ARITHMETIC SUMS WITH MULTIPLICATIVE COEFFICIENTS

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(Joint work with Professor Chaohua Jia)

0. Introduction

- Let $f(n)$ be a multiplicative function satisfying $|f(n)| \leq 1$.
- $q \in \mathbb{Z}^+$, $a \in \mathbb{Z}$, $\gcd(a,q) = 1$.
- $e(x) = e^{2i\pi x}$, $x \in \mathbb{R}$.
- $t(\cdot)$ be the divisor function.
- $\overline{n}$ denotes the multiplicative inverse of n: $n\overline{n} \equiv 1 \pmod q$.

Goal: Estimate nontrivially the weighted sums

$$\sum_{\substack{n \leq N \\ (a,q)=1}} f(n) e\left(\frac{a\overline{n}}{q}\right),$$

$$\sum_{n \leq N} f(x)\chi_q(n+a), \quad \chi \neq \chi_0 \pmod q, \quad \underline{q \text{ prime}},$$

or more generally, for $t \geq 2$, $a_1, \ldots, a_t$ being pairwise distinct integers modulo a prime q, $\chi \neq \chi_0 \pmod q$,

$$\sum_{n \leq N} f(x)\chi_q((n+a_1)\cdots(n+a_t)).$$

Main tool: a modification of Kátai / Bourgain–Sarnak–Ziegler fintite version of Vinogradov's inequality (sieve).

1. Motivation & Main Results

- f multiplicative function with $|f| \leq 1$, $\alpha \in \mathbb{R}$.

$$S(\alpha) = \sum_{n \leq N} f(n)e(\alpha n).$$

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Daboussi (1974): If

$$|\alpha - \frac{a}{q}| \leq \frac{1}{q^2}, \quad (a,q)=1, \quad 3 \leq q \leq \left(\frac{N}{\log N}\right)^{1/2},$$

then

$$S(\alpha) \ll \frac{N}{\sqrt{\log\log q}}$$

uniformly for f.

Montgomery & Vaughan (1977): suppose $q \leq N$, $(a,q)=1$, then

$$S\left(\frac{a}{q}\right) \ll N\left(\frac{1}{\log 2N} + \frac{1}{\sqrt{\varphi(q)}} + \sqrt{\frac{q}{N}}\left(\log\frac{2N}{q}\right)^{\frac{3}{2}}\right)$$

uniformly for f. Assume GRH, they obtained: $\forall\, \chi \neq \chi_0 \pmod q$, $\forall\, N$,

$$\sum_{n \leq N} \chi(n) \ll \sqrt{q}\log\log q.$$

• $f = \mu$ the Möbius function

Hajela, Pollington & Smith (1998):

$$M(a) := \sum_{\substack{n \leq N \\ (a,q)=1}} \mu(n) e\left(\frac{a\overline{n}}{q}\right) \ll_\varepsilon Nq^\varepsilon \left(\frac{(\log N)^{\frac{5}{2}}}{q^{\frac{1}{2}}} + \frac{q^{\frac{3}{10}}(\log N)^{\frac{11}{5}}}{N^{\frac{1}{5}}}\right),$$

which gives a nontrivial estimate in the range $(\log N)^{5+10\varepsilon} \ll q \ll N^{\frac{2}{3}-3\varepsilon}$.

Wang & Zheng (1998), Deng (1999) independently:

$$M(a) \ll N\tau(q)\left(\frac{(\log N)^{\frac{5}{2}}}{q^{\frac{1}{2}}} + \frac{q^{\frac{1}{5}}(\log N)^{\frac{13}{5}}}{N^{\frac{1}{5}}}\right)$$

for $(\log N)^{5+\varepsilon} \ll q \ll N^{1-\varepsilon}$. They remarked that, if assume GRH, we have

$$M(a) \ll_\varepsilon q^{\frac{1}{2}} N^{\frac{1}{2}+\varepsilon}.$$

• For the case of von Mangoldt function, i.e. weighted sums over prime variables, there are many works, see Fouvry & Michel (1998), Garaev (2010), Fouvry & Shparlinski (2011), Baker (2012), ...

Theorem 1. *Let $f(n)$ be a multiplicative function satisfying $|f(n)| \le 1$, q $(\le N^2)$ be a positive integer and a be an integer with $(a, q) = 1$. Then*

$$\sum_{\substack{n\le N\\(n,q)=1}} f(n)e\left(\frac{a\overline{n}}{q}\right) \ll \sqrt{\frac{\tau(q)}{q}}N\log\log(6N) + q^{\frac{1}{4}+\frac{\varepsilon}{2}}N^{\frac{1}{2}}(\log(6N))^{\frac{1}{2}} + \frac{N}{\sqrt{\log\log(6N)}},$$

where $\overline{n}$ is the multiplicative inverse of n such that $\bar{n}n \equiv 1 \pmod q$.

Remarks. 1) The estimate is nontrivial for $(\log\log(6N))^{2+\varepsilon} \ll q \ll N^{2-5\varepsilon}$.

2) If we use the result of Bourgain & Garaev (2014), when q is prime, the upper bound in the range can be extended to $q \ll N^A$ for A being any given large constant.

Notation. We assume N is sufficiently large. Denote

$$d_0 = \sqrt{\log\log(6N)}, \qquad D_0 = e^{d_0} = \exp(\sqrt{\log\log(6N)}),$$
$$d_1 = d_0^2 = \log\log(6N), \qquad D_1 = e^{d_1} = \log(6N).$$

Let p denote a prime number, and ε be a sufficiently small positive constant.

2. Sketch of the proof

Write

$$S = \{n : 1 \le n \le N,\ n \text{ has a prime factor in } [D_0, D_1)\}$$
$$T = \{n : 1 \le n \le N,\ n \text{ has no prime factor in } [D_0, D_1)\}.$$

Lemma 1. We have

$$|T| \ll \frac{N}{\sqrt{\log\log(6N)}}.$$

Proof. Let

$$P(N) = \prod_{D_0 \le p < D_1} p.$$

We have

$$\begin{aligned}
|T| &= \sum_{\substack{n\le N\\(n,\,P(N))=1}} 1 = \sum_{n\le N}\sum_{d|(n,\,P(N))} \mu(d)\\
&= \sum_{d|P(N)} \mu(d) \sum_{\substack{n\le N\\ d|n}} 1 = \sum_{d|P(N)} \mu(d)\Big(\frac{N}{d} + O(1)\Big)\\
&= N\sum_{d|P(N)} \frac{\mu(d)}{d} + O\Big(2^{\pi(D_1)}\Big) = N\prod_{D_0\le p<D_1}\Big(1-\frac{1}{p}\Big) + O\Big(2^{\frac{2D_1}{\log D_1}}\Big)\\
&\ll N\frac{\log D_0}{\log D_1} + O\Big(2^{2\frac{\log(6N)}{\log\log(6N)}}\Big) \ll \frac{N}{\sqrt{\log\log(6N)}}. \qquad \square
\end{aligned}$$

By Lemma 1, we have

$$\sum_{\substack{n\le N\\(n,q)=1}} f(n)e\left(\frac{a\overline{n}}{q}\right) = \sum_{\substack{n\le N\\ n\in S\\(n,q)=1}} f(n)e\left(\frac{a\overline{n}}{q}\right) + O\left(\frac{N}{\sqrt{\log\log(6N)}}\right).$$

Let

$$P_r = \{p:\, e^r \le p < e^{r+1}\}, \qquad \text{if } [d_0] \le r \le [d_1].$$

Then

$$\bigcup_{r=[d_0]+1}^{[d_1]-1} P_r \subseteq \{p:\, D_0 \le p < D_1\} \subseteq \bigcup_{r=[d_0]}^{[d_1]} P_r.$$

The prime number theorem yields

$$|P_r| \ll \frac{e^r}{r}.$$

Write

$$S' = \{n:\, 1\le n\le N,\, n \text{ has a prime factor in } \bigcup_{r=[d_0]}^{[d_1]} P_r\},$$

$$S'' = \{n:\, 1\le n\le N,\, n \text{ has a prime factor in } \bigcup_{r=[d_0]+1}^{[d_1]-1} P_r\}.$$

Then

$$S'' \subseteq S \subseteq S'.$$

Hence

$$\begin{aligned} |S\setminus S''| \le |S'\setminus S''| &\ll \sum_{p\in P_{[d_0]}} \frac{N}{p} + \sum_{p\in P_{[d_1]}} \frac{N}{p} \ll N\left(\frac{|P_{[d_0]}|}{e^{[d_0]}} + \frac{|P_{[d_1]}|}{e^{[d_1]}}\right) \\ &\ll \frac{N}{d_0} = \frac{N}{\sqrt{\log\log(6N)}}. \end{aligned}$$

We note that

$$\begin{aligned} &|\{n:\, 1\le n\le N,\, n \text{ has at least two prime factors in the} \\ &\qquad \text{same one of } P_r\text{'s } ([d_0]+1\le r\le [d_1]-1)\}| \\ &\ll \sum_{r=[d_0]+1}^{[d_1]-1} \sum_{p\in P_r}\sum_{p'\in P_r} \frac{N}{pp'} \ll N \sum_{r=[d_0]+1}^{[d_1]-1} \left(\frac{|P_r|}{e^r}\right)^2 \\ &\ll N \sum_{r=[d_0]+1}^{[d_1]-1} \frac{1}{r^2} \ll \frac{N}{d_0} = \frac{N}{\sqrt{\log\log(6N)}}. \end{aligned}$$

Therefore for

$$S''' = \{n:\ 1 \le n \le N,\ n \text{ has exact one prime factor} \\ \text{in one of } P_r\text{'s } ([d_0]+1 \le r \le [d_1]-1)\},$$

we have $S''' \subseteq S''$ and $|S'' \setminus S'''| \ll \frac{N}{\sqrt{\log\log(6N)}}$.

The set S''' can be decomposed as

$$S''' = \bigcup_{r=[d_0]+1}^{[d_1]-1} S_r,$$

where

$$S_r = \{n:\ 1 \le n \le N,\ n \text{ has exact one prime factor in } P_r \\ \text{and has no prime factor in } \bigcup_{i<r} P_i\}.$$

By the prime number theorem, it is easy to see that each S_r $(r = [d_0]+1, \cdots, [d_1]-1)$ is not empty. The sets S_r are disjoint from each other. Every element $n \in S_r$ can be written in exact one way as $n = py$, where $p \in P_r$, y has no prime factor in $\bigcup_{i\le r} P_i$, $py \le N$.

From the above discussion, we get

$$\begin{aligned}
&\sum_{\substack{n\le N\\(n,\,q)=1}} f(n)e\left(\frac{a\overline{n}}{q}\right)\\
&= \sum_{\substack{n\le N\\ n\in S'''\\(n,\,q)=1}} f(n)e\left(\frac{a\overline{n}}{q}\right) + O\Big(\frac{N}{\sqrt{\log\log(6N)}}\Big)\\
&= \sum_{r=[d_0]+1}^{[d_1]-1}\ \sum_{\substack{n\le N\\ n\in S_r\\(n,\,q)=1}} f(n)e\left(\frac{a\overline{n}}{q}\right) + O\Big(\frac{N}{\sqrt{\log\log(6N)}}\Big)\\
&= \sum_{r=[d_0]+1}^{[d_1]-1}\ \sum_{\substack{e^r\le p<e^{r+1}\\(p,\,q)=1}}\ \sum_{\substack{y\le \frac{N}{p}\\ y \text{ has no prime factor in } \bigcup_{i\le r} P_i\\(y,\,q)=1}} f(py)e\left(\frac{a\overline{p}\,\overline{y}}{q}\right) + O\Big(\frac{N}{\sqrt{\log\log(6N)}}\Big)\\
&= \sum_{r=[d_0]+1}^{[d_1]-1}\ \sum_{\substack{y\le \frac{N}{p}\\ y \text{ has no prime factor in } \bigcup_{i\le r} P_i\\(y,\,q)=1}} f(y) \sum_{\substack{e^r\le p<e^{r+1}\\ p\le \frac{N}{y}\\(p,\,q)=1}} f(p)e\left(\frac{a\overline{p}\,\overline{y}}{q}\right) + O\Big(\frac{N}{\sqrt{\log\log(6N)}}\Big)\\
&\ll \sum_{r=[d_0]+1}^{[d_1]-1}\ \sum_{\substack{y\le \frac{N}{e^r}\\(y,\,q)=1}} \Big|\sum_{\substack{e^r\le p<e^{r+1}\\ p\le \frac{N}{y}\\(p,\,q)=1}} f(p)e\left(\frac{a\overline{p}\,\overline{y}}{q}\right)\Big| + O\Big(\frac{N}{\sqrt{\log\log(6N)}}\Big).
\end{aligned}$$

Let

$$Y = \frac{N}{e^r}.$$

We shall estimate the sum

$$\Sigma_1 = \sum_{\substack{y \le Y \\ (y,\, q)=1}} \Big| \sum_{\substack{e^r \le p < e^{r+1} \\ p \le \frac{N}{y} \\ (p,\, q)=1}} f(p) e\left(\frac{a\overline{p}\,\overline{y}}{q}\right) \Big|.$$

Lemma 2. *For the positive integer q and the integer b, we have*

$$\sum_{\substack{X<n\le Z \\ (n,q)=1}} e\left(\frac{b\overline{n}}{q}\right) \ll \left(\frac{Z-X}{q}+1\right)(b,q) + q^{\frac{1}{2}+\varepsilon}. \tag{16}$$

Proof. Lemma 2.1 in Fouvry–Shparlinski states that

$$\sum_{\substack{X<n\le Z \\ (n,q)=1}} e\left(\frac{b\overline{n}}{q}\right) \ll \mu^2\left(\frac{q}{(b,q)}\right)\left(\frac{Z-X}{q}+1\right) \cdot \frac{\varphi(q)}{\varphi\left(\frac{q}{(b,q)}\right)} + \tau(q)\tau((b,q))\log(2q) q^{\frac{1}{2}}.$$

Then the bounds

$$\frac{\varphi(q)}{\varphi\left(\frac{q}{(b,q)}\right)} = q\prod_{p|q}\left(1-\frac{1}{p}\right)\cdot\left(\frac{q}{(b,q)}\prod_{p|\frac{q}{(b,q)}}\left(1-\frac{1}{p}\right)\right)^{-1} = (b,q)\prod_{\substack{p|q \\ p\nmid \frac{q}{(b,q)}}}\left(1-\frac{1}{p}\right) \le (b,q)$$

and $\tau(q) \ll q^{\frac{\varepsilon}{4}}$ produce the conclusion in Lemma 2. □

Using Cauchy inequality, we can estimate Σ_1 by Lemma 2. For more details, see K. Gong, C. Jia, *Kloosterman sums with multiplicative coefficients.* arXiv:1401.4556.

3. Shifted character sums

Let χ be a non-principal Dirichlet character modulo a prime q, and a be an integer with $(a,\, q) = 1$.

I. M. Vinogradov (1930s–1950s): character sums over shifted primes

$$\sum_{n\le N} \Lambda(n)\chi(n+a),$$

with he best known is a nontrivial estimate for the range $N^\varepsilon \le q \le N^{\frac{4}{3}-\varepsilon}$, which lies deeper than the direct consequence of Generalized Riemann Hypothesis.

A. A. Karatsuba (1970): widen the range to $N^\varepsilon \le q \le N^{2-\varepsilon}$.

Theorem 2. *Let $f(n)$ be a multiplicative function satisfying $|f(n)| \le 1$, q $(\le N^2)$ be a prime number, χ be a Dirichlet character modulo q, $(a,q)=1$. Then*

$$\sum_{\substack{n\le N\\ (n,q)=1}} f(n)\chi(n+a) \ll \frac{N}{q^{\frac14}}\log\log(6N) + N^{\frac12}q^{\frac14}\log(6N) + \frac{N}{\sqrt{\log\log(6N)}}.$$

The estimate is nontrivial for

$$(\log\log(6N))^{4+\varepsilon} \ll q \ll \frac{N^2}{(\log(6N))^{4+\varepsilon}},$$

which should be compared with the conjectural range as indicated by A. A. Karatsuba (1970).

Theorem 3. *We assume f, q, χ the same as Theorem 2. For $t \ge 2$ and pairwise distinct integers $a_1, a_2, \dots, a_t$ modulo q, we have*

$$\sum_{\substack{n\le N\\ (n,q)=1}} f(n)\chi((n+a_1)\cdots(n+a_t)) \ll \frac{N}{q^{\frac14}}\log\log(6N) + N^{\frac12}q^{\frac14}\log(6N) + \frac{N}{\sqrt{\log\log(6N)}}.$$

Remarks. 1) Taking $f=\mu$ to be the Möbius function in the Theorem 3, we obtain an example for the *Möbius Randomness Law*[1]. Such an example answers, in a special case, a *problematic* issue posed by Sarnak (2010).

2) The $t=2$ case of Theorem 3 corresponds to Karatsuba (1978). While for any $t\ge 3$, our result should be compared with a conditional result of Karatsuba (DAN, 1978), which relies on a conjectural upper bound for a kind of character sums in two variables. □

Indeed, Karatsuba's Conjecture states that, if $f(x)\in\mathbb{Z}[x]$ is not a complete square modulo q, the integers $a_1,\dots,a_t$ are pairwise non-congruent modulo q, $t\ge 2$, and $F(x,y) = f(x+a_1y)\cdots f(x+a_ty)$, then the estimate

$$\left|\sum_{x=1}^{q}\sum_{y=1}^{q}\left(\frac{F(x,y)}{q}\right)\right| \ll q$$

holds, where the constant implied in $\ll$ depends only on t and $\deg f$.

In the same way as that of §2, we can bound the original sum by

$$\sum_{\substack{n\le N\\ (n,q)=1}} f(n)\chi(n+a) \ll \sum_{r=[d_0]+1}^{[d_1]-1}\sum_{\substack{y\le \frac{N}{e^r}\\ (y,q)=1}}\left|\sum_{\substack{e^r\le p<e^{r+1}\\ p\le \frac{N}{y}\\ (p,q)=1}} f(p)\chi(py+a)\right| + \frac{N}{\sqrt{\log\log(6N)}} + \frac{N}{q}.$$

[1]Möbius Randomness Law. The Möbius function μ changes sign randomly so that for any "reasonable" sequence of complex numbers $\mathcal{A}=(a_m)$ the twisted sum

$$M(\mathcal{A},x) = \sum_{m\le x}\mu(m)a_m$$

is relatively small due to the cancellation of its terms. — Iwaniec & Kowalski

Lemma 3. *Let q be a prime number, $\chi_1, \dots, \chi_r$ be Dirichlet characters modulo q, at least one of which is non-principal. Let $f(X) \in \mathbb{F}_q[X]$ be an arbitrary polynomial of degree d. Then for pairwise distinct $a_1, \dots, a_r \in \mathbb{F}_q$, we have*

$$\left| \sum_{x \in \mathbb{F}_q} \chi_1(x+a_1) \cdots \chi_r(x+a_r) e\left(\frac{f(x)}{q}\right) \right| \le (r+d) q^{\frac{1}{2}}.$$

Lemma 4. *Let q be a prime number, χ be a non-principal Dirichlet character modulo q. Then for an arbitrary integer h with $1 \le h \le q$ and distinct $s, t \in \mathbb{F}_q$,*

$$\sum_{x=1}^{h} \chi\left(\frac{x+s}{x+t}\right) = O(q^{\frac{1}{2}} \log q)$$

holds true.

Lemma 5. *Let q be a prime number, χ be a non-principal Dirichlet character modulo q, $(a, q) = 1$. Then for two primes p_1, p_2 with $(p_1, q) = (p_2, q) = 1$, $p_1 \not\equiv p_2 \pmod q$, we have*

$$\sum_{X < y \le Z} \chi\left(\frac{p_1 y + a}{p_2 y + a}\right) \ll \frac{Z - X}{q} \sqrt{q} + \sqrt{q} \log q.$$

holds true.

For more details in this part, see the preprint
K. Gong, C. Jia, *Shifted character sums with multiplicative coefficients.* arXiv:1404.2204.