Cluster algebras and subtraction-free computations (joint work with S. Fomin, G. Koshevoy)

Dima Grigoriev (Lille)

CNRS

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Complexity of rational functions with respect to a subset of arithmetic operations

Let $M \subset \{+, -, \times, /\}$. Complexity $C_M(f)$ for a rational function $f \in \mathbb{Q}(X_1, \ldots, X_n)$ is defined as the minimal number of operations from M necessary to compute f, provided it is finite.

Question. For given $M \subset M_1 \subset \{+, -, \times, /\}$ how big can be $C_M(f)$ in comparison with $C_{M_1}(f)$?

This question is non-trivial just for three pairs of $M \subset M_1$.

Theorem

$C_{+,-,\times}(f) \leq O(C_{+,-,\times,/}(f) \cdot \deg(f))$
*for a polynomial f (**V. Strassen, 1973**)*

Corollary

$C_{+,-,\times}(\text{det}) \leq O(n^4)$.

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Subtraction-free computations

We show that

- $C_{+,\times}(f)$ can be exponential bigger than $C_{+,\times,/}(f)$;
- $C_{+,\times,/}(f)$ can be exponentially bigger than $C_{+,-,\times,/}(f)$.

Subtraction-free computations have a meaning in numerical analysis: if a computation deals with positive numbers c given with a relative error, i. e. $(1-\epsilon)\cdot b < c < (1+\epsilon)\cdot b$ where ϵ is a relative error then one can estimate easily relative error after operations $+, \times, /$.

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Cluster algebras

Consider $n \times n$ matrix $X = (X_{ij})$ with variable entries. For a set $I \subset \{1, \dots, n\}$ of size $|I| = k$ denote by Δ_I the determinant of $k \times k$ submatrix of X formed by first k rows and by columns $i \in I$. Then the **flag algebra** $F[\{\Delta_I\}_I]$ is the ring of regular functions on the flag variety.

Clusters are special families of flag minors Δ_I (being bases of the flag algebra). To describe clusters define relation $I \prec J$ for $I, J \subset \{1, \dots, n\}$ if for any pair $i \in I \setminus J$, $j \in J \setminus I$ we have $i < j$. We say that I, J are **strongly separated** if either $I \prec J$ or $J \prec I$. Cluster is a maximal (wrt inclusion) family of pairwise strongly separated flag minors (excluding $I = \{1, \dots, n\}$). Each cluster contains $n(n+1)/2 - 1$ flag minors (**Fomin-Zelevinsky**).

Shorthand: for $j \notin I$ denote $Ij =: I \cup \{j\}$.

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Cluster graph

Plücker relations: $\Delta_{Iik} \cdot \Delta_{Ij} = \Delta_{Iij} \cdot \Delta_{Ik} + \Delta_{Ijk} \cdot \Delta_{Ii}$, $i, j, k \notin I$; $i < j < k$

Clusters form a directed acyclic **cluster graph**: there is an edge from one cluster to another if they are, respectively, of the types

$\{\Delta_{Iik}, \Delta_{Iij}, \Delta_{Ik}, \Delta_{Ijk}, \Delta_{Ii}, \dots\}$;

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Cluster graph is graded according to the sum of sizes $|I|$. A birational (with integer positive coefficients) transformation from one cluster to another (along an edge or back) is called a **flip**. Cluster graph has the unique minimal element corresponding to I being all the proper intervals of $\{1, \dots, n\}$ and the unique maximal element corresponding to all the complements of the proper intervals.

Fomin-Zelevinsky: any non-minimal cluster contains a 5-tuple $\Delta_{Iik}, \Delta_{Iij}, \Delta_{Ik}, \Delta_{Ijk}, \Delta_{Ii}$, therefore, a flip can be applied to the cluster. Any flag minor belongs to some cluster. Thus, any flag minor can be computed with $O(n^3)$ flips starting with the interval flag minors, and a flag minor is a Laurent polynomial in the interval flag minors with integer positive coefficients.

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Cluster graph is graded according to the sum of sizes $|I|$. A birational (with integer positive coefficients) transformation from one cluster to another (along an edge or back) is called a **flip**. Cluster graph has the unique minimal element corresponding to I being all the proper intervals of $\{1, \dots, n\}$ and the unique maximal element corresponding to all the complements of the proper intervals.

Fomin-Zelevinsky: any non-minimal cluster contains a 5-tuple $\Delta_{Iik}, \Delta_{Iij}, \Delta_{Ik}, \Delta_{Ijk}, \Delta_{Ii}$, therefore, a flip can be applied to the cluster. Any flag minor belongs to some cluster. Thus, any flag minor can be computed with $O(n^3)$ flips starting with the interval flag minors, and a flag minor is a Laurent polynomial in the interval flag minors with integer positive coefficients.

Cluster graph

Plücker relations: $\Delta_{Iik} \cdot \Delta_{Ij} = \Delta_{Iij} \cdot \Delta_{Ik} + \Delta_{Ijk} \cdot \Delta_{Ii},\ i, j, k \notin I;\ i < j < k$

Clusters form a directed acyclic **cluster graph**: there is an edge from one cluster to another if they are, respectively, of the types
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Totally positive matrices

As a consequence from cluster transformations we conclude that for a real matrix if all the interval flag minors are positive then all flag minors are positive as well.

A real matrix is called *totally positive* if all its minors are positive. Cluster transformation entail that for total positivity it suffices to verify positivity of minors formed by sets of rows I and of columns J for all pairs of intervals $I, J \subset \{1, \ldots, n\}$.

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Cubic subtraction-free complexity of Schur polynomials

Substitute for the entries of the matrix $X_{ij} = x_i^j$. Then $\Delta := \Delta_{\{1,\dots,k\}} = \prod_{1 \le i < j \le k}(x_j - x_i)$ is Vandermond determinant. The quotient $S_I = \Delta_I / \Delta$ is a **Schur polynomial** having integer positive coefficients (called **Kostka numbers** being $\#P$-hard to compute).

Since for an interval $I = [i, i+1, \dots, j]$ the interval Schur polynomial is the monomial $S_I = (x_1 \cdots x_{j-i+1})^i$ being easy to compute, we get

Corollary

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This does not yet imply an exponential gap between $C_{+,\times,/}$ and $C_{+,\times}$ because we don't know a lower bound on the complexity $C_{+,\times}(S_I)$. To establish this gap we proceed to another class of cluster transformations.

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Star-mesh transformations

Let edges of a complete graph G with n vertices (viewed as an electrical circuit) be endowed with **conductances** $x_{i,j}$. Denote by $cond_{i,j}(G)$ the conductance of the circuit between vertices i, j.

For a spanning tree T of G denote by X^T the monomial being the product of $x_{i,j}$ for all edges i, j of T. The generating polynomial $f(G)$ is the sum of the monomials X^T over all spanning trees T.

Example: for G with 3 vertices $f(G) = x_{1,2} \cdot x_{2,3} + x_{1,2} \cdot x_{1,3} + x_{2,3} \cdot x_{1,3}$.

For vertices v, w of G denote by $G_{v,w}$ the graph obtained from G by gluing v, w into a vertex u with new conductances $x_{i,u} := x_{i,v} + x_{i,w}$.

Kirchhoff (1847): $cond_{i,j}(G) = f(G)/f(G_{i,j})$.

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Example: $x^2 - x + 1 = (x^3 + 1)/(x + 1)$.

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For any representation $g_n := (1 - x_1)^2 + (x_1 - 2x_2)^2 + (x_2^2 - x_3)^2 + (x_3^2 - x_4)^2 + \cdots + (x_{n-1}^2 - x_n)^2 + 4x_n^2 \cdot x_1 = P/Q$ with P, Q being polynomials with positive coefficients we have $\deg(Q) > 2^{2^n}$.

Corollary

$C_{+,\times,/}(g_n) > 2^n$ (evidently, $C_{+,-,\times,/}(g_n) < O(n)$).

Subtraction can be exponentially powerful

Polya (1923): any polynomial $h \in \mathbb{R}[x_1, \dots, x_n]$ positive everywhere on the positive orthant $\mathbb{R}^n_{>0}$ can be represented as a fraction $P/(1 + x_1 + \cdots + x_n)^N$ for suitable integer N and a polynomial P with positive coefficients.

Example: $x^2 - x + 1 = (x^3 + 1)/(x + 1)$.

Corollary

For a polynomial $h \in \mathbb{R}[x_1, \dots, x_n]$ its complexity $C_{+,\times,/}(h)$ is finite iff h is positive everywhere on $\mathbb{R}^n_{>0}$.

Lemma

For any representation $g_n := (1 - x_1)^2 + (x_1 - 2x_2)^2 + (x_2^2 - x_3)^2 + (x_3^2 - x_4)^2 + \cdots + (x_{n-1}^2 - x_n)^2 + 4x_n^2 \cdot x_1 = P/Q$ with P, Q being polynomials with positive coefficients we have $\deg(Q) > 2^{2^n}$.

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$C_{+,\times,/}(g_n) > 2^n$ *(evidently, $C_{+,-,\times,/}(g_n) < O(n)$).*