Universal stratifications (joint work with P. Milman)

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06/05/2013, St.Petersbourg

Stratifications

Stratification of an (algebraic) variety $V = \sqcup_i S_i \subset \mathbb{C}^n$ is a decomposition where *strata* $S_i \cap S_j = \emptyset$. Usually, some extra properties are imposed on strata S_i, e. g. each S_i to be smooth, irreducible, open in its closure, in addition to satisfy Whitney or Thom conditions. Thus, we fix a class of stratifications satisfying certain conditions.

Relation of coarseness of stratifications

We say that stratification $V = \sqcup_i S_i$ is **coarser** than stratification $V = \sqcup_j R_j$ if for every i there exists j such that $S_i \cap R_j$ is dense in both S_i and in R_j. Informally, stratification $\{S_i\}_i$ is coarser than $\{R_j\}_j$ when each stratum $S_i = \sqcup_l R_{j_l}$ for a suitable subfamily $\{R_{j_l}\}_l$. Among strata $\{R_{j_l}\}_l$ there is a unique stratum dense in S_i.

Stratification $\{S_i\}_i$ is **universal** in a fixed class of stratifications if it is the coarsest in this class.

If a universal stratification $V = \sqcup_i S_i$ of V does exist then it is natural to define an intrinsic complexity of V in terms of $\{S_i\}_i$, say as $\sum_i \deg(S_i)$.

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Whitney condition on stratifications

Stratification $\{S_i\}_i$ fulfils **Whitney-(a) condition** if for any sequence of points $\{x_k\}_k \subset S_i$ with existing limits $\lim_k x_k = x \in S_j$ and of tangent spaces $T = \lim_k T_{x_k}(S_i)$ we have $T \supset T_x(S_j)$.

Theorem (**Whitney** 1965). Any variety admits a stratification satisfying Whitney-(a) condition.

Thom condition on stratifications

Let $f : \mathbb{C}^n \to \mathbb{C}$ be a polynomial $f \in \mathbb{Z}[X_1, \dots, X_n]$ with 0 being a *critical value*. A stratification of the set of critical points $\mathrm{Crit}(f) := \{x \in \mathbb{C}^n \, : \, f(x) = \frac{\partial f}{\partial X_1}(x) = \cdots = \frac{\partial f}{\partial X_n}(x) = 0\} = \sqcup_i S_i$ fulfils **Thom condition** if for any sequence $\{x_k\}_k \subset \mathbb{C}^n$ with the limits $\lim_k x_k = x \in S_i$ and $\lim_k \mathrm{grad}_{x_k}(f) = w$, we have $w \perp T_x(S_i)$.

Theorem (**Hironaka** 1976). For any polynomial f there exists a stratification with Thom condition of $\mathrm{Crit}(f)$.

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Gauss map

Gauss map on a constructible set $S \subset \mathbb{C}^n$ sends each non-singular point $x \in S$ to the tangent space $T_x(S)$. If Gauss map can be extended continuously on the whole S then S is called **Gauss regular**. If a continuous extension does exist it is unique.

If S is smooth it is Gauss regular. The inverse is not true: the plane curve (cusp) $x^2 = y^3$ is Gauss regular, but not smooth at point $(0, 0)$.

Thus, we consider a class of stratifications of $\mathrm{Crit}(f) = \sqcup_i S_i$ with S_i being Gauss regular and satisfying Whitney-(a) and Thom conditions.

Question: when $\mathrm{Crit}(f)$ admits a universal stratification?

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Glaeser closure of a bundle of vector spaces

Let $Q \subset V \times W$ where W be a vector space. Consider a bundle of vector spaces $Q^{(1)} \subset V \times W$ whose each fiber $Q_v^{(1)}$ is the linear hull of the fiber $\overline{Q}_v = \{w : (v, w) \in \overline{Q}\}$ of the closure $\overline{Q}$ (in Zariski topology). Perhaps, $Q^{(1)}$ is not close. Applying this construction to $Q^{(1)}$ we get a bundle of linear spaces $Q^{(2)} \subset V \times W$.

Continuing we get an increasing chain of bundles $Q \subset Q^{(1)} \subset Q^{(2)} \subset \cdots \subset V \times W$. This chain stabilizes after at latest of $r = 2 \cdot \dim W$ iterations, $Q^{(r)} = Q^{(r+1)} = \cdots$. The resulting *closed* bundle of vector spaces $Gl(Q) = Q^{(r)}$ is called the **Glaeser closure** of Q.

Apply this construction to the set $Q = \{(x, \lambda \cdot \mathrm{grad}_x(f))\}_{\lambda \in \mathbb{C}} \subset \mathbb{C}^{2n}$. As a result we obtain a closed bundle of vector spaces $G_f = Gl(Q)|_{\mathrm{Crit}(f) \times \mathbb{C}^n} \subset \mathrm{Crit}(f) \times \mathbb{C}^n$.

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Apply this construction to the set $Q = \{(x, \lambda \cdot \mathrm{grad}_x(f))\}_{\lambda \in \mathbb{C}} \subset \mathbb{C}^{2n}$. As a result we obtain a closed bundle of vector spaces $G_f = Gl(Q)|_{\mathrm{Crit}(f) \times \mathbb{C}^n} \subset \mathrm{Crit}(f) \times \mathbb{C}^n$.

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Dual bundle of vector spaces for a stratification

For a stratification $\mathrm{Crit}(f) = \sqcup_i S_i$ with Gauss regular strata S_i define a **dual bundle** of vector spaces $B(\{S_i\}_i) \subset \mathrm{Crit}(f) \times \mathbb{C}^n$ such that for any point $x \in S_i$ the fiber $(B(\{S_i\}_i))_x = \{w \in \mathbb{C}^n : (x, w) \in B(\{S_i\}_i)\}$ of $B(\{S_i\}_i)$ at $x \in S_j$ equals the orthogonal complement $(T_x(S_j))^{\perp}$.

Lemma

- *Stratification $\{S_i\}_i$ satisfies Thom condition iff the dual bundle $B(\{S_i\}_i) \supset G_f$.*
- *Stratification $\{S_i\}_i$ satisfies Whitney-(a) condition iff $B(\{S_i\}_i)$ is closed.*

Lemma

Stratification $\{S_i\}_i$ is coarser than stratification $\{R_j\}_j$ iff $B(\{S_i\}_i) \subset B(\{R_j\}_j)$.

Thus, the question on existence of a universal stratification is equivalent in the language of dual bundles of vector spaces to whether among the dual bundles containing bundle G_f, there is the minimal one.

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For a stratification $\mathrm{Crit}(f) = \sqcup_i S_i$ a dual bundle $B(\{S_i\}_i) \subset \mathrm{Crit}(f) \times \mathbb{C}^n$ was constructed above. Conversely, with a closed bundle of vector spaces $B \subset V \times \mathbb{C}^n$ we associate *quasistrata*

$$B_k := \{x \in V \,:\, \dim(\{w \,:\, (x, w) \in B\}) = k\},\ 0 \le k \le n.$$

We call B **Lagrangian** if for any Gauss regular point $x \in B_k$ fiber $\{w \,:\, (x, w) \in B\}$ is the orthogonal complement of the tangent space $T_x(B_k)$.

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- *If bundle $G := G_f$ is Lagrangian then there exists a universal stratification of $\mathrm{Crit}(f)$ satisfying Whitney-(a) and Thom conditions.*
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Existence of universal stratifications in terms of Lagrangian bundles

Converse also holds

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If $\mathrm{Crit}(f)$ admits a universal stratification then bundle G_f is Lagrangian.

Conjecture. Dimension of every irreducible component of G_f equals n.

If the conjecture was true one could treat $G_f \subset \mathbb{C}^{2n}$ as a nonsmooth analogue of a Lagrangian manifold.

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Complexity issues

Let a polynomial $f \in \mathbb{Z}[X_1, \dots, X_n]$, $\deg(f) \leq d$ and the bit-sizes of the coefficients of f do not exceed M. The complexity of constructing (and thereby, the size of) bundle G_f and the quasistrata G_k, $0 \leq k \leq n$ can be bounded by a polynomial in $M, d^{2^{O(n)}}$ (so, double-exponential). The reason is that each iteration in the construction of the Glaeser closure increases the complexity by a polynomial, and there could be at most $2n$ iterations.

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Examples

Stratification of quadratic forms

Let $f = \sum_{1 \le i \le j \le n} A_{i,j} \cdot X_i \cdot X_j \in \mathbb{Z}[\{A_{i,j}\}, \{X_i\}]$ be a generic quadratic form, so polynomial in $n(n+3)/2$ variables.

Then $\mathrm{Crit}(f) = \{\{a_{i,j}\}, \{0\}\}$.

The Glaeser closure is constructed in a single iteration. Bundle G_f is Lagrangian.

Thus, the quasistrata (thereby, strata) $G_{k(q)} = \{(\{a_{i,j}\}, \{0\}) : \mathrm{rk}(a_{i,j}) = q\}$ where $k(q) = n + (n-q) \cdot (n-q+1)/2$, constitute a universal stratification.

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A variety without a universal stratification

We give an example of a polynomial $f = A \cdot X^2 + 2 \cdot B^2 \cdot X \cdot Y + C \cdot Y^2 \in \mathbb{Z}[A, B, C, X, Y]$ for which $\mathrm{Crit}(f)$ does not admit a universal stratification. $\mathrm{Crit}(f) = \{x = y = 0\} \subset \mathbb{C}^5$ is a 3-dimensional linear space. Its quasistrata are

- $G_2 = \{x = y = 0,\ a \cdot c - b^4 \neq 0\}$;
- $G_3 = \{x = y = 0,\ a \cdot c - b^4 = 0,\ (a, c) \neq 0\}$;
- $G_4 = \{0\}$.

Quasistrata G_2, G_3 are Lagrangian, while G_4 is not Lagrangian (since $\dim G_4 = 0 < 5 - 4$).

To illustrate this example note that $\mathrm{Crit}(f) = G_2 \sqcup G_3 \sqcup G_4$ is a stratification with smooth strata, satisfying Whitney-(a) and Thom conditions.

Consider a (rational) curve $K := \{x = y = 0,\ a = c = t^2,\ b = t\}_{t \in \mathbb{C}}$. Then $\mathrm{Crit}(f) = G_2 \sqcup (G_3 \setminus K) \sqcup K$

provides another stratification of $\mathrm{Crit}(f)$ which is neither coarser nor finer than the stratification $\mathrm{Crit}(f) = G_2 \sqcup G_3 \sqcup G_4$. There is no stratification of $\mathrm{Crit}(f)$ being coarser than both ones.

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