eta \, \sinh\eta \, |\mathcal{A}|^2.}$$

The integral can be done to obtain

$$n_0 = 2\sum_{p=1}^{\infty}\left(\tanh\frac{\eta_0}{2}\right)^{2Np} \simeq \frac{1}{2N}e^{\eta_0} - 1 + \mathcal{O}(\eta_0)\,.$$

The number of zero modes is the $\mathcal{O}(1)$ term

$$\boxed{n_0 = -1}$$

- This is exactly how we defined degeneracy.
- It should be: $n_0 =$ the 'degeneracy of the zero eigenvalue'.

$\mathcal{N}=4$ Supergravity

- The bosonic fields contribute

$$K^B_{\mathbb{Z}_N}(0;t) = \frac{1}{N}K^B(0;t) + 2\left(\frac{N^4 - 65N^2 + 135N - 71}{45N}\right)$$

- The fermionic fields contribute

$$K^F_{\mathbb{Z}_N}(0;t) = \frac{1}{N}K^F(0;t) - 2\left(\frac{N^4 - 65N^2 + 180N - 116}{45N}\right)$$

- The total contribution to the log term is then

$$K_{\mathbb{Z}_N}(0;t) = \frac{1}{N}K(0;t) - 2 + \frac{2}{N}$$

Further, $K(0;t) = -2$, so

$$K_{\mathbb{Z}_N}(0;t) = -2.$$

The net zero mode contribution is

$$\tilde{n} = \sum_{\phi} n^0_{\phi}\left(\beta_{\phi} - 1\right) = +2$$

The coefficient of $\ln a$ in $\ln \mathcal{Z}_{str}$, the log term is then

$$\boxed{K_{\mathbb{Z}_N}(0;t) + \tilde{n} = 0.}$$

$\Rightarrow$ perfect match with microscopic counting.

$\mathcal{N} = 8$ Supergravity

$\mathcal{N} = 8 \Rightarrow \mathcal{N} = 4$ fields and additional fields. No extra zero modes

- The contribution of the $\mathcal{N} = 4$ fields already vanishes.
- Consider the contribution of the extra fields.

The final results are:

- The Bosonic fields contribute

$$K^{B}_{\mathbb{Z}_N}(0;t) = \frac{1}{N}K^{B}(0;t) + 8\left(\frac{N^4 - 20N^2 + 19}{45N}\right)$$

- The fermionic fields contribute

$$K^{F}_{\mathbb{Z}_N}(0;t) = \frac{1}{N}K^{F}(0;t) - 8\left(\frac{-26 + 45N - 20N^2 + N^4}{45N}\right)$$

- The total contribution is

$$K_{\mathbb{Z}_N}(0;t) = \frac{1}{N}K(0;t) - 8 + \frac{8}{N}.$$

Further, $K(0;t) = -8$, so

$$\boxed{K_{\mathbb{Z}_N}(0;t) = -8.}$$

$\Rightarrow$ perfect match with microscopic counting.

$\mathcal{N} = 2$ Supergravity

- We can also calculate for black holes in $\mathcal{N} = 2$ Supergravity.
- Here the microscopic answer is not known. $\Rightarrow$ prediction?
- Suppose we have n_H hypermultiplets and n_V vector multiplets.

Then

$$\ln \mathcal{Z}_{str} = \frac{A_H}{4N} + \left(2 - N\frac{\chi}{24}\right) \ln A_H .$$

Here $\chi = 2\,(n_V - n_H + 1)$ is the Euler character of the CY that the string theory is compactified on.

- This is puzzling. If $N \simeq \sqrt{A_H}$ then the 1-loop correction is bigger than the classical answer. What does this mean?
- In general the N dependence is interesting. It does not appear for $\mathcal{N} = 4$ and $\mathcal{N} = 8$. Can we reproduce this growth from the microscopic side?

Conclusions

- The QEF computes all possible corrections to Bekenstein-Hawking entropy of extremal black holes.
- We can test this against the string answer for $\mathcal{N} = 4$ and $\mathcal{N} = 8$ black holes.
- We find a perfect match with asymptotic expansion for the string theory answer.
- To compute this expression we developed new techniques for evaluating the heat kernel on AdS spaces.
- In particular, we generalised the Plancherel Formula to quotients of AdS spaces.
- We also obtained the corresponding answer for $\mathcal{N} = 2$ black holes.
- The answer has curious properties. It would be interesting to better understand them.

Thank You