

# Quantum Hall criticality and localization in graphene with short-range impurities

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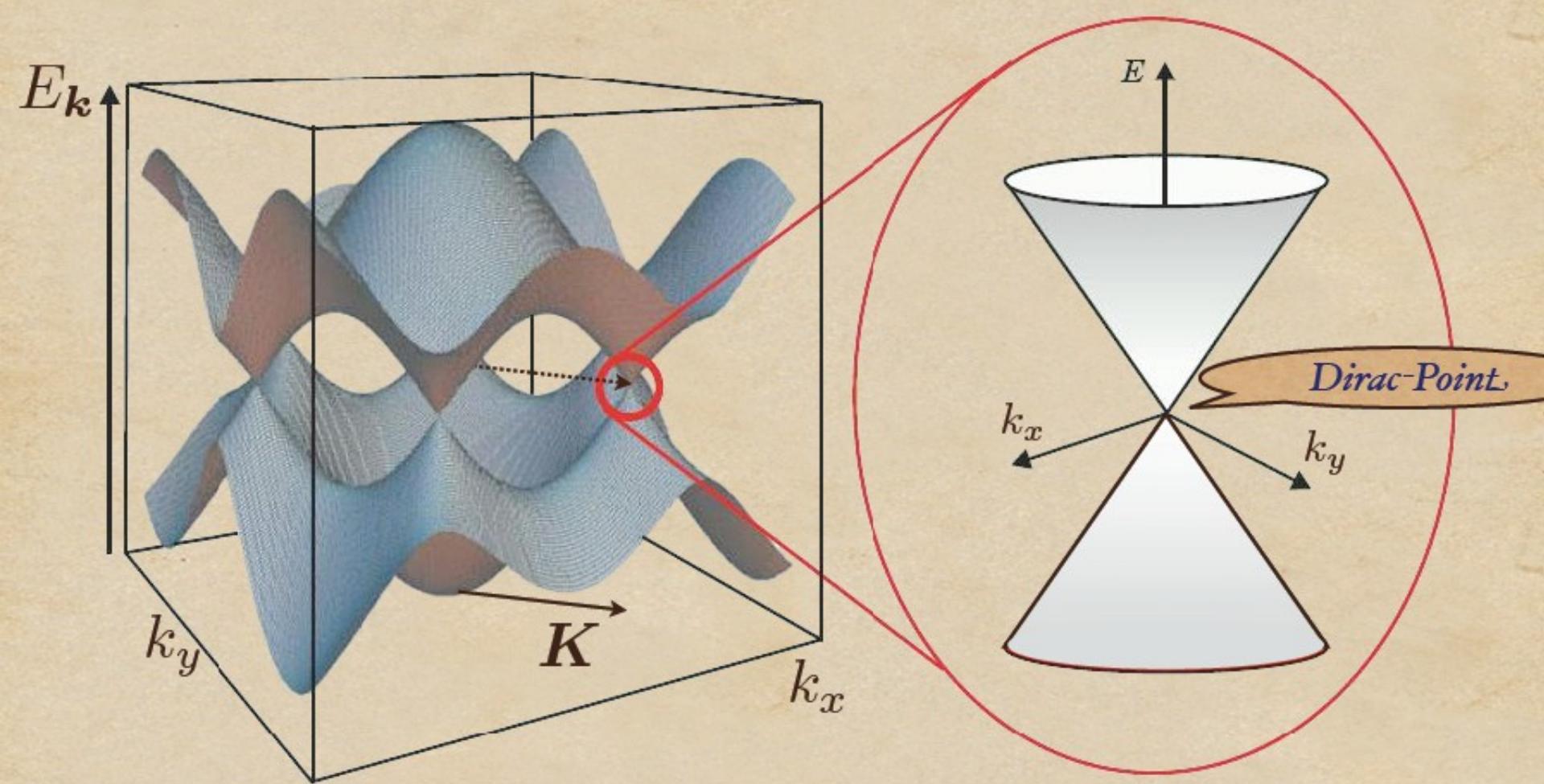
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# Outline

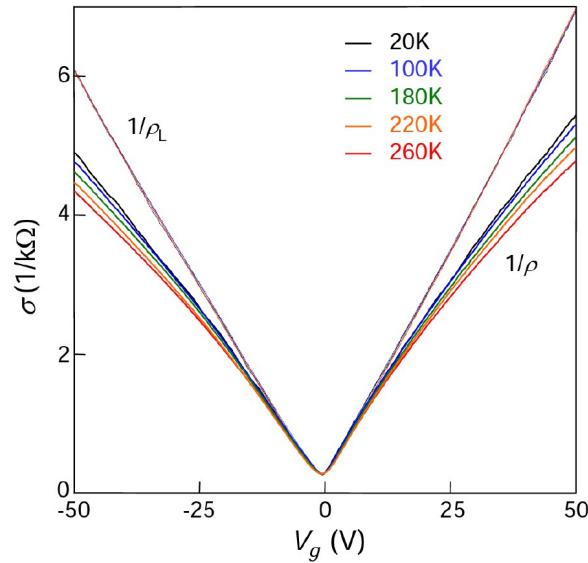
- Graphene: Dirac point at  $B=0$ 
  - Various types of criticality
- Point-like scatterers:
  - „Unfolded“ scattering theory
- Dirac point at finite  $B$ :
  - Quantum Hall criticality & localization
- Point-like scatterers in strong  $B$ :
  - „Level condensation“: ballistic transport

# Clean graphene: band structure

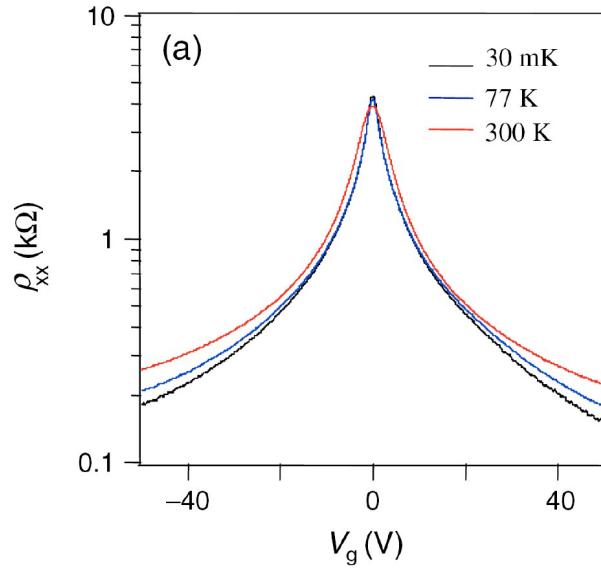


# Experiments on conductivity

## Density dependence



Novoselov, Geim et al '08



Zhang, Tan, Stormer, Kim '07

Conductivity is linear in density:

- long-range Coulomb impurities
- corrugations (ripples)

... or resonant scatterers (strong impurities, vacancies)?

## Conductivity at $n_e = 0$ (Dirac point)

Generic disorder:

Anderson localization will drive  $\sigma \sim e^2/h$  to  $\sigma \rightarrow 0$  with lowering  $T$ .

BUT

Experiment:  $\sigma \approx 4 \times e^2/h$  independent of  $T$  (Absence of localization !)

Can one have non-zero conductivity at Dirac point?

Yes, if disorder either

(i) preserves one of chiral symmetries (e.g., ripples, dislocations, vacancies)

or

(ii) is of long-range character  $\longrightarrow$  does not mix the valleys

(Coulomb impurities, Coulomb impurities + ripples)

# Disordered electronic systems: Symmetry classification

Altland, Zirnbauer '97

## Conventional (Wigner-Dyson) classes

	T	spin	rot.	chiral	p-h	symbol
GOE	+	+	-	-	-	AI
GUE	-	+/-	-	-	-	A
GSE	+	-	-	-	-	AII

## Chiral classes

	T	spin	rot.	chiral	p-h	symbol
ChOE	+	+	+	+	-	BDI
ChUE	-	+/-	-	+	-	AIII
ChSE	+	-	-	+	-	CII

$$H = \begin{pmatrix} 0 & t \\ t^\dagger & 0 \end{pmatrix}$$

## Bogoliubov-de Gennes classes

	T	spin	rot.	chiral	p-h	symbol
	+	+	-	-	+	CI
	-	+	-	-	+	C
	+	-	-	-	+	DIII
	-	-	-	-	+	D

$$H = \begin{pmatrix} h & \Delta \\ -\Delta^* & -h^T \end{pmatrix}$$

# Mechanisms of Anderson criticality in 2D

“Common wisdom”: all states are localized in 2D

In fact, in 9 out of 10 symmetry classes the system can escape localization!

→ variety of critical points

Mechanisms of delocalization & criticality in 2D:

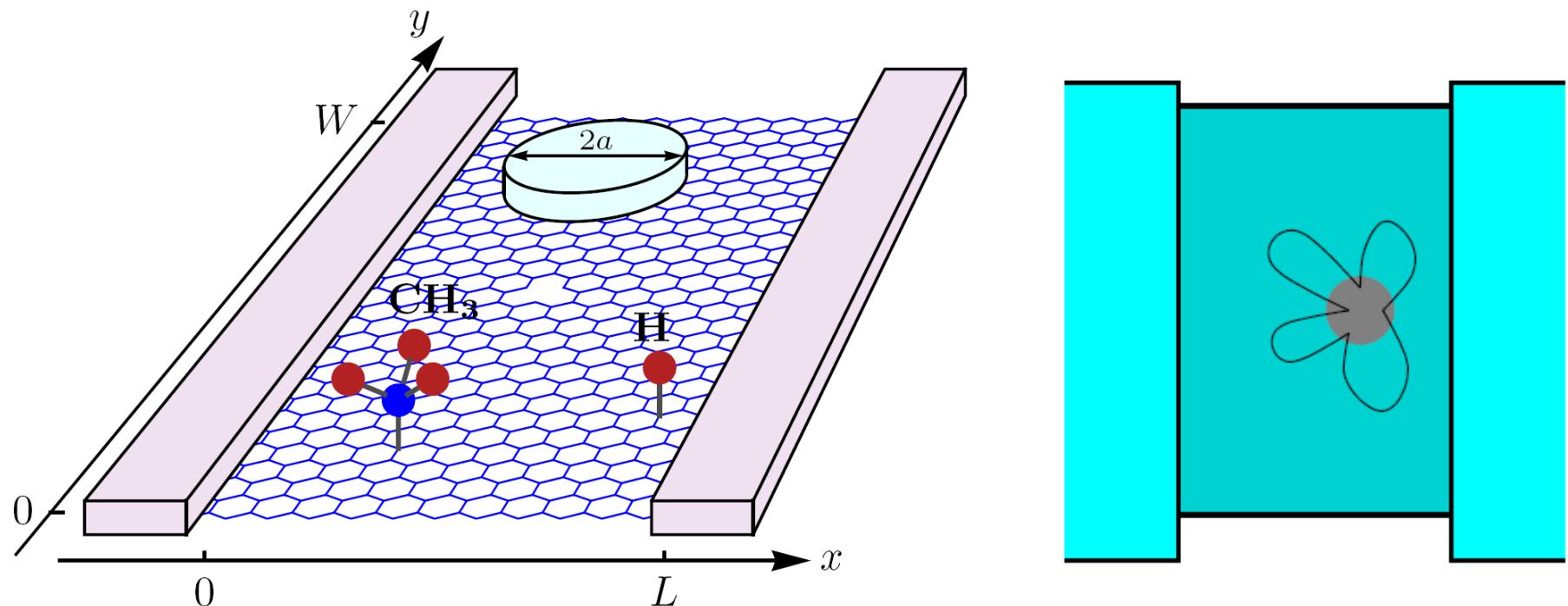
- broken spin-rotation invariance → antilocalization, metallic phase, MIT classes AII, D, DIII
- topological term  $\pi_2(\mathcal{M}) = \mathbb{Z}$  (quantum-Hall-type)  
classes A, C, D : IQHE, SQHE, TQHE
- topological term  $\pi_2(\mathcal{M}) = \mathbb{Z}_2$   
classes AII, CII
- chiral classes: vanishing  $\beta$ -function, line of fixed points  
classes AIII, BDI, CII
- Wess-Zumino term (random Dirac fermions, related to chiral anomaly)  
classes AIII, CI, DIII

# Anderson delocalization in graphene

Disorder	Symmetries	Class	Field-theory	Conductivity
Vacancies, strong on-site impurities	$C_z, T_0$	BDI	Gade	$\approx 4e^2/\pi h$
Vacancies + RMF		AIII	Gade	$\approx 4e^2/\pi h$
$\sigma_3\tau_{1,2}$ disorder		CII	Gade	$\approx 4e^2/\pi h$
Dislocations	$C_0, T_0$	CI	WZW	$4e^2/\pi h$
Dislocations + RMF		AIII	WZW	$4e^2/\pi h$
Ripples, RMF	$\Lambda_z, C_0$	$2 \times$ AIII	WZW	$4e^2/\pi h$
Smooth resonant impurities	$\Lambda_z, C, T$	$2 \times$ DIII	WZW	$\infty$
Charged impurities	$\Lambda_z, T_\perp$	$2 \times$ AII	$\theta = \pi$	$\sim 4e^2/h$ or $\infty$
Random Dirac mass: $\sigma_3\tau_0, \sigma_3\tau_3$	$\Lambda_z, CT_\perp$	$2 \times$ D	$\theta = \pi$	$4e^2/\pi h$
Charged impurities + ripples	$\Lambda_z$	$2 \times$ A	$\theta = \pi$	$4\sigma_U^*$

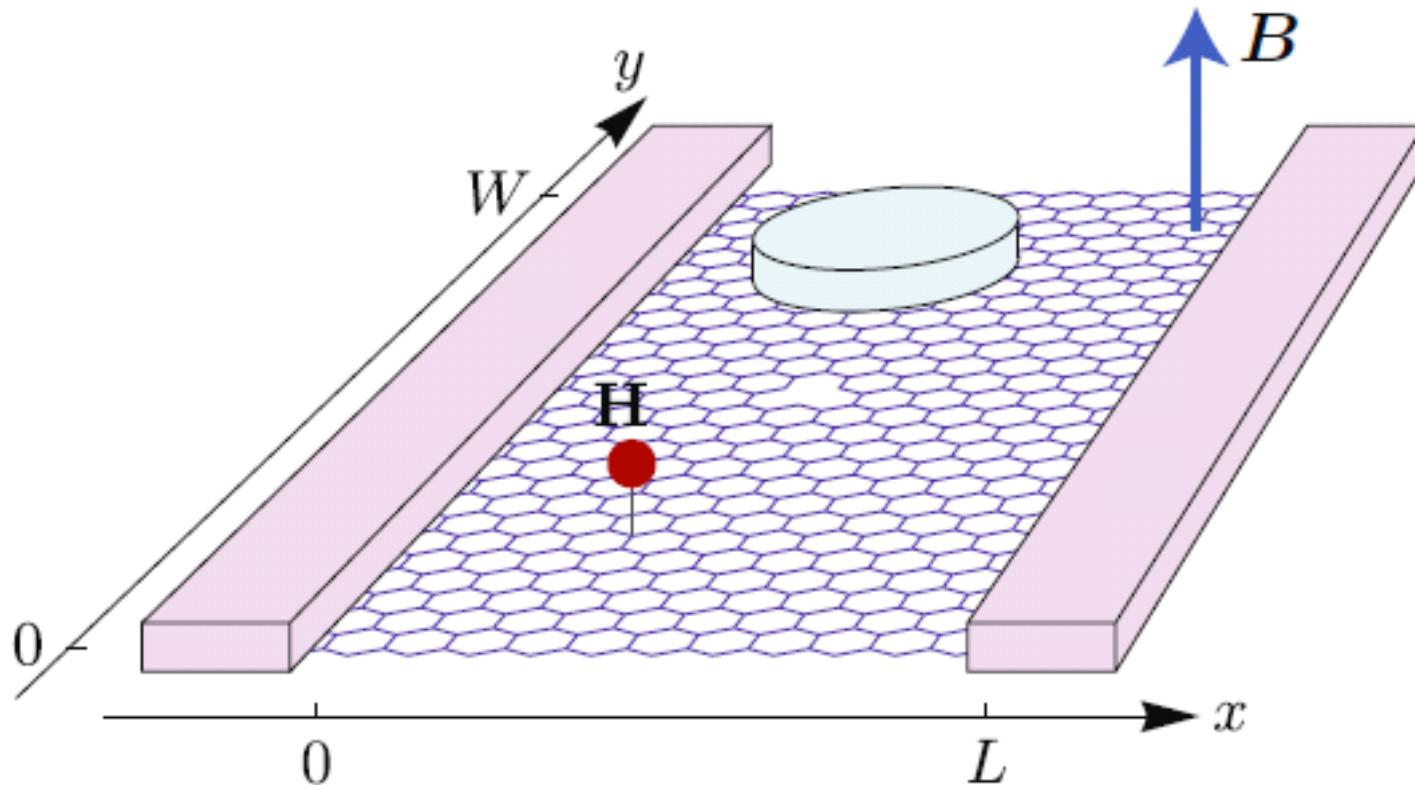
„Unfolded“ scattering theory

# Setup



- rectangular sample with dimensions  $L \times W$
- large aspect ratio:  $W \gg L$   
     $\Rightarrow$  boundary conditions (edge modes) irrelevant
- zero energy (Dirac point)
- ideal contacts
- perfect metallic leads

# Point-like impurities



- scalar impurities - smooth on atomic scales
- adatoms - on-site potential impurities
- vacancy - infinitely strong on-site potential
- resonant scalar impurity - creates a quasibound state at the Dirac point

# Methodology

Generating function:  $\mathcal{F}(\phi) = \text{Tr} \ln G_\phi^{-1}$

$$\text{Conductance} = \frac{4e^2}{h} \left. \frac{\partial^2 \mathcal{F}}{\partial \phi^2} \right|_{\phi=0}$$

Dyson:  $\mathcal{F} = \mathcal{F}_0 + \delta\mathcal{F}$

$$\mathcal{F}_0 = \text{Tr} \ln G_0^{-1} = W\phi^2/2\pi L \quad - \text{ballistic}$$

$$\delta\mathcal{F} = \text{Tr} \ln(1 - VG_0) \quad - \text{impurity contribution}$$

# Unfolding to impurity space

Use the identity:  $\delta\mathcal{F} = \text{Tr} \ln(1 - VG_0) = \text{Tr} \ln(1 - \hat{V}\hat{G}_0)$

where  $\hat{V} = \text{diag}(V_1, \dots, V_{N_{\text{imp}}})$  instead of  $V = \sum_{n=1}^{N_{\text{imp}}} V_n$  and  $(\hat{G}_0)_{ij} = G_0$

Green's function of the ballistic sample:  $G_0(\mathbf{r}_1, \mathbf{r}_2)$

Includes metal boundary conditions in x direction!

May be defined in Keldysh space and include counting fields

$$G(\mathbf{r}, \mathbf{r}') = G_0(\mathbf{r}, \mathbf{r}') + \sum_{nm} G_0(\mathbf{r}, \mathbf{r}_n) \hat{\mathbf{T}}_{nm} G_0(\mathbf{r}_m, \mathbf{r}')$$

Full T-matrix:  $\mathbf{T} = \frac{1}{1 - V\circled{G}_0} V$

Green's function of almost coinciding arguments

# Unfolding to impurity space

Decomposition to the singular and regular parts

$$\lim_{\mathbf{r}' \rightarrow \mathbf{r}} G_0(\mathbf{r}, \mathbf{r}') = g(\mathbf{r}, \mathbf{r}') + G_{\text{reg}}(\mathbf{r})$$

$g(\mathbf{r}, \mathbf{r}')$  is the Green's function of an infinite ballistic system (no leads, no counting fields)

Use the limit of short-range impurities

$$\Rightarrow \delta \mathcal{F} = \text{Tr} \ln(1 - \hat{T}\hat{G}_{\text{reg};\phi})$$

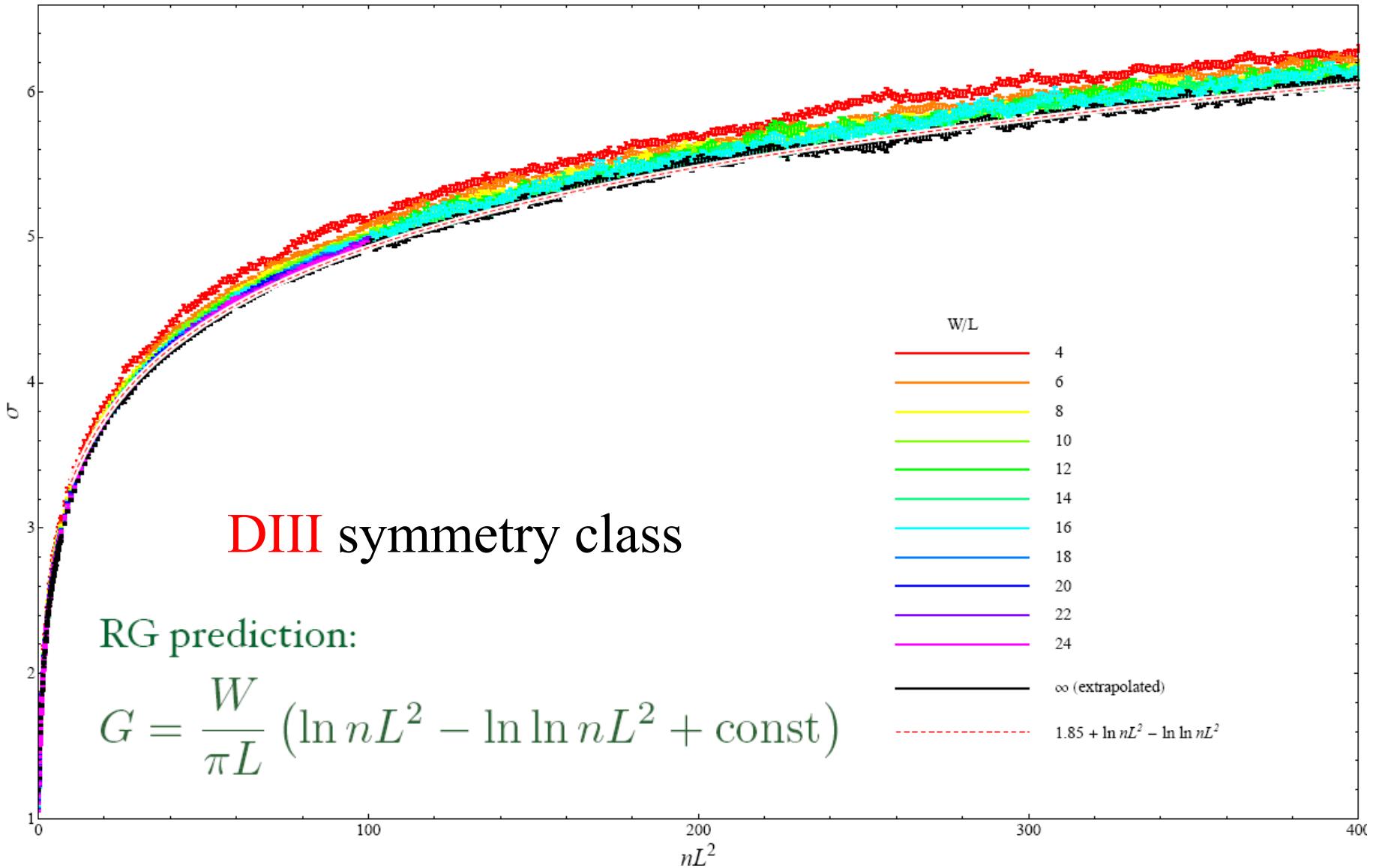
$2N \times 2N$  determinant!

Standard matrix product instead of operator convolution

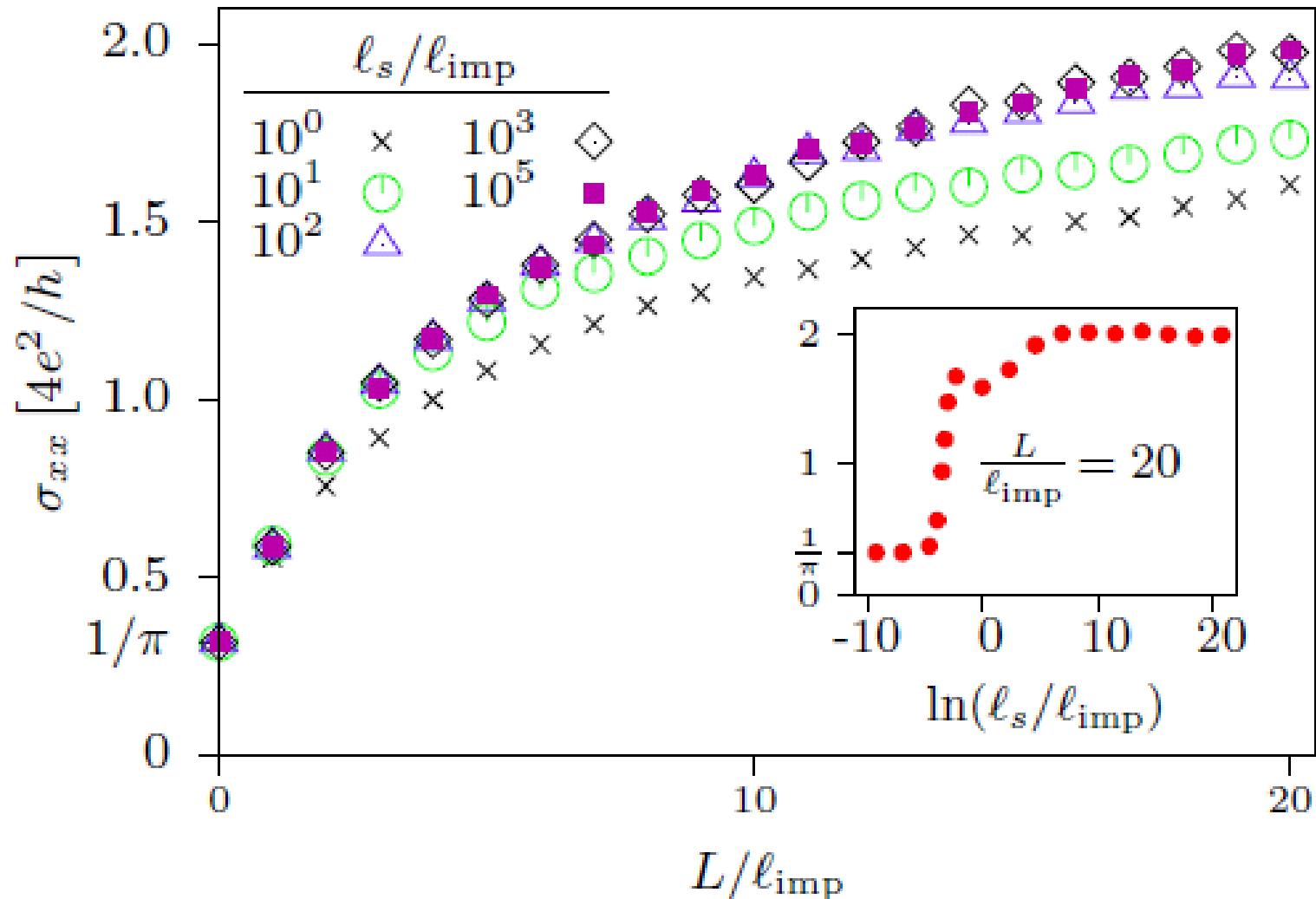
Extremely efficient numerics:  $\mathcal{O}(N^2)$  operations per realization

# Scalar impurities

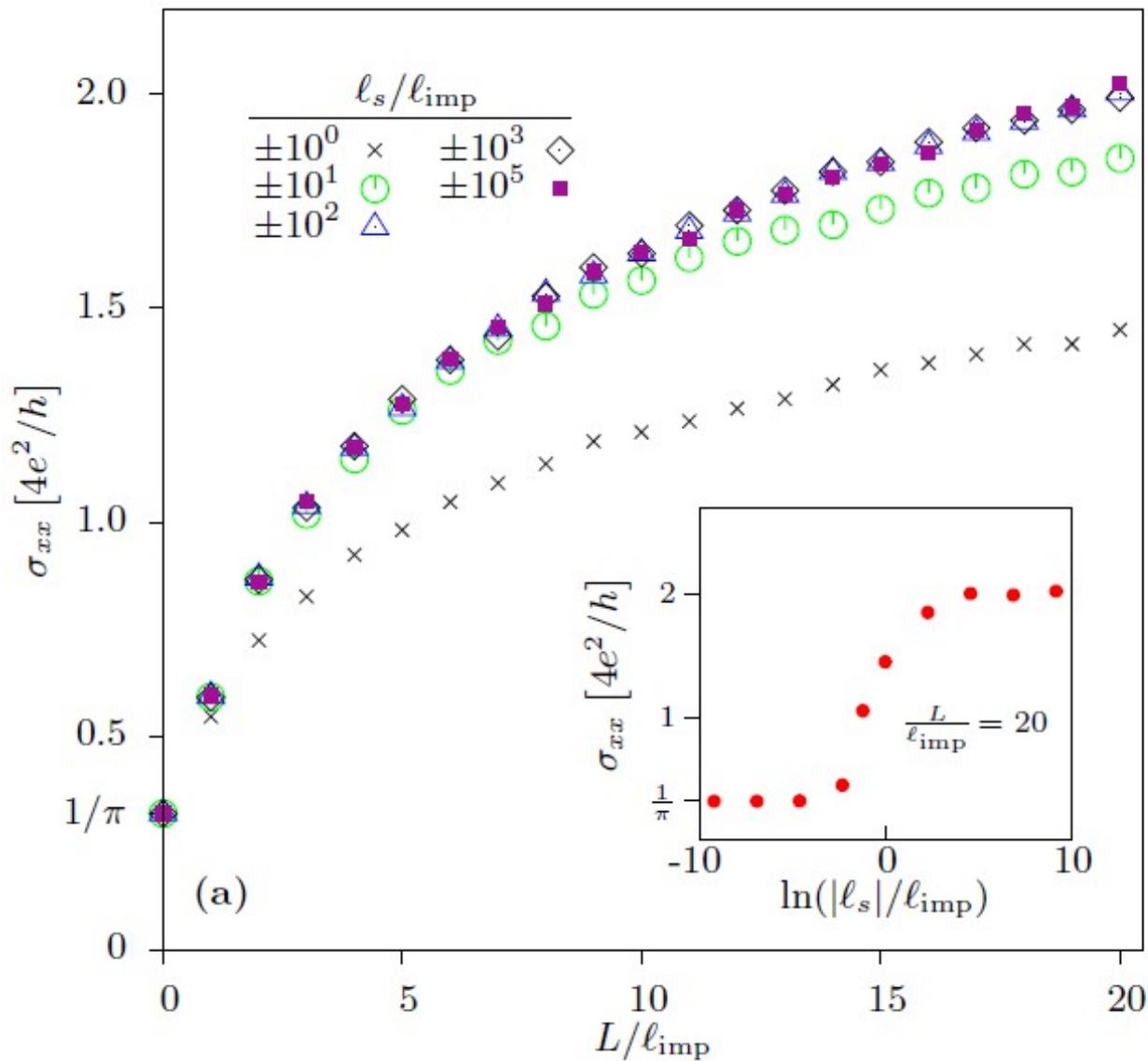
# Resonant scalar impurities



# Scalar impurities: Same sign



# Scalar impurities: Random sign

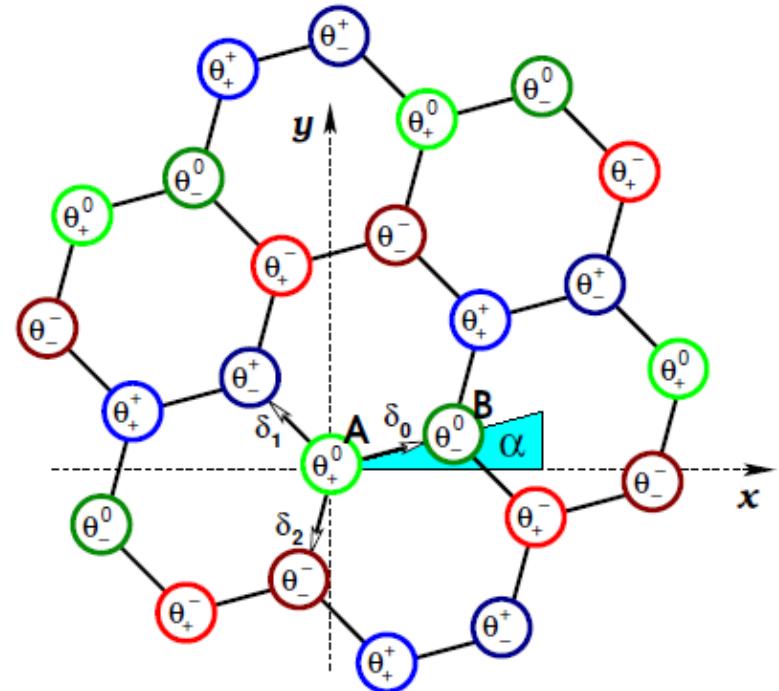


# Vacancies and adatoms

# Site coloring

Bloch functions:

$$\langle u_i | = \begin{cases} (e^{i\theta_+/2}, 0, 0, e^{-i\theta_+/2}), & \mathbf{r}_i \in A, \\ (0, ie^{i\theta_-/2}, ie^{-i\theta_-/2}, 0), & \mathbf{r}_i \in B. \end{cases}$$



$$T_+ = \frac{\ell_a}{2} \begin{pmatrix} 1 & 0 & 0 & e^{-i\theta_+^c} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ e^{i\theta_+^c} & 0 & 0 & 1 \end{pmatrix}_{\sigma\tau}$$

$$T_- = \frac{\ell_a}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & e^{-i\theta_+^c} & 0 \\ 0 & e^{i\theta_+^c} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{\sigma\tau}$$

Each on-site potential comes in one of 6 “colors”:

$$\theta_n^c = \zeta_n \alpha + 2K r_n = \pm \alpha + \frac{4\pi c}{3} \quad c = -1, 0, 1$$

# Vacancies: formalism

## ■ Conductance

$$G = \frac{4e^2}{\pi h} \left\{ \frac{W}{L} + \pi \operatorname{Tr}[K, Y](K + K^T)^{-1}[K^T, Y](K + K^T)^{-1} \right\}$$

$$K_{mn} = \frac{e^{\frac{i}{2}(\theta_m - \theta_n)}}{\sin \frac{\pi}{2L} [\zeta_m x_m + \zeta_n x_n + i(y_m - y_n)]} \quad K = K^\dagger$$

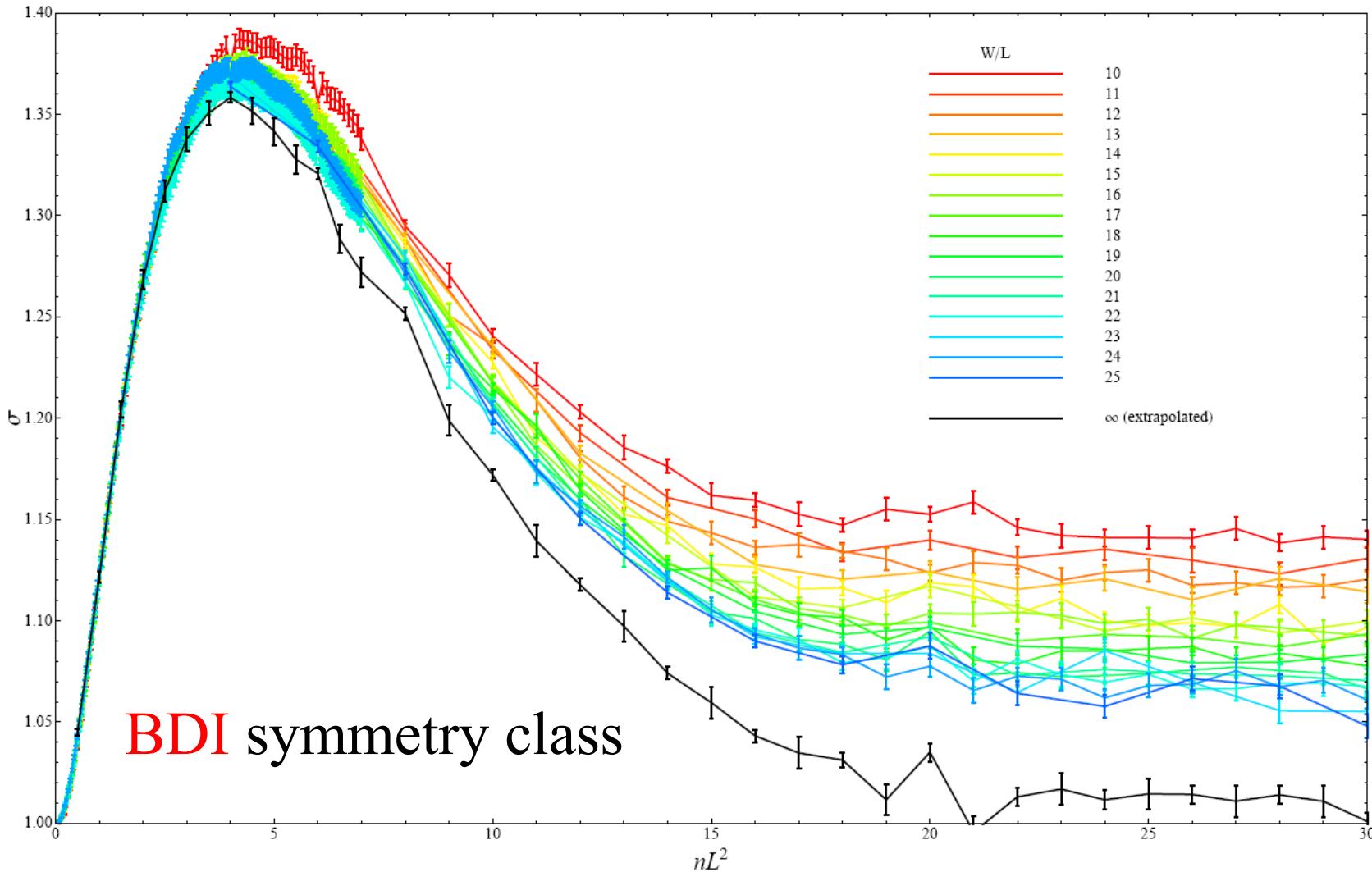
$$Y = L^{-1} \operatorname{diag}\{y_1, y_2, \dots, y_N\}$$

$\zeta_i = \pm 1$  and  $\theta_i$  are sublattice and color of  $i$ th vacancy

## ■ Symmetry class BDI [Gade & Wegner '91]

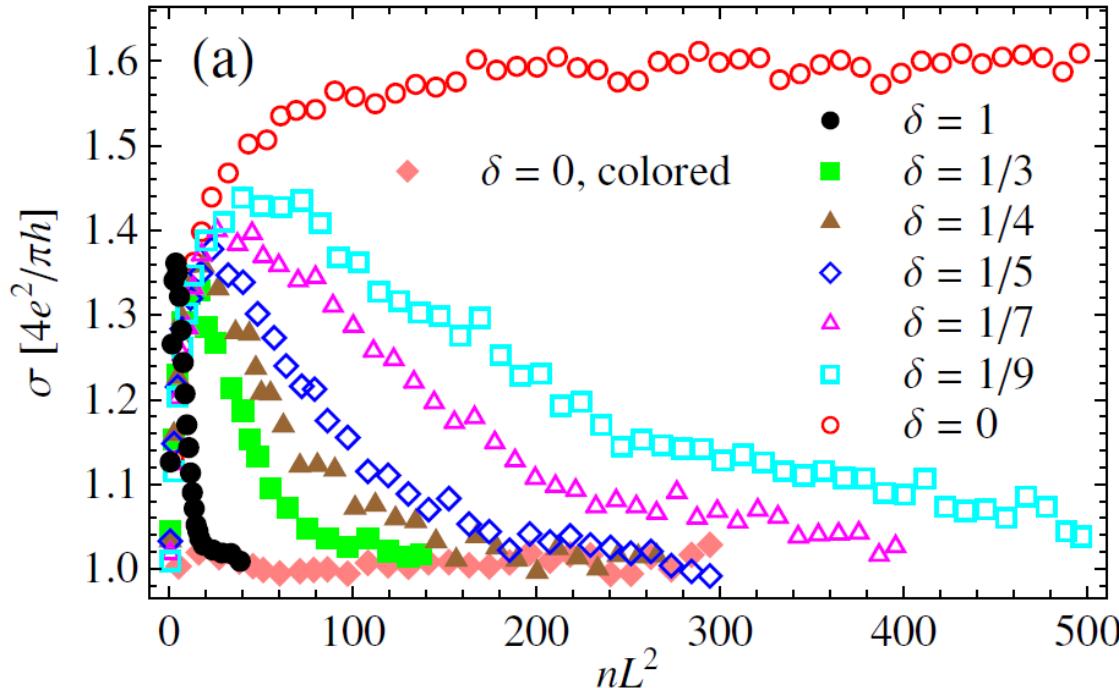
$$\frac{d\bar{\sigma}}{d \log L} = 0 \quad \text{perturbatively to ALL orders in } 1/\bar{\sigma}$$

# A-vacancies: conductance vs L



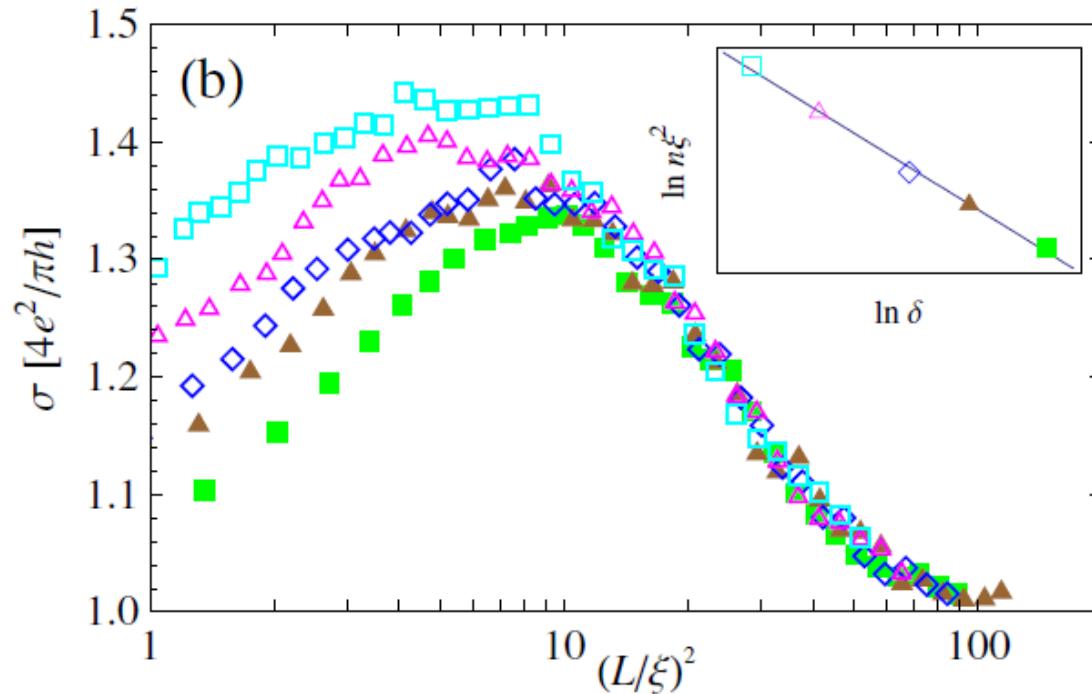
**BDI:** beta-function is zero → conductance saturates

# A + B vacancies



- Single color, armchair boundary ( $\alpha = 0$ )
- Sublattice imbalance  $\delta = (n_A - n_B)/n$
- Unstable fixed point for  $\delta = 0$  (conductivity saturates at  $\sigma \approx 2e^2/h$ )
- Stable fixed point for  $n_B \neq n_A$  with  $\sigma \approx \frac{4e^2}{\pi h}$

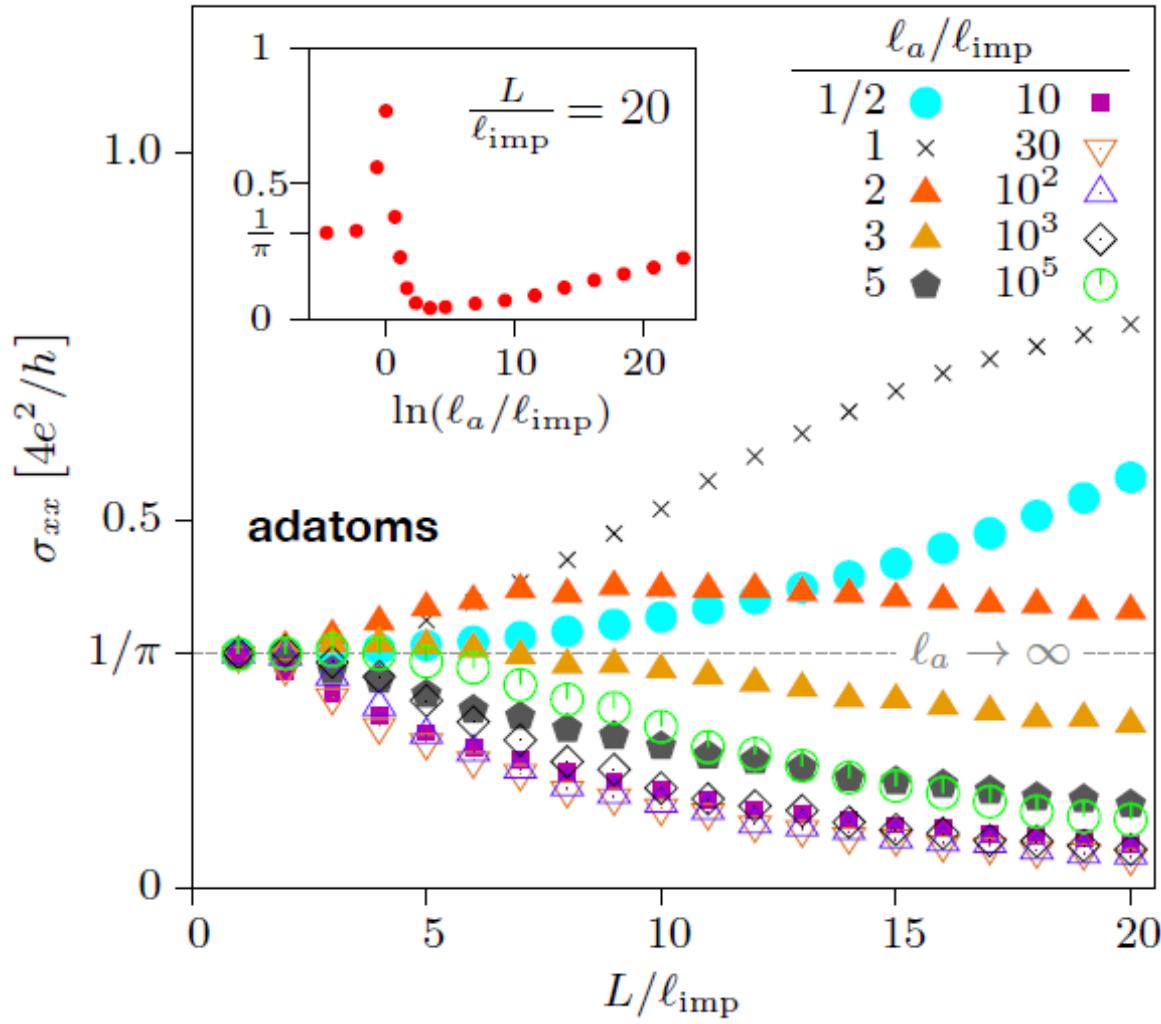
# A + B vacancies: scaling



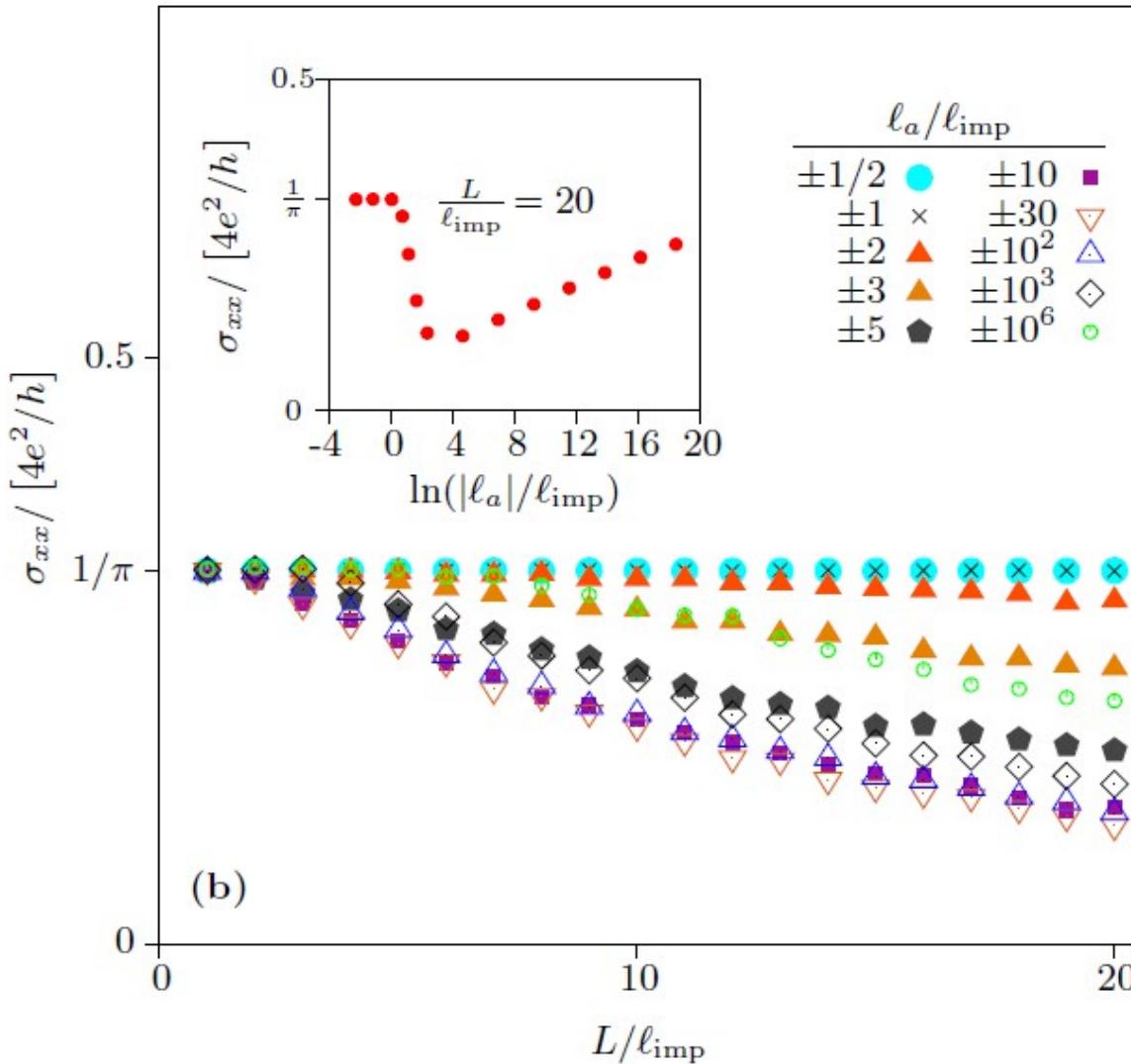
- Crossover curves collapse in units of  $L/\xi$
- Power law scaling  $n\xi^2 \sim \delta^{0.72}$

Novel strong-coupling criticality in class BDI beyond sigma model

# Adatoms: Same sign



# Adatoms: Random sign



# Results for B=0

- scalar: All (symplectic,TR+,SR-)

$$\sigma = \frac{4e^2}{\pi h} \ln nL^2 \rightarrow \infty$$

- resonant scalar: DIII (<sup>with WZNW</sup> BdG,TR+,SR-)

$$\sigma = \frac{4e^2}{\pi h} (\ln nL^2 - \ln \ln nL^2)$$

- adatoms: AI (orthogonal,TR+,SR+)

$$\sigma = \frac{4e^2}{\pi h} e^{-L/\xi_i} \rightarrow 0$$

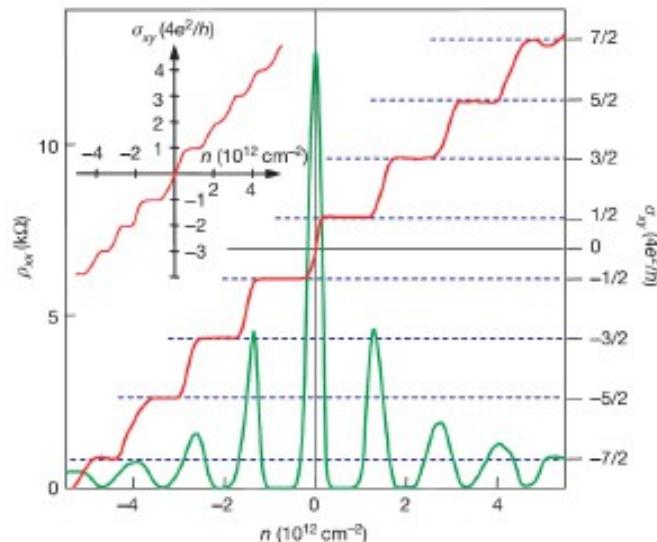
- vacancies: BDI (chiral orthogonal,TR+,SR+)

$$\sigma \rightarrow \frac{4e^2}{h} \times \frac{1}{\pi}$$

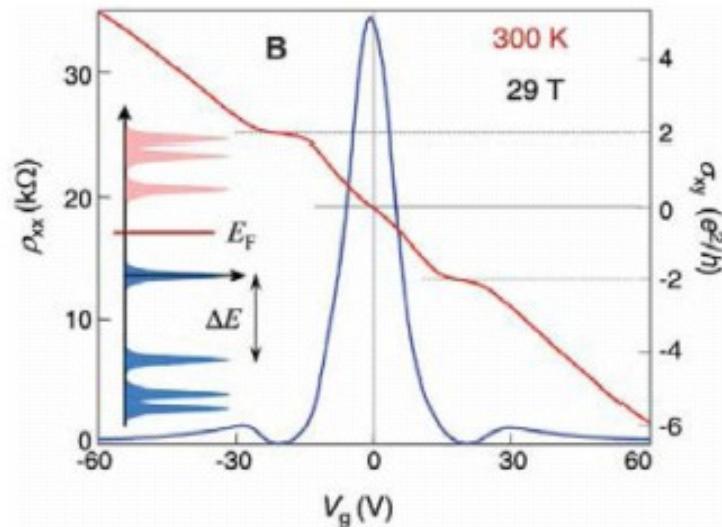
# Graphene in magnetic field

# Anomalous QHE in graphene

## Experiments on QHE



Novoselov, Geim et al '05



Novoselov, Geim, Stormer, Kim '07

### Anomalous quantum Hall effect

- only odd plateaus:  $\sigma_{xy} = (2n + 1)2e^2/h$
- QHE transition at zero concentration
- visible up to room temperature!

# Anomalous QHE in graphene

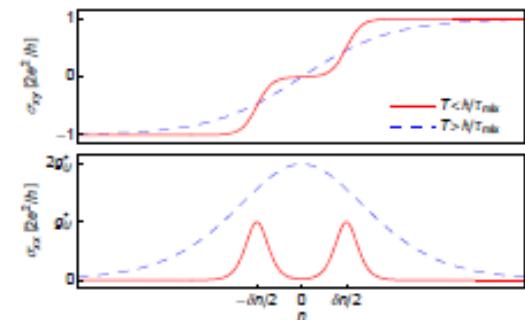
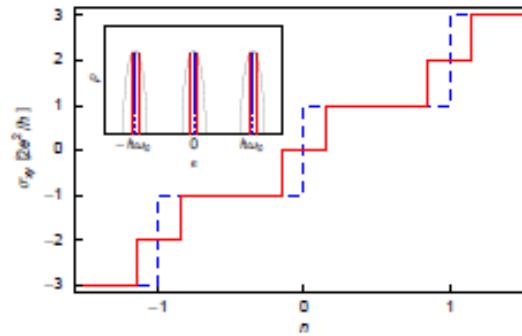
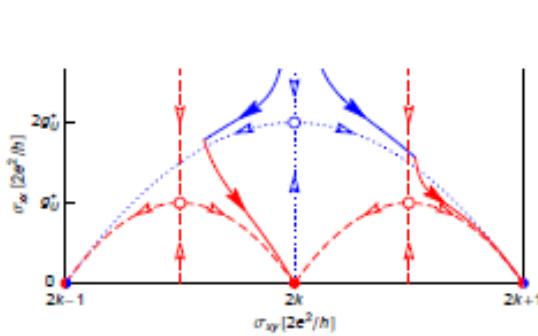
## Effective field theory: $\sigma$ -model

Single valley (unitary  $\sigma$ -model with topological term  $\theta = 2\pi\sigma_{xy} + \pi$ ):

$$S[Q] = \frac{1}{4} \text{Str} \left[ -\frac{\sigma_{xx}}{2} (\nabla Q)^2 + \left( \sigma_{xy} + \frac{1}{2} \right) Q \nabla_x Q \nabla_y Q \right]$$

Weakly mixed valleys:

$$S[Q_K, Q_{K'}] = S[Q_K] + S[Q_{K'}] + \frac{\rho}{\tau_{\text{mix}}} \text{Str} Q_K Q_{K'}$$



Even plateau width  $\sim (\tau/\tau_{\text{mix}})^{0.45}$ , visible at  $T < T_{\text{mix}} \sim \hbar/\tau_{\text{mix}}$

Estimate for Coulomb scatterers: even plateaus 5%,  $T_{\text{mix}} \sim 100$  mK,

# Anomalous QHE in graphene

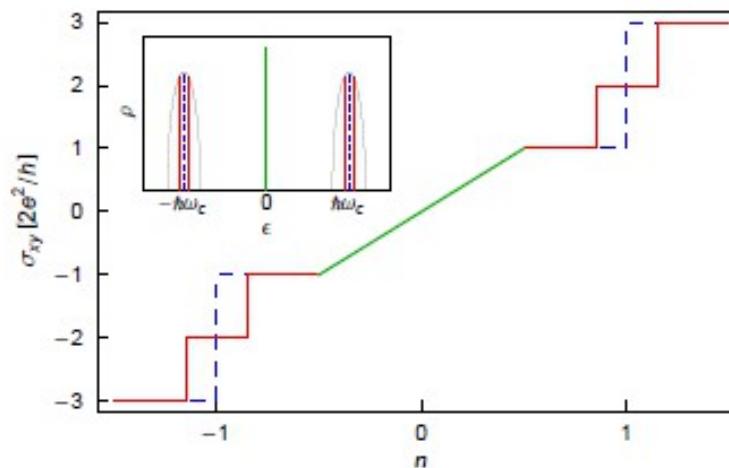
## Chiral disorder: “Classical” quantum Hall effect

- Ripples  $\Leftrightarrow$  Abelian random vector potential
- Dislocations  $\Leftrightarrow$  non-Abelian random vector potential

Atiyah–Singer theorem: Zero Landau level remains **degenerate**

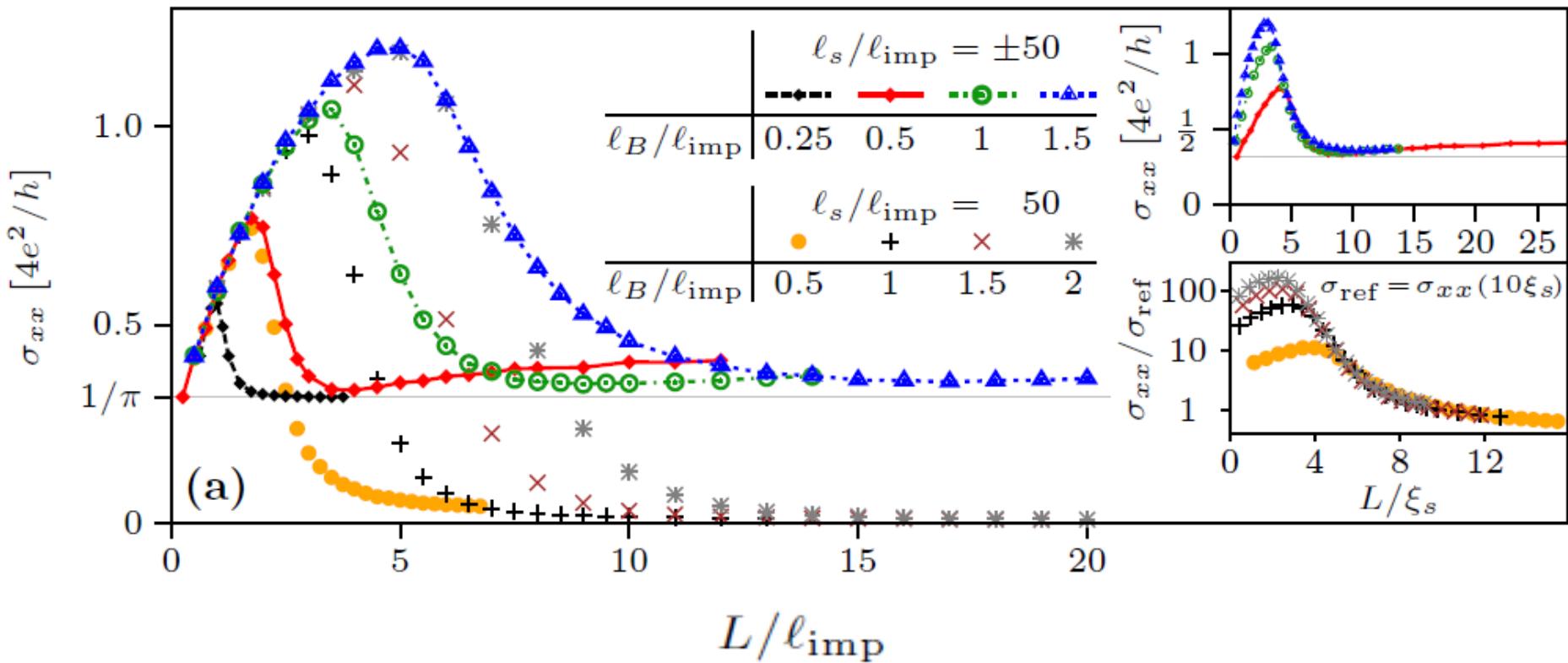
$\implies$  no localization

Aharonov, Casher '79



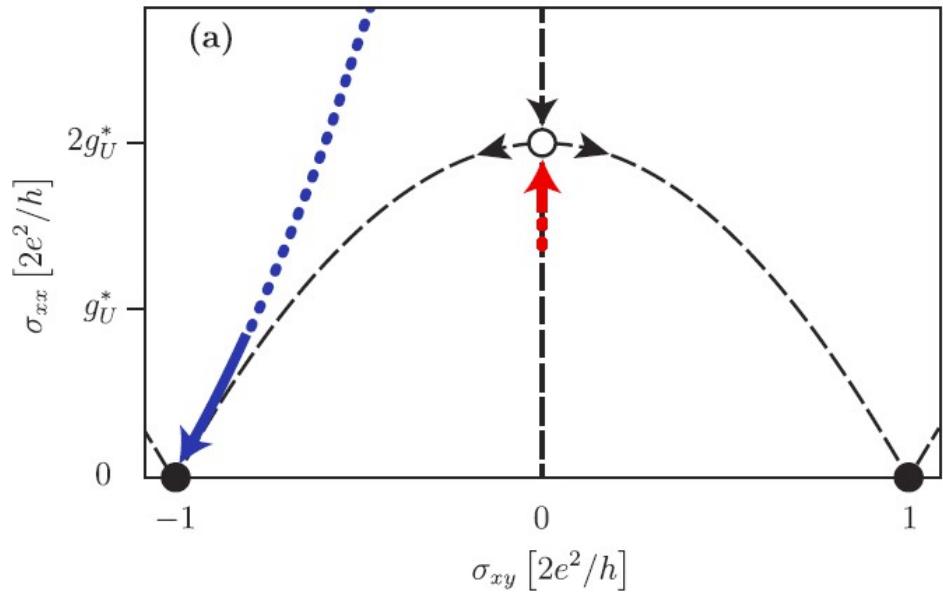
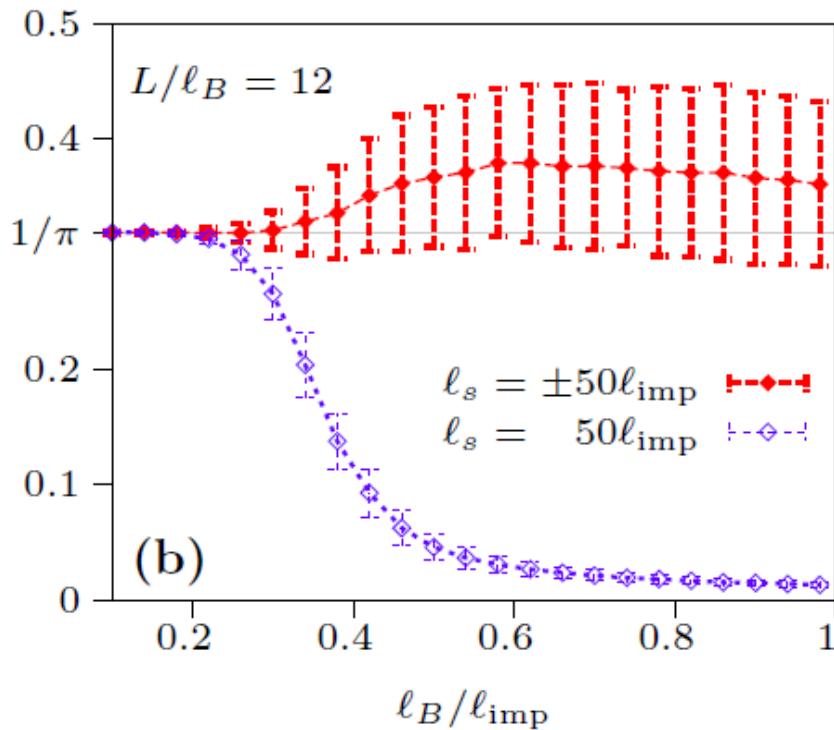
- Ripples: odd plateaus
- Ripples + Dislocations: all non-zero plateaus

# Finite B: Scalar impurities



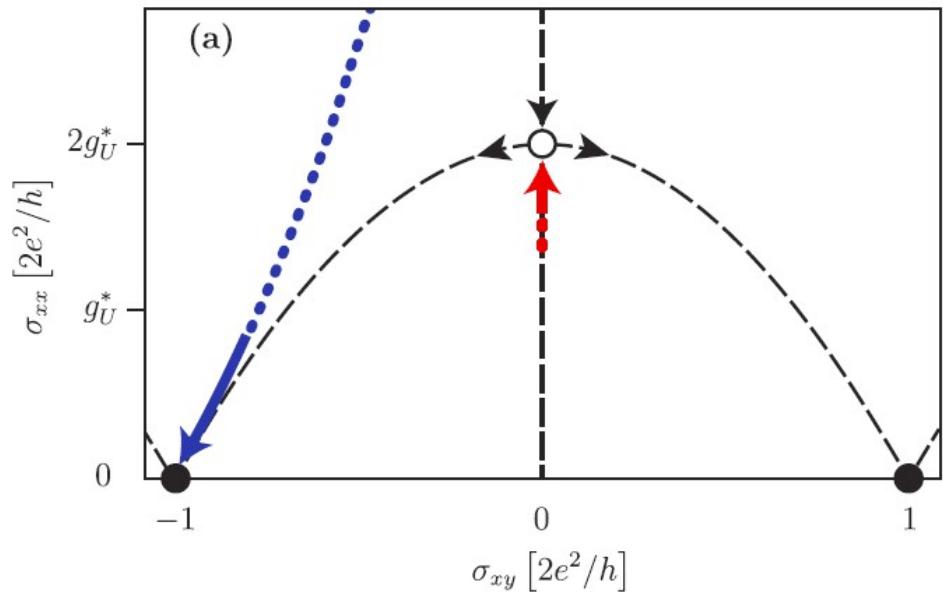
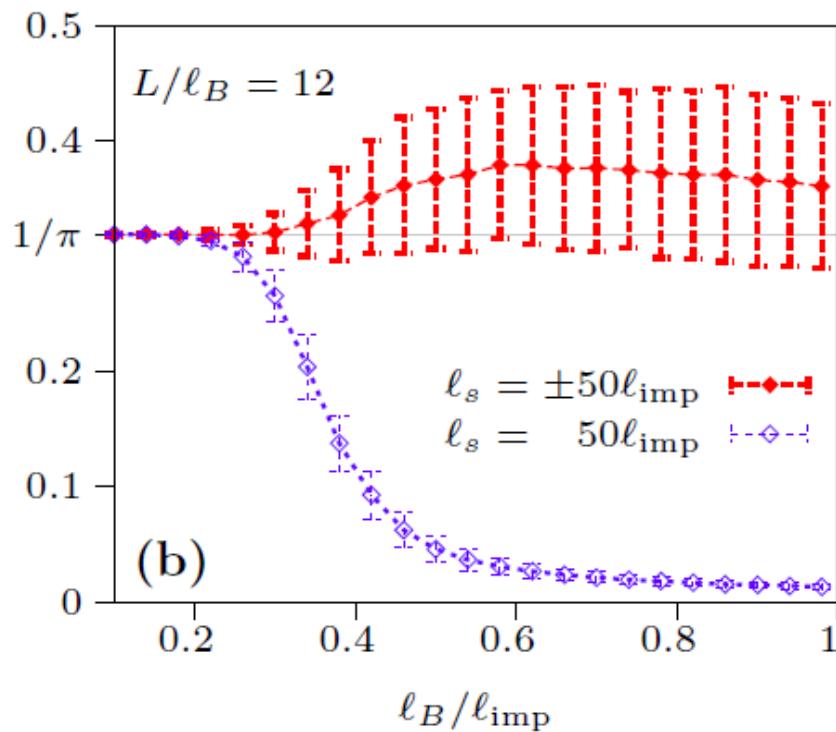
DIII $\rightarrow$ AII $\rightarrow$ A for weaker  $B$  and DIII $\rightarrow$ AIII $\rightarrow$ A for stronger  $B$

# Finite B: Scalar impurities



**Scaling with the system size** agrees with NLsM RG  
Approach to the **quantum-Hall critical point** from below  
at the half-filling  $\ell_s = \pm 50\ell_{\text{imp}}$

# Finite B: Scalar impurities



**Anderson localization** away from the half-filling  $\ell_s = 50\ell_{\text{imp}}$

**Transition to level condensation**  $\ell_B < 0.2\ell_{\text{imp}}$

# Finite B: Scalar impurities

Finite magnetic field  $B$  :  $\ell_{\text{imp}} \ll \ell_B \ll L$

*at half filling*       $\sigma \rightarrow \frac{4e^2}{h} \times 0.4$

- scalar: A (unitary,TR-,SR-)

*away from half filling*       $\sigma = \frac{4e^2}{\pi h} e^{-L/\xi_B} \rightarrow 0$

- resonant scalar: AIII (chiral unitary,TR-,SR-)      *with WZNW*

$$\sigma \rightarrow \frac{4e^2}{h} \times \frac{1}{\pi}$$

Strong magnetic field: level condensation  $B$  :

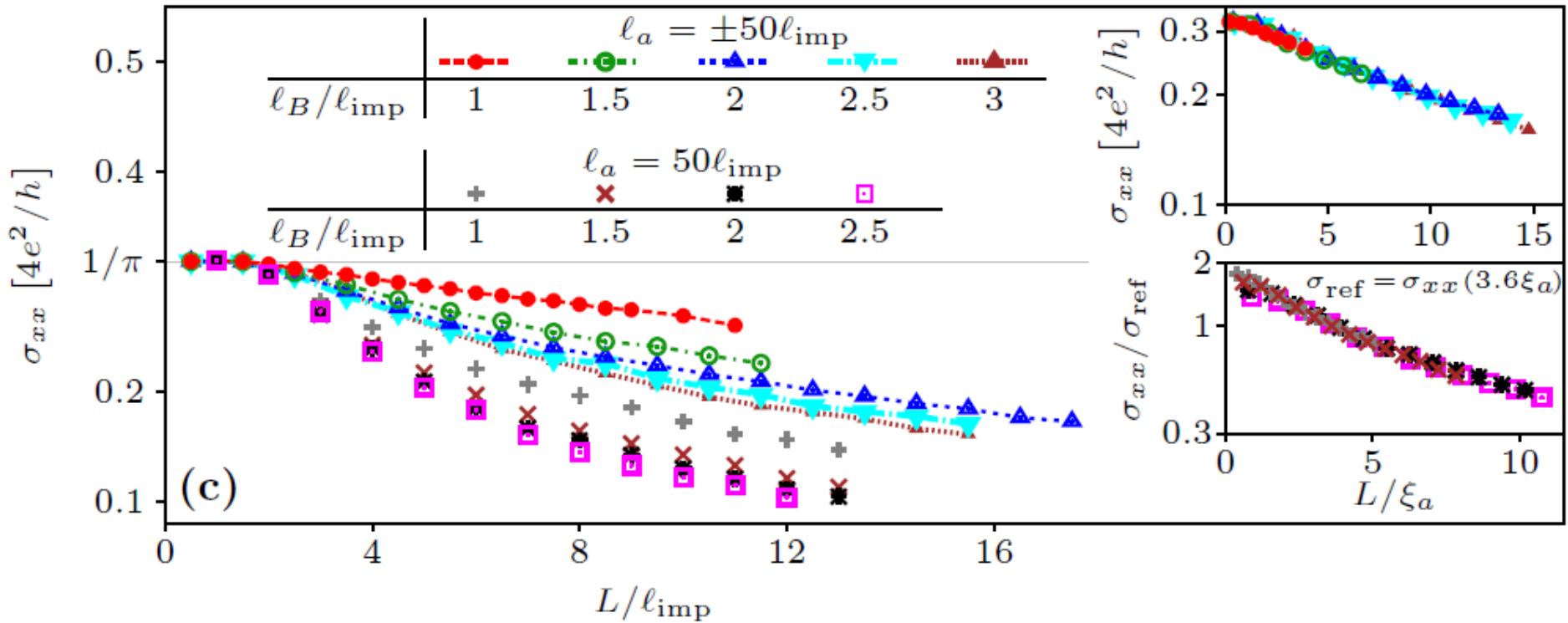
$$\ell_B \ll \ell_{\text{imp}}$$

$$\sigma = \text{ballistic value} = \frac{4e^2}{\pi h}$$

conductivity is independent of the impurity positions!

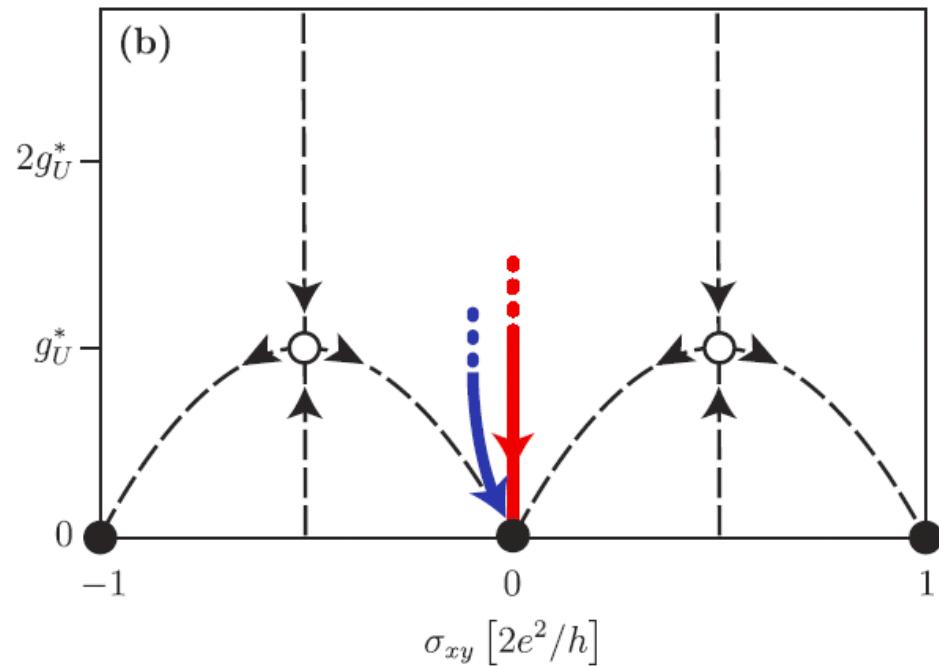
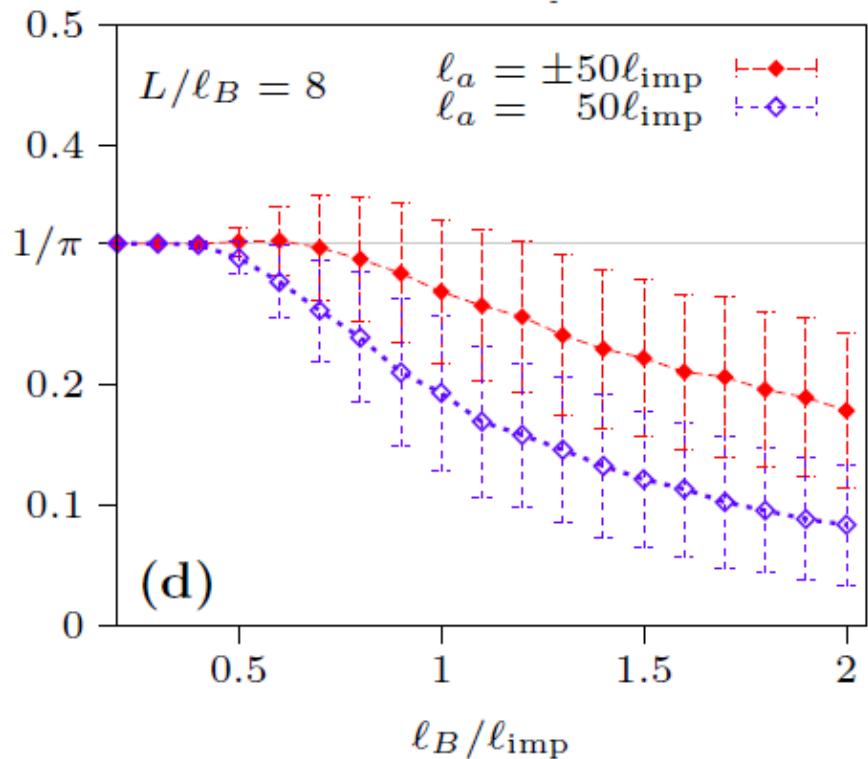
**No fluctuations!**

# Finite B: Adatoms



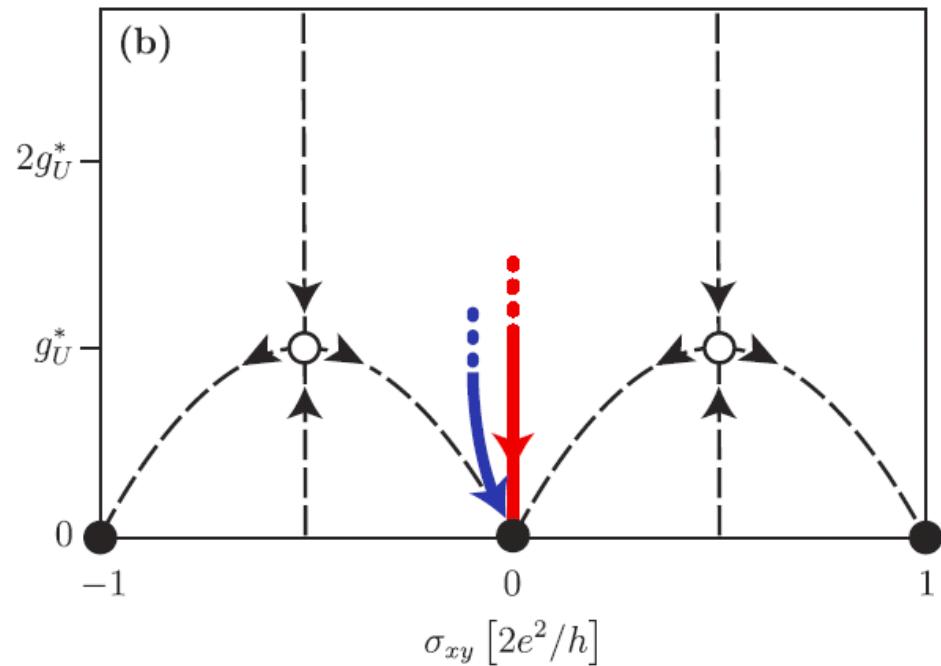
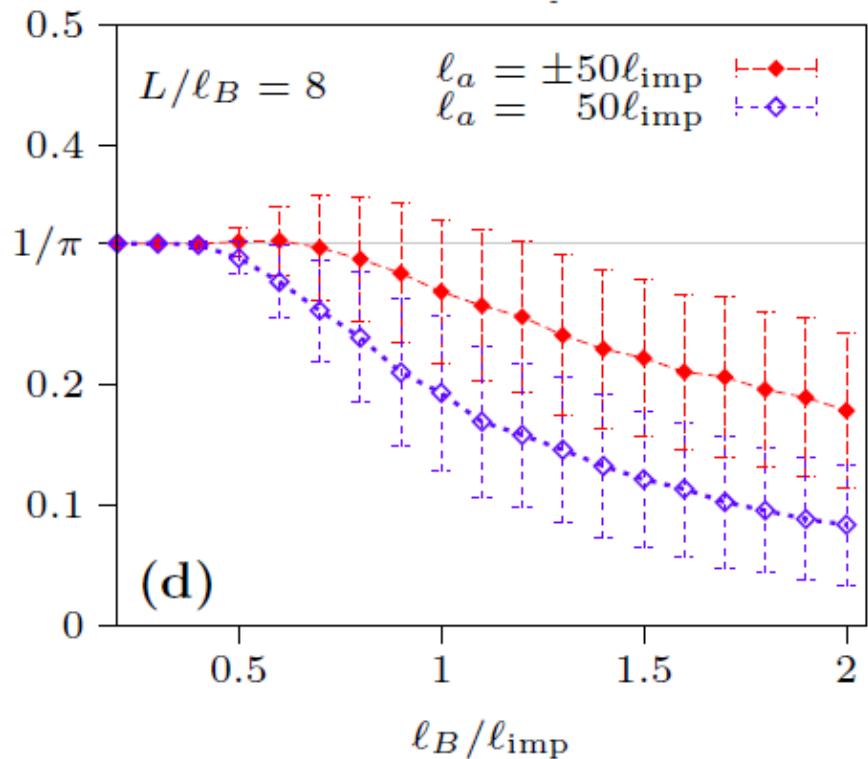
BDI  $\rightarrow$  AI  $\rightarrow$  A for weaker  $B$  and BDI  $\rightarrow$  AIII  $\rightarrow$  A for stronger  $B$

# Finite B: Adatoms



**Scaling with the system size** agrees with NLsM RG  
**Anderson localization** at the half-filled zeroth Landau level  
in contrast to the scalar impurity case

# Finite B: Adatoms



**Transition to level condensation**

$$\ell_B < 0.4 \ell_{\text{imp}} \text{ or } N_{\text{imp}} < N_{\Phi}$$

# Finite B: Adatoms

**Finite magnetic field  $B$ :**  $\ell_{\text{imp}} \ll \ell_B \ll L$

- adatoms: A (unitary,TR-,SR+)

$$\sigma = \frac{4e^2}{\pi h} e^{-L/\xi_i} \rightarrow 0$$

- vacancies: AlII (chiral unitary,TR-,SR+)

$$\sigma \rightarrow \frac{4e^2}{h} \times \frac{1}{\pi}$$

**Strong magnetic field: level condensation  $B$ :**  $\ell_B \ll \ell_{\text{imp}}$

$$\sigma = \text{ballistic value} = \frac{4e^2}{\pi h}$$

conductivity is independent of the impurity positions!

**No fluctuations!**

# Summary

- Graphene: Dirac point at  $B=0$ 
  - Various types of criticality
- Point-like scatterers:
  - „Unfolded“ scattering theory
- Dirac point at finite  $B$ :
  - Quantum Hall criticality & localization
- Point-like scatterers in strong  $B$ :
  - „Level condensation“: ballistic transport