

Algebraic geometry for shallow capillary-gravity waves

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1 Shallow capillary-gravity waves

- Travelling waves
- Phase-space analysis

2 Extended phase-space analysis

Outline

1 Shallow capillary-gravity waves

- Travelling waves
- Phase-space analysis

2 Extended phase-space analysis

Serre–(Green–Naghdi) equations

Shallow water equations

Credits:

- Lord RAYLEIGH (1876) [1] (only steady version)
- F. SERRE (1953) [2]
- C. SU & C. GARDNER (1969) [3]
- A. GREEN & P. NAGHDI (1976) [4]
- E. PELINOVSKY & ZHELEZNYAK (1985) [5]



Modern derivations:

- Asymptotic methods: Green & Naghdi (1976) [4]
- Variational methods:
 - Particle description: Miles & Salmon (1985) [6]
 - Eulerian description: Clamond & DD (2012) [7]



Paul M. Naghdi

Serre's equations with surface tension

1D case: the governing equations

- Governing equations (the mass conservation + momentum):

$$h_t + [h\bar{u}]_x = 0,$$

$$\bar{u}_t + \bar{u}\bar{u}_x + gh_x + \frac{1}{3}h^{-1}\partial_x\left[h^2\tilde{\gamma}\right] = \tau\left[h_x\left(1+h_x^2\right)^{-1/2}\right]_{xx}$$

- Vertical acceleration:

$$\tilde{\gamma} = h(\bar{u}_x^2 - \bar{u}_{xt} - \bar{u}\bar{u}_{xx}) = 2h\bar{u}_x^2 - h[\bar{u}_t + \bar{u}\bar{u}_x]_x$$

- Conservative form:

$$[h\bar{u}]_t + \left[h\bar{u}^2 + \frac{1}{2}gh^2 + \frac{1}{3}h^2\tilde{\gamma} - \tau R\right]_x = 0$$

- Surface tension:

$$R = hh_{xx}\left(1+h_x^2\right)^{-3/2} + \left(1+h_x^2\right)^{-1/2},$$

Serre's equations: the conservation laws

- Momentum conservation:

$$[h\bar{u} - \frac{1}{3}(h^3\bar{u}_x)_x]_t + \left[h\bar{u}^2 + \frac{1}{2}gh^2 - \frac{1}{3}2h^3\bar{u}_x^2 - \frac{1}{3}h^3\bar{u}\bar{u}_{xx} - h^2h_x\bar{u}\bar{u}_x - R \right]_x = 0$$

- Tangential velocity at the free surface:

$$\left[\bar{u} - \frac{(h^3\bar{u}_x)_x}{3h} \right]_t + \left[\frac{1}{2}\bar{u}^2 + gh - \frac{1}{2}h^2\bar{u}_x^2 - \frac{\bar{u}(h^3\bar{u}_x)_x}{3h} - \frac{\tau h_{xx}}{(1+h_x^2)^{3/2}} \right]_x = 0$$

- Energy conservation:

$$\begin{aligned} & \left[\frac{1}{2}h\bar{u}^2 + \frac{1}{6}h^3\bar{u}_x^2 + \frac{1}{2}gh^2 + \tau\sqrt{1+h_x^2} \right]_t + \\ & \left[\left(\frac{1}{2}\bar{u}^2 + \frac{1}{6}h^2\bar{u}_x^2 + gh + \frac{1}{3}h\gamma - \frac{\tau R}{h} \right)h\bar{u} + \tau\bar{u}\sqrt{1+h_x^2} + \frac{\tau hh_x\bar{u}_x}{\sqrt{1+h_x^2}} \right]_x = 0 \end{aligned}$$

Serre–CG equations: travelling waves

$$\text{Fr} = c^2/gd, \text{Bo} = \tau/gd^2, \text{We} = \text{Bo}/\text{Fr} = \tau/c^2 d$$

- Mass conservation: $\bar{u} = -cd/h, d = \langle h \rangle = \frac{1}{2\ell} \int_{-\ell}^{\ell} h dx$

- Momentum conservations lead:

$$\frac{\text{Fr} d}{h} + \frac{h^2}{2 d^2} + \frac{\tilde{\gamma} h^2}{3gd^2} - \frac{\text{Bo} h h_{xx}}{(1+h_x^2)^{\frac{3}{2}}} - \frac{\text{Bo}}{(1+h_x^2)^{\frac{1}{2}}} = \text{Fr} + \frac{1}{2} - \text{Bo} + K_1$$

- Tangential velocity:

$$\frac{\text{Fr} d^2}{2 h^2} + \frac{h}{d} + \frac{\text{Fr} d^2 h_{xx}}{3 h} - \frac{\text{Fr} d^2 h_x^2}{6 h^2} - \frac{\text{Bo} d h_{xx}}{(1+h_x^2)^{\frac{3}{2}}} = \frac{\text{Fr}}{2} + 1 + \frac{\text{Fr} K_2}{2}$$

$$\tilde{\gamma}/g = \text{Fr} d^3 h_{xx}/h^2 - \text{Fr} d^3 h_x^2/h^3$$

- Integration constants:

$$K_2 = \left\langle \frac{(3+h_x^2) d^2}{3 h^2} - 1 \right\rangle$$

$$K_1 + \frac{1}{2} + \text{Fr} - \text{Bo} = \left\langle \frac{\text{Fr} d^2}{h^2} + \frac{1}{2} - \frac{\text{Bo} d/h}{(1+h_x^2)^{\frac{1}{2}}} \right\rangle / \left\langle \frac{d}{h} \right\rangle.$$

Serre–CG equations: travelling waves

Particular emphasis on solitary waves

- Combination of two equations:

$$\frac{\text{Fr } d}{2 h} - \frac{h^2}{2 d^2} - \frac{\text{Fr } d h_x^2}{6 h} - \frac{\text{Bo}}{(1 + h_x^2)^{\frac{1}{2}}} + \frac{(\text{Fr} + 2 + \text{Fr } K_2) h}{2 d} = \text{Cnst}$$

- Consider solitary waves ($K_1 = K_2 \equiv 0$):

$$F(h, h') \equiv \frac{\text{Fr } h'^2}{3} + \frac{2 \text{Bo } h/d}{(1 + h'^2)^{\frac{1}{2}}} - \text{Fr} + \frac{(2\text{Fr} + 1 - 2\text{Bo}) h}{d} - \frac{(\text{Fr} + 2) h^2}{d^2} + \frac{h^3}{d^3} = 0$$

- Property: $h(x) \equiv h(-x)$
- At the crest of a **regular** wave: $h(0) = d + a, h'(0) = 0$
 $\text{Fr} = 1 + a/d$

Serre–CG equations: travelling waves

Particular emphasis on solitary waves

- Combination of two equations:

$$\frac{\text{Fr } d}{2 h} - \frac{h^2}{2 d^2} - \frac{\text{Fr } d \, h_x^2}{6 h} - \frac{\text{Bo}}{(1 + h_x^2)^{\frac{1}{2}}} + \frac{(\text{Fr} + 2 + \text{Fr } K_2) h}{2 d} = \text{Cnst}$$

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- We are interested in solutions: $h(\infty) = d$, $h'(\infty) = 0$

Serre–CG equation: Limiting cases I

Pure gravity and pure capillary waves

- Pure gravity waves: $\text{Bo} \rightarrow 0$

$$h = d + a \operatorname{sech}^2(\kappa x/2), \quad (\kappa d)^2 = 3a/(d+a), \quad \text{Fr} = 1 + a/d$$

- Pure capillary waves: $\text{Fr} \rightarrow \infty, \text{Bo} \rightarrow \infty, \text{We} = \text{Const}$

$$\frac{h'^2}{3} + \frac{2 \text{We } h/d}{\left(1 + h'^2\right)^{\frac{1}{2}}} - 1 + \frac{2(1 - \text{We})h}{d} - \frac{h^2}{d^2} = 0$$

In the limit $\text{We} \rightarrow 0$:

- Capillary wave equation reads: $h' = \pm \sqrt{3}(1 - h/d)$
- Solitary wave with angular crest:

$$h = d + a \exp(-\sqrt{3}|x|)$$

Serre–CG equation: Limiting cases II

Introduce a simplification to equations

- Small slope approximation [8]:

$$(1 + h'^2)^{-\frac{1}{2}} \approx 1 - \frac{1}{2} h'^2 + \dots$$

- Master equation becomes:

$$\left(\frac{\text{Fr}^2}{3} - \frac{\text{Bo} h}{d}\right) h'^2 = \text{Fr}^2 - \frac{(2\text{Fr}^2 + 1) h}{d} + \frac{(\text{Fr}^2 + 2) h^2}{d^2} - \frac{h^3}{d^3}$$

- Change of independent variables:

$$d\xi = |1 - 3 \text{We} h / d|^{-\frac{1}{2}} dx \quad (*)$$

Analytical solution (\equiv pure gravity case):

- $h(\xi) = d + a \operatorname{sech}^2(\kappa\xi/2)$
- $(\kappa d)^2 = 3a / (d + a)$, $\text{Fr} = 1 + a/d$
- $x(\xi)$ can be found from $(*)$

Asymptotic analysis

Following McCOWAN (1891) [9]

- Exponentially decaying solitary waves:

$$h(x) \sim d + a \exp(-\kappa x), \quad x \rightarrow +\infty, \quad \kappa > 0$$

'Dispersion' relation:

$$\text{Fr}^2 = \frac{3 - 3 \text{Bo} (\kappa d)^2}{3 - (\kappa d)^2} \quad \text{or} \quad (\kappa d)^2 (\text{Fr} - 3 \text{Bo}) = 3 (\text{Fr}^2 - 1)$$

- κ can be only real **or** purely imaginary
 - \Rightarrow Solitary waves **or** Periodic waves
- Critical values:
 - $\text{Fr} = 1, \text{Bo} = 1/3, \kappa d = \sqrt{3}$ or $\text{Bo} = \frac{1}{3}\text{Fr}$

- Algebraic decay:

$$h(x) \sim d + a(\kappa x)^{-\alpha}, \quad x \rightarrow +\infty, \quad \alpha > 1$$

- Then necessarily $\Rightarrow \text{Fr} \equiv 1$
- $\alpha = 1 \Rightarrow \text{Bo} \neq \frac{1}{3}, \quad \alpha > 2 \Rightarrow \text{Bo} = \frac{1}{3}$

Phase-space analysis: local behaviour

Nonlinear autonomous ODE analysis

- Two-parameter (Fr, Bo) family of real algebraic curves in \mathbb{R}^2 :

$$\begin{aligned} F_{\text{Fr}, \text{Bo}}(h, k) := & \frac{\text{Fr}}{3}k^2 + 2 \frac{\text{Bo} h}{(1+k^2)^{\frac{1}{2}}} + \\ & - \text{Bo} + (2\text{Fr} - 2\text{Bo} + 1)h - (\text{Fr} + 2)h^2 + h^3 = 0 \end{aligned}$$

- With asymptotic behaviour at $x \rightarrow \infty$ ($k := h'$):

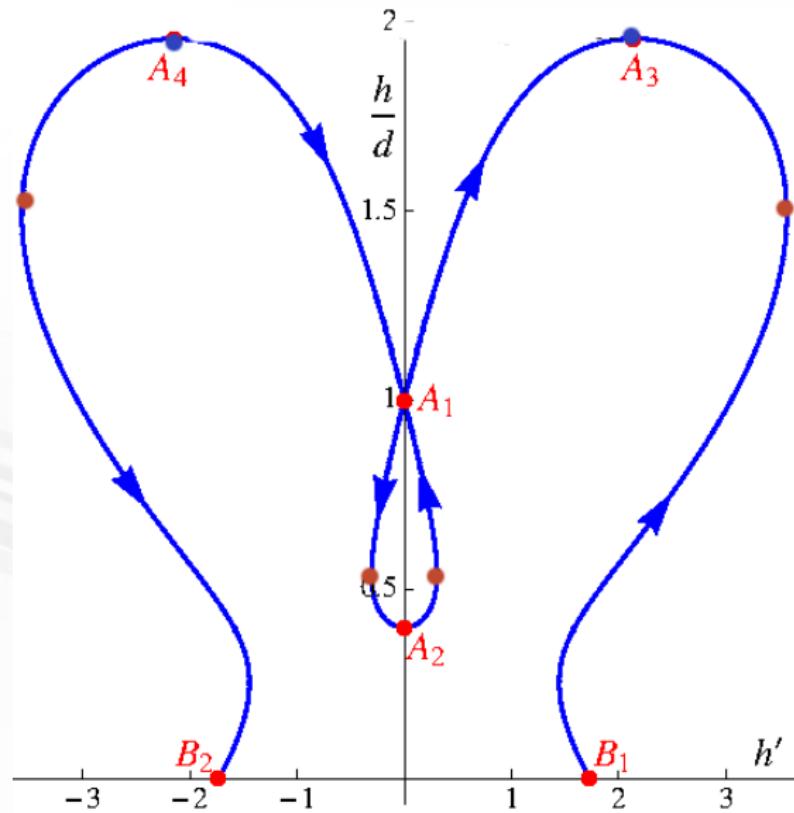
$$h(\infty) = 1, \quad k(\infty) = 0$$

Typical workflow with a parametrized curve:

- Find multiple points (with horizontal tangent): $\partial_k F_{\text{Fr}, \text{Bo}} = 0$
- Find points with vertical tangent: $\partial_h F_{\text{Fr}, \text{Bo}} = 0$
- Decompose it into oriented branches ($k \gtrless 0, h \nearrow \searrow 0$)
- Study the family of algebraic curves $F_{\text{Fr}, \text{Bo}}(h, k) \in \mathbb{R}^2$ with certified topology methods [10]

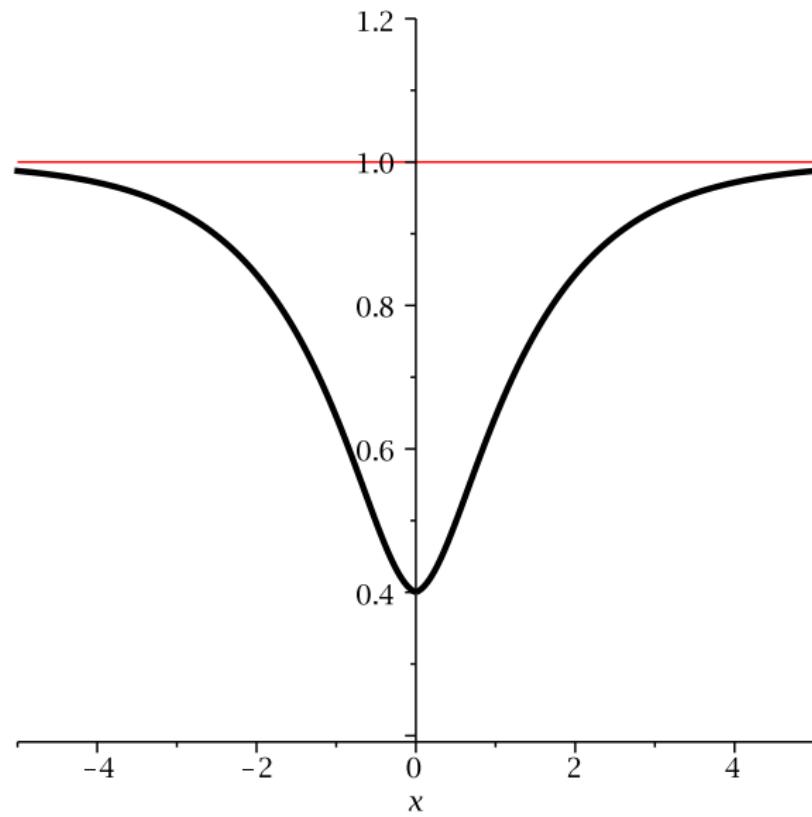
Phase-space analysis: depression wave

A particular example for $\text{Fr} = 0.4$, $\text{Bo} = 0.9 > 1/3$



Phase-space analysis: depression wave

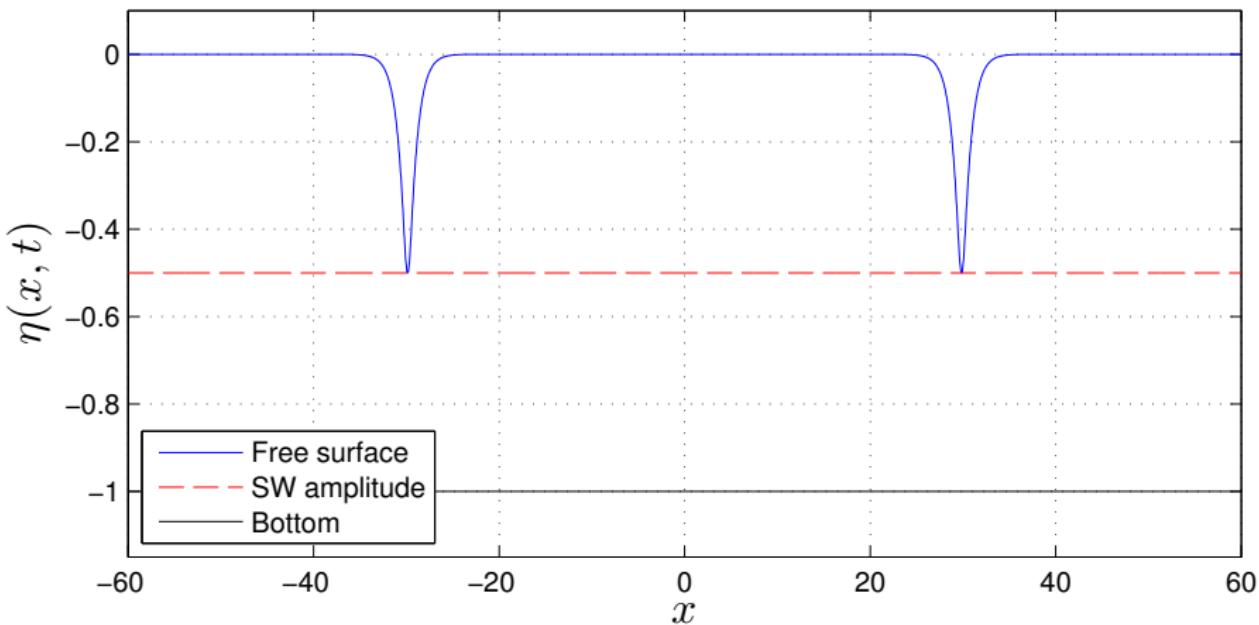
A particular example for $\text{Fr} = 0.4$, $\text{Bo} = 0.9 > 1/3$



Solitary waves collision

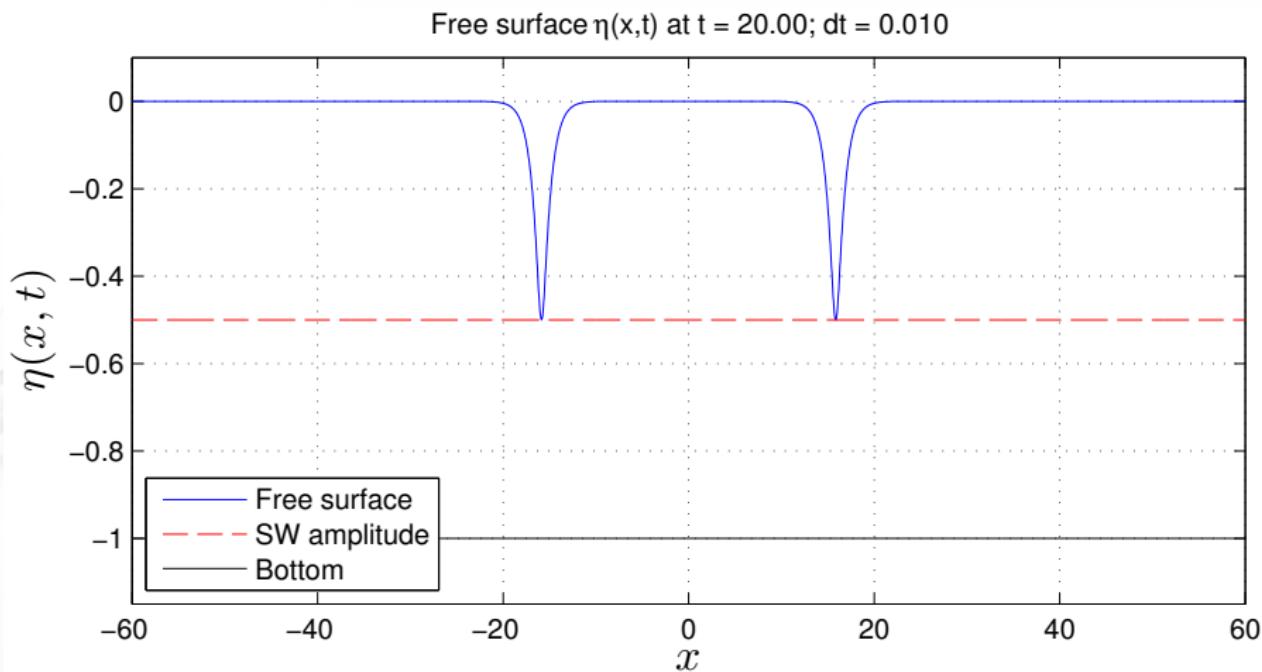
Solitary waves of depression: $a/d = -1/2$, $\text{Fr} = 1/2$, $\text{Bo} = 1/2$

Free surface $\eta(x,t)$ at $t = 0.20$; $dt = 0.010$



Solitary waves collision

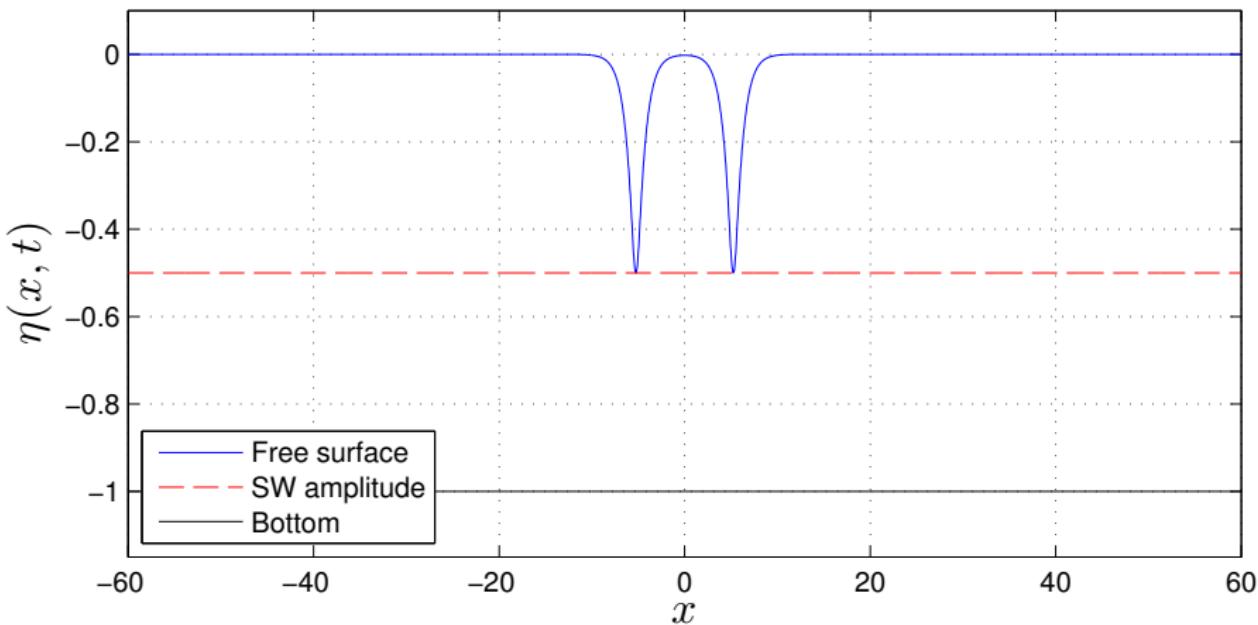
Solitary waves of depression: $a/d = -1/2$, $\text{Fr} = 1/2$, $\text{Bo} = 1/2$



Solitary waves collision

Solitary waves of depression: $a/d = -1/2$, $\text{Fr} = 1/2$, $\text{Bo} = 1/2$

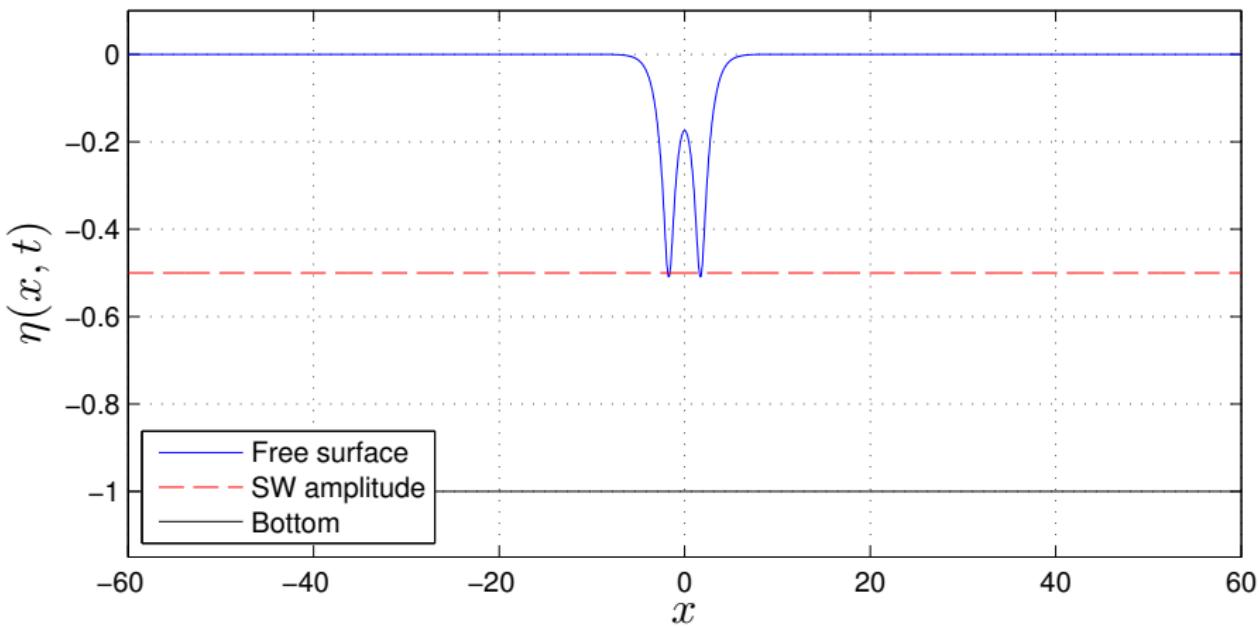
Free surface $\eta(x,t)$ at $t = 35.00$; $dt = 0.010$



Solitary waves collision

Solitary waves of depression: $a/d = -1/2$, $\text{Fr} = 1/2$, $\text{Bo} = 1/2$

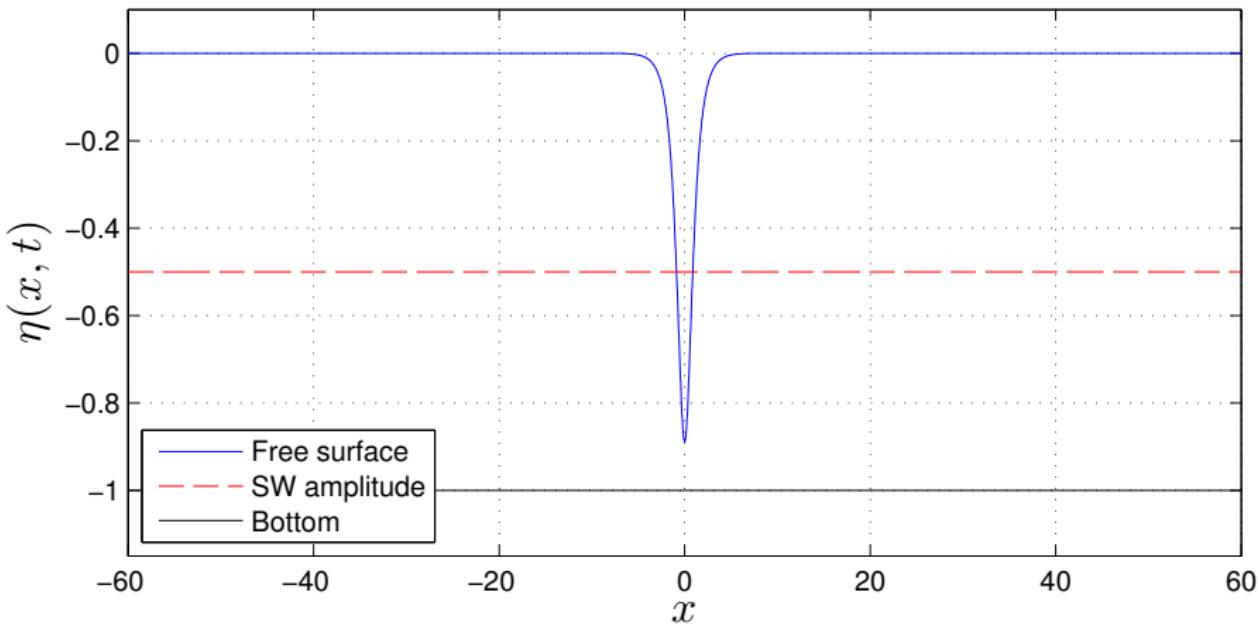
Free surface $\eta(x,t)$ at $t = 40.00$; $dt = 0.010$



Solitary waves collision

Solitary waves of depression: $a/d = -1/2$, $\text{Fr} = 1/2$, $\text{Bo} = 1/2$

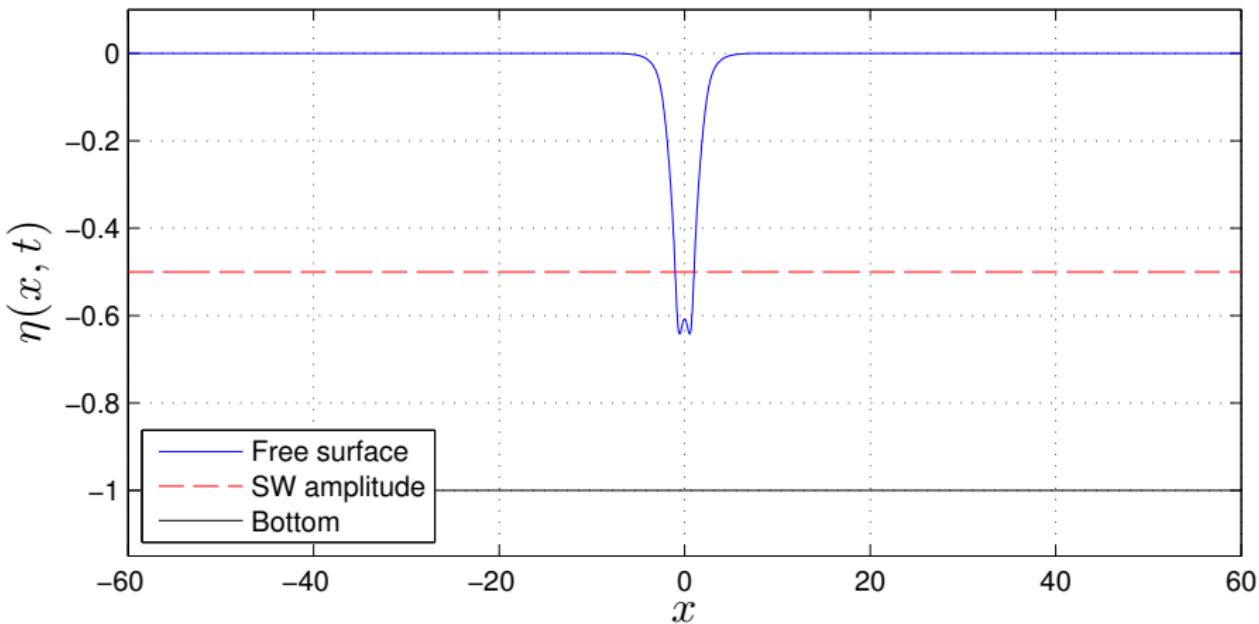
Free surface $\eta(x,t)$ at $t = 42.20$; $dt = 0.010$



Solitary waves collision

Solitary waves of depression: $a/d = -1/2$, $\text{Fr} = 1/2$, $\text{Bo} = 1/2$

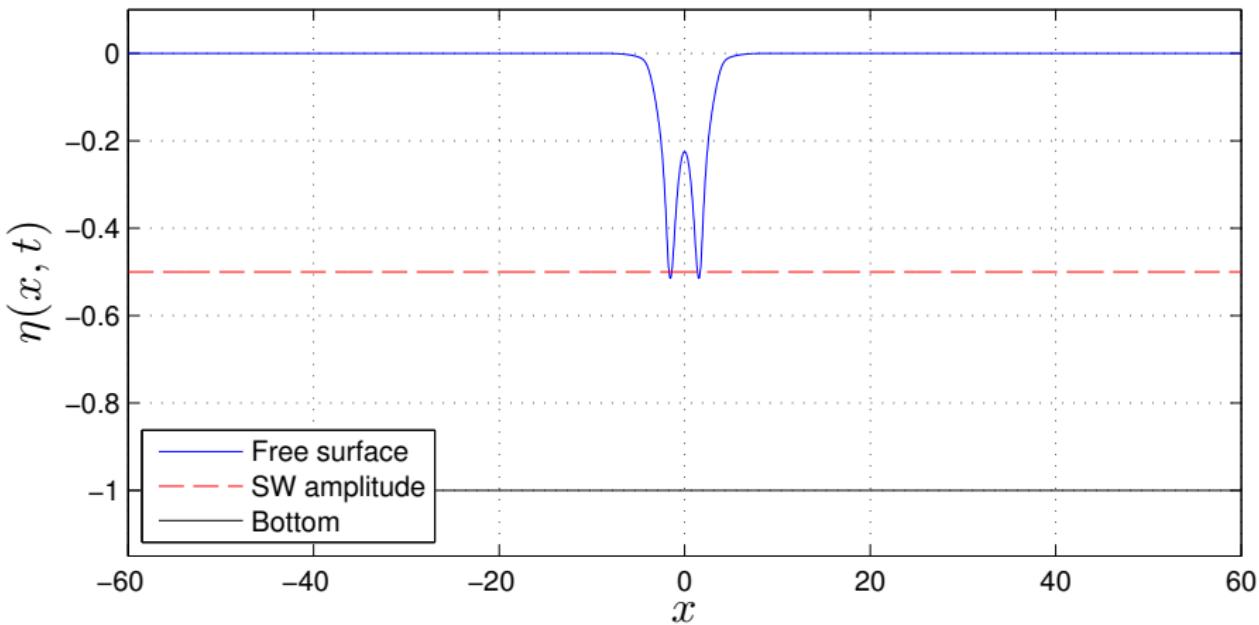
Free surface $\eta(x,t)$ at $t = 43.00$; $dt = 0.010$



Solitary waves collision

Solitary waves of depression: $a/d = -1/2$, $\text{Fr} = 1/2$, $\text{Bo} = 1/2$

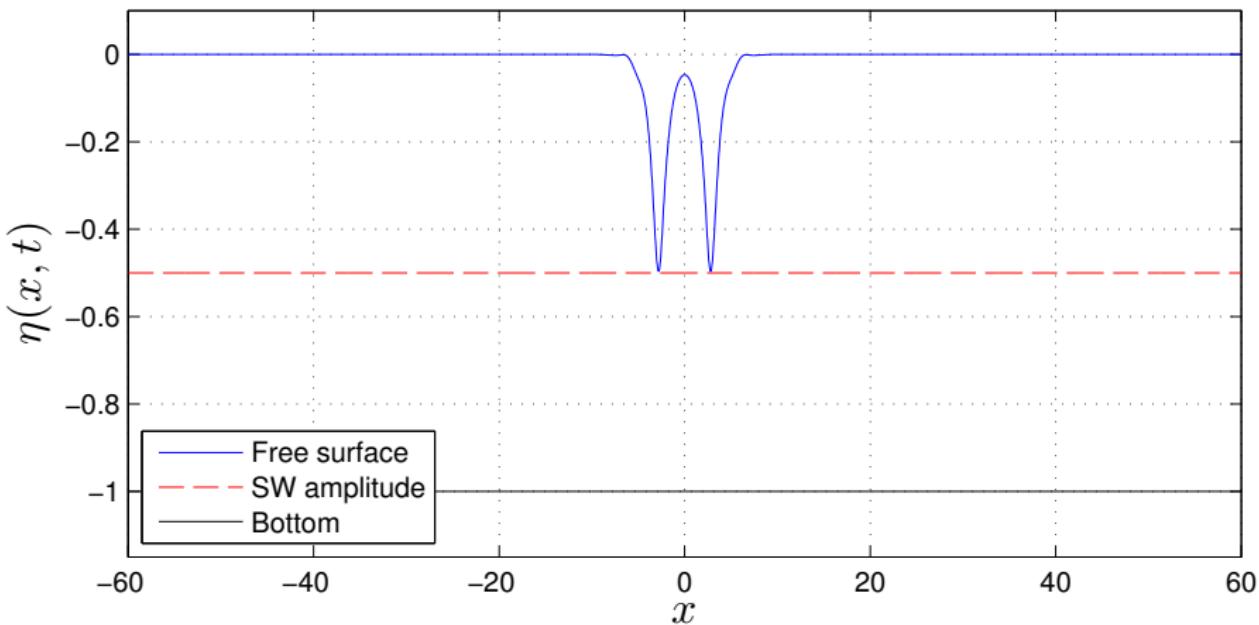
Free surface $\eta(x,t)$ at $t = 44.20$; $dt = 0.010$



Solitary waves collision

Solitary waves of depression: $a/d = -1/2$, $\text{Fr} = 1/2$, $\text{Bo} = 1/2$

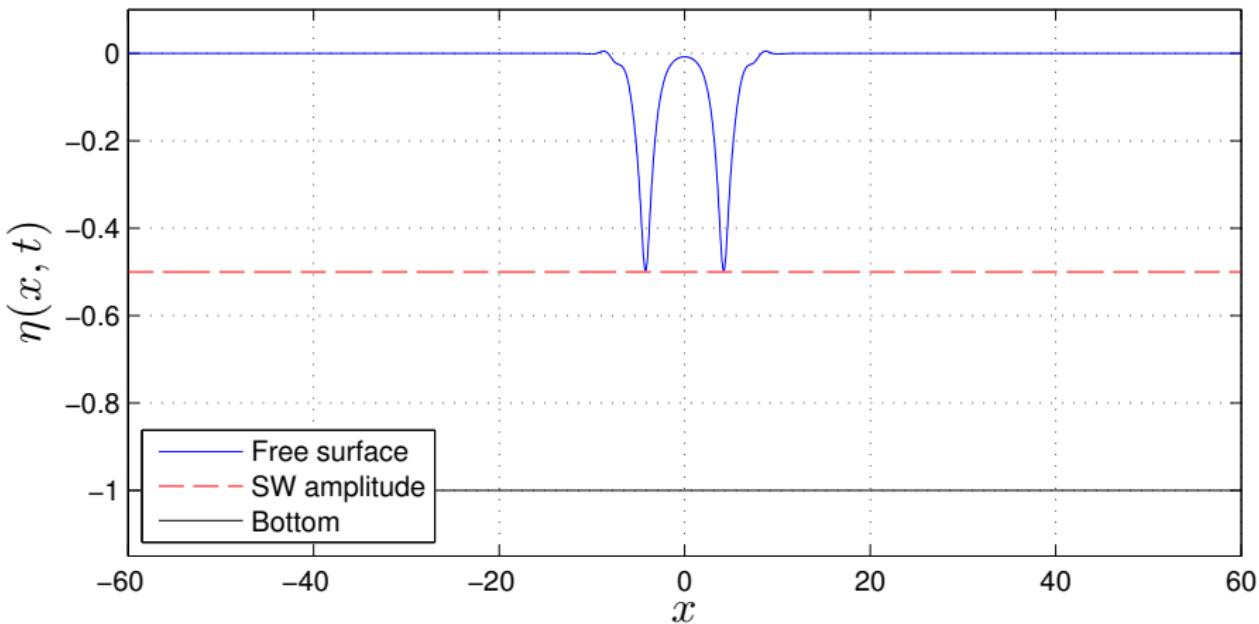
Free surface $\eta(x,t)$ at $t = 46.00$; $dt = 0.010$



Solitary waves collision

Solitary waves of depression: $a/d = -1/2$, $\text{Fr} = 1/2$, $\text{Bo} = 1/2$

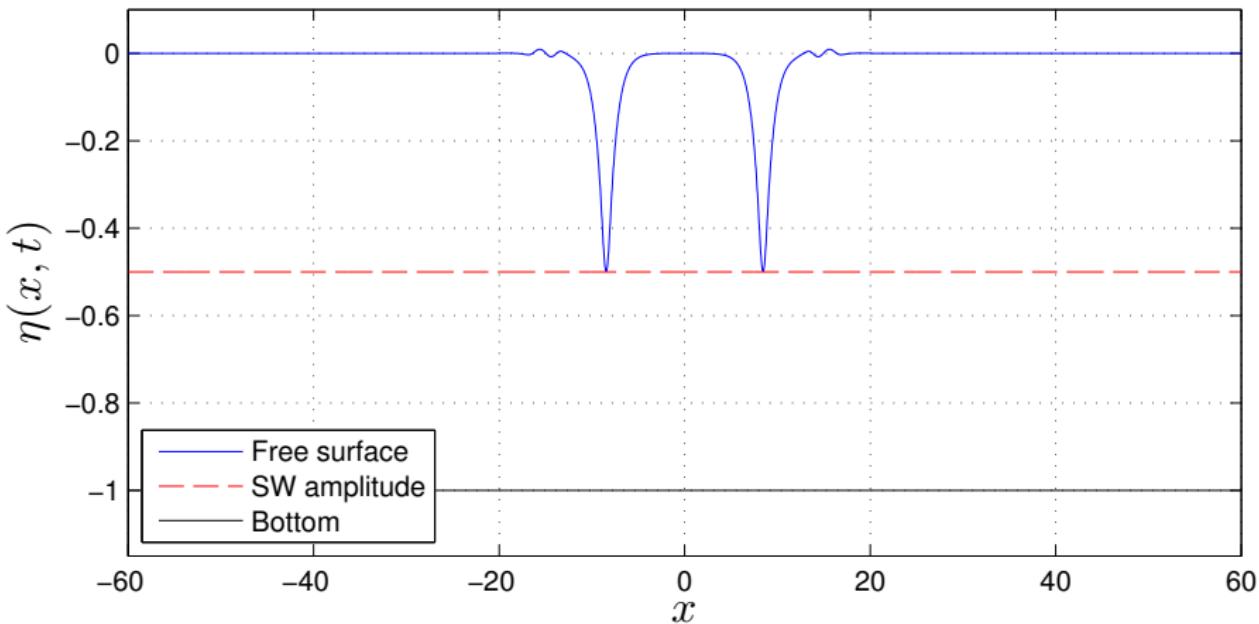
Free surface $\eta(x,t)$ at $t = 48.00$; $dt = 0.010$



Solitary waves collision

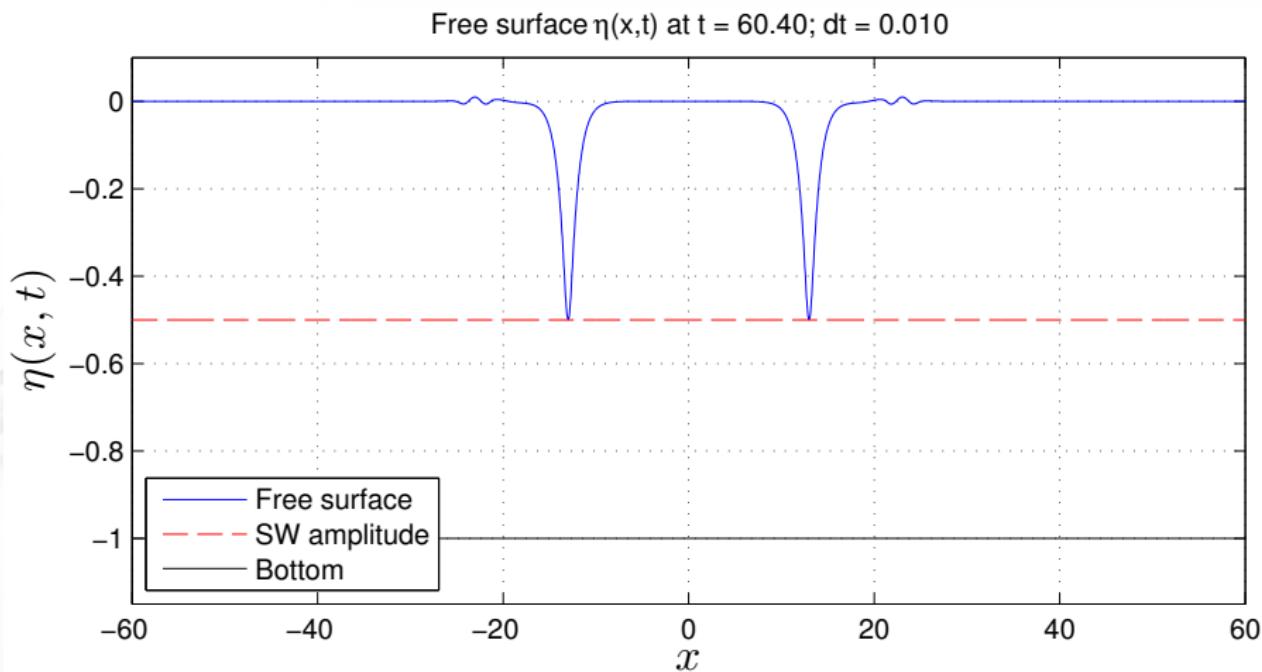
Solitary waves of depression: $a/d = -1/2$, $\text{Fr} = 1/2$, $\text{Bo} = 1/2$

Free surface $\eta(x,t)$ at $t = 54.00$; $dt = 0.010$



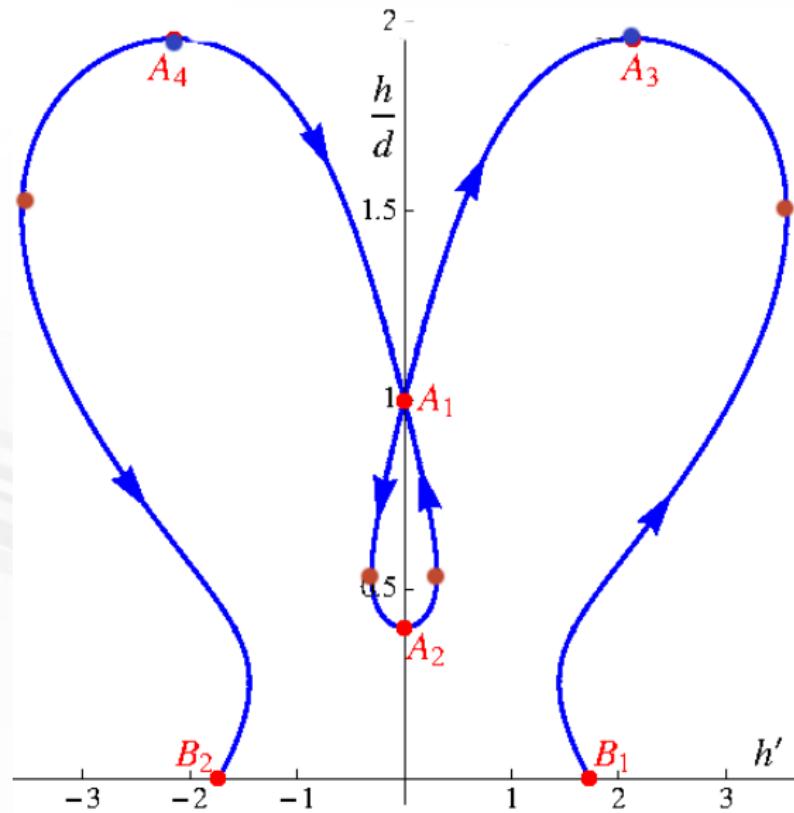
Solitary waves collision

Solitary waves of depression: $a/d = -1/2$, $\text{Fr} = 1/2$, $\text{Bo} = 1/2$



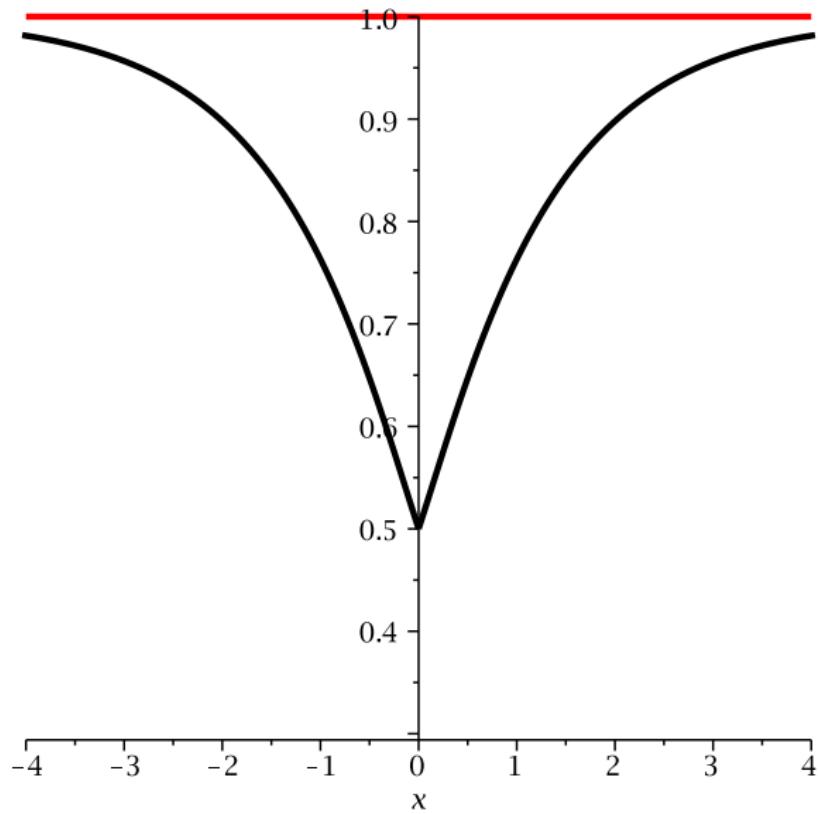
Phase-space analysis: peakon of depression

A particular example for $\text{Fr} = 0.4$, $\text{Bo} = 0.9 > 1/3$



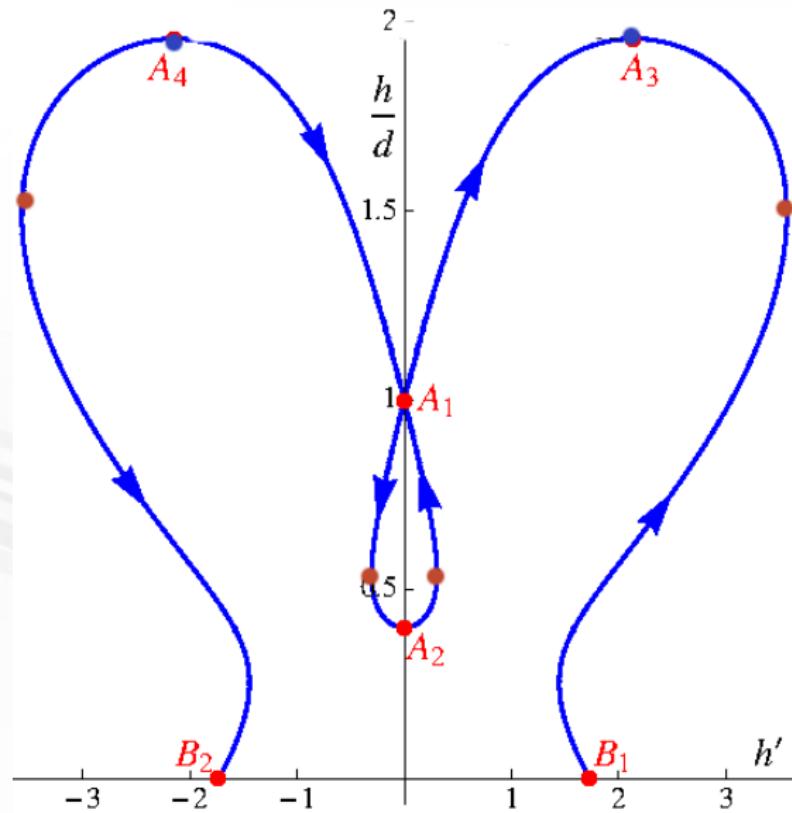
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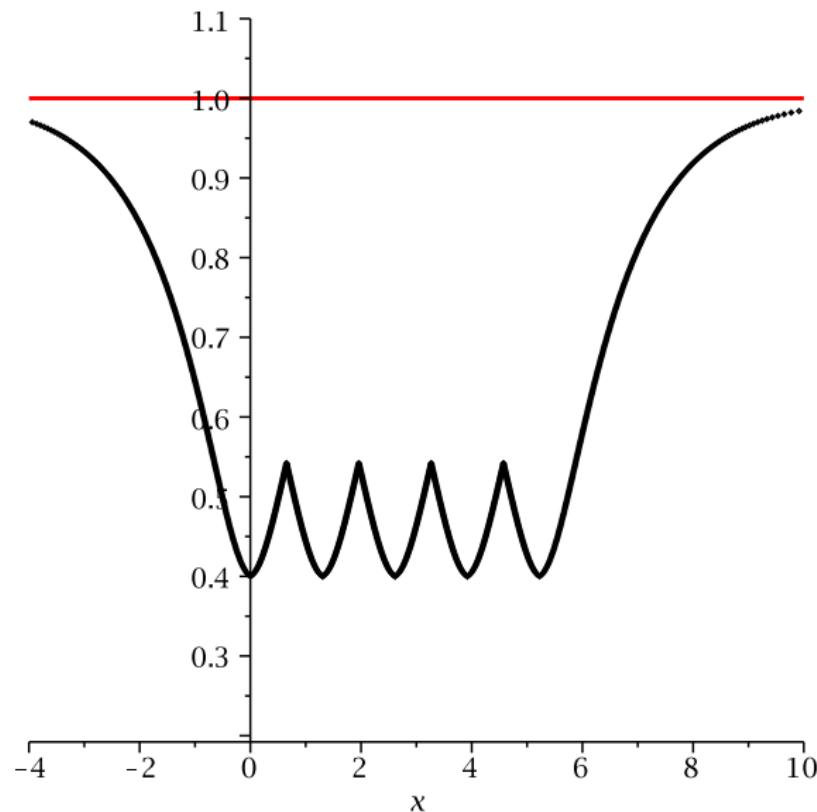
Phase-space analysis: multi-peakon of depression

A particular example for $\text{Fr} = 0.4$, $\text{Bo} = 0.9 > 1/3$



Phase-space analysis: multi-peakon of depression

A particular example for $\text{Fr} = 0.4$, $\text{Bo} = 0.9 > 1/3$



1 Shallow capillary-gravity waves

- Travelling waves
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2 Extended phase-space analysis

Phase space for roots of polynomials

Geometric interpretation for quadratic equations

- Polynomial quadratic equation: $x^2 + ax + b = 0$
- Coefficients as parameters:

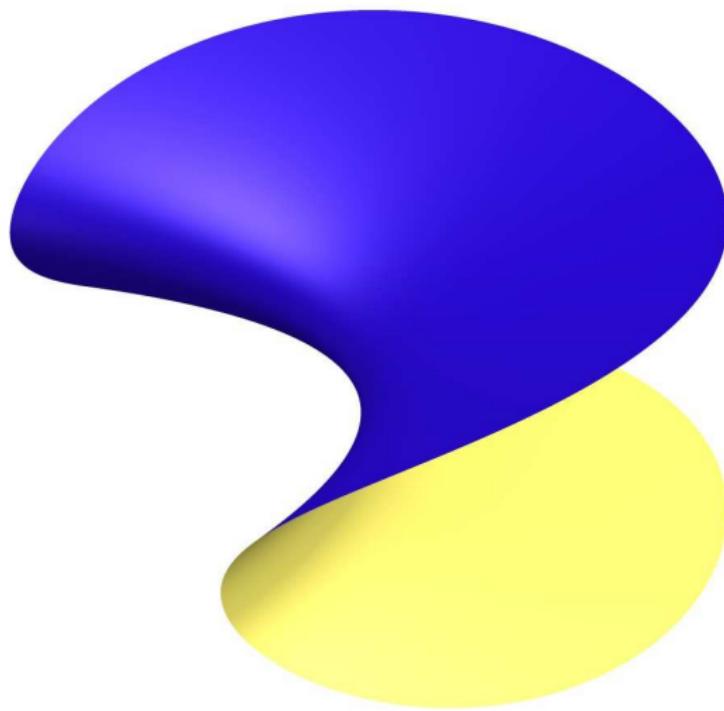
$$\begin{aligned}\mathcal{P} : \mathbb{R}^3 &\mapsto \mathbb{R} \\ (a, b, x) &\mapsto x^2 + ax + b\end{aligned}$$

- Consider an algebraic surface $\mathcal{S} \subseteq \mathbb{R}^3$:
$$(a, b, x) \in \mathcal{S} \iff \mathcal{P}(a, b, x) = x^2 + ax + b = 0$$
- Consider the fibers of the map:
$$F : (a, b) \mapsto x \in \mathcal{S}, \quad F(a, b) := \{x \mid (a, b, x) \in \mathcal{S}\}$$
- Count the cardinals:

$$|F(a, b)| \in \{0, 1, 2\}$$

Corresponding algebraic surface

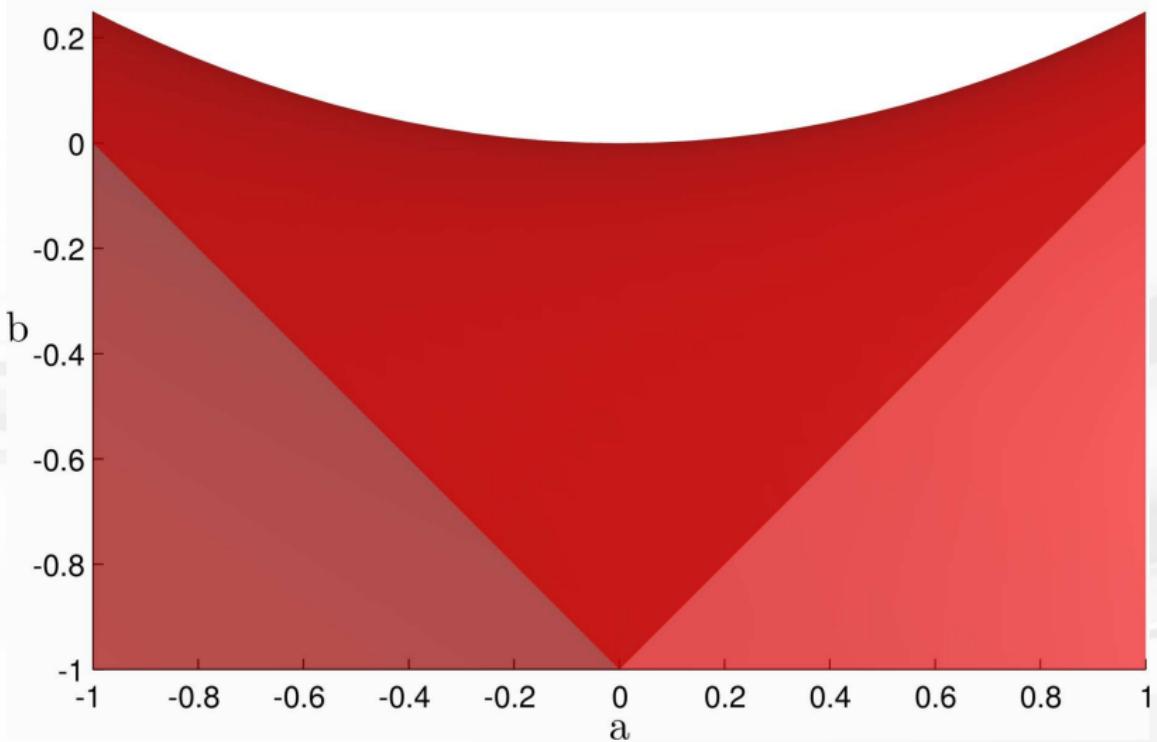
Quadratic polynomial: $x^2 + ax + b = 0$



Corresponding algebraic surface

Quadratic polynomial: $x^2 + ax + b = 0$

Quadratic equation: $x^2 + ax + b = 0$



Cubic equations

Geometric interpretation for cubic equations

- Consider a generic cubic polynomial equation:

$$x^3 + rx^2 + sx + t = 0$$

- Obtain the *reduced form* by substitution $x \rightarrow x - \frac{1}{3}r$

$$x^3 + ax + b = 0,$$

$$a := s - \frac{r^2}{3}, \quad b := \frac{2r^3}{27} - \frac{sr}{3} + t.$$

- By following Girolamo CARDANO (1501 – 1576), the discriminant is

$$\mathcal{D}^2 := \frac{b^2}{4} + \frac{a^3}{27}$$

Corresponding algebraic surface

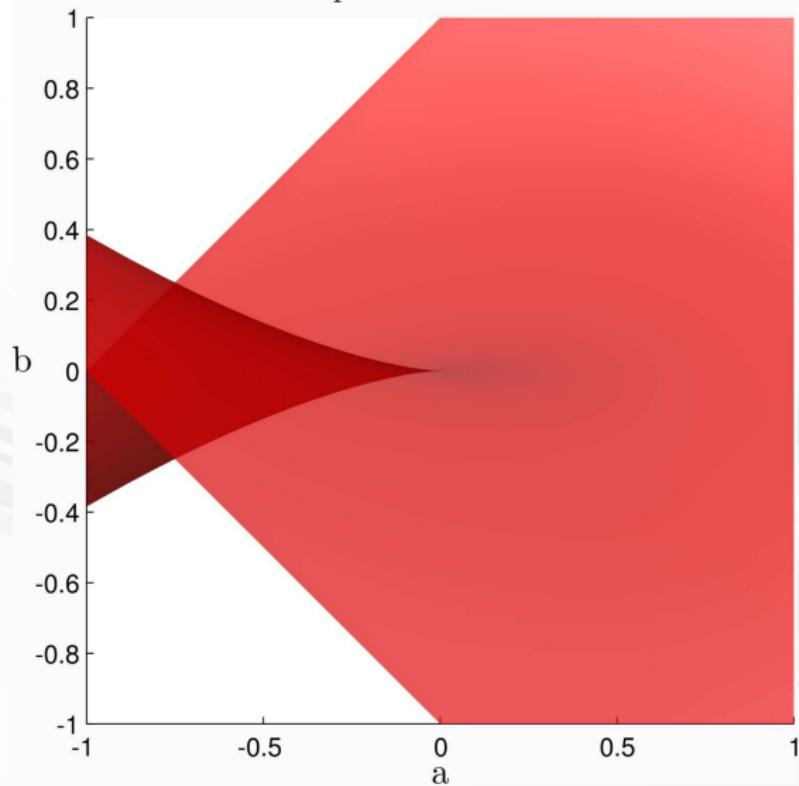
Cubic polynomial: $x^3 + ax + b = 0$



Corresponding algebraic surface

Cubic polynomial: $x^3 + ax + b = 0$

Cubic equation: $x^3 + ax + b = 0$



Phase space partition for higher order polynomials

The notion of the resultant of two polynomials

- Consider two polynomials $P, Q \in \mathbb{R}[x]$:

$$\deg P = n, \quad \deg Q = m$$

$$P(x) = a_0 x^n + \dots, \quad Q(x) = b_0 x^m + \dots$$

- Resultant of two polynomials is

$$\mathbf{R}(P, Q) := a_0^m b_0^n \prod_{i=1}^n \prod_{j=1}^m (r_i - s_j)$$

where r_i are zeros of $P(x)$ and s_j are zeros of $Q(x)$.

- Resultant is the determinant of the Sylvester matrix!
- Discriminant of a polynomial equation $P(x) = 0$ is

$$\mathcal{D}(P) := (-1)^{\frac{n(n-1)}{2}} \mathbf{R}(P, P')$$

Phase space analysis: global behaviour

Detection of multiple points

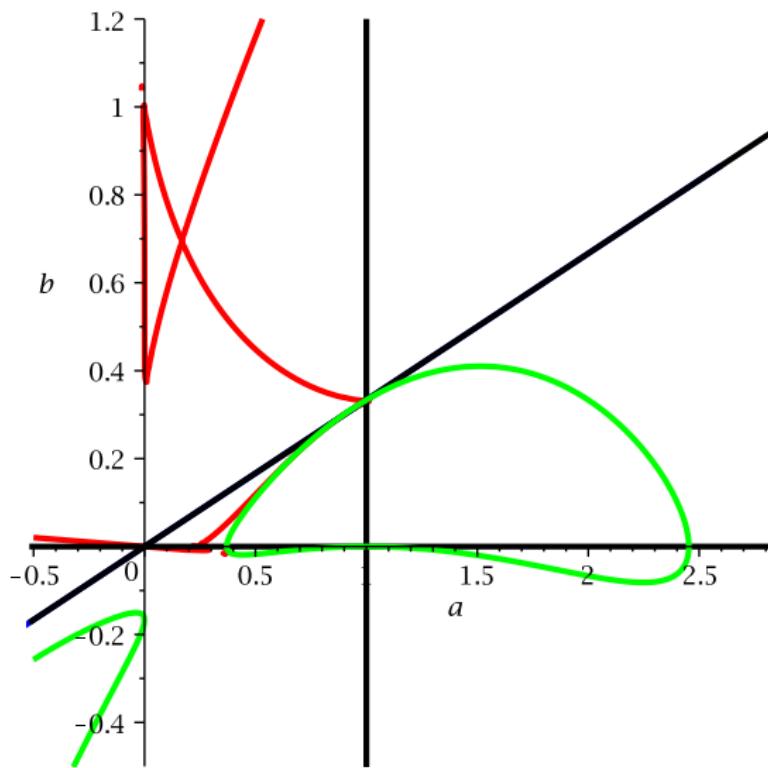
- Points with horizontal tangent satisfy:
 - $F_{Fr, Bo}(k, h) = 0$
 - $\partial_k F_{Fr, Bo}(k, h) = 0$
- The 2nd equation can be solved analytically:
 - $k = 0$
 - $Fr(k^2 + 1)^3 = 9Bo^2h^2$
- To avoid cubic roots, change of variables:
 - $k^2 = y^2 - 1, y \geq 1$
 - Wave height can be expressed as $h = \frac{Fr}{3Bo} Y^3$
- Polynomial equation in y :

$$f := Fr^2 y^9 - (3Fr - 2)Fr Bo y^6 + \\ 9Bo^2(1 + 2Fr - 2Bo)y^3 + 27Bo^3y^2 - 36Bo^3$$

- Adapted tool to describe the real roots in (Fr, Bo) space:
discriminant locus! $\mathcal{D} = (Fr - 3Bo)^2 \times \mathbb{P}_{10}$

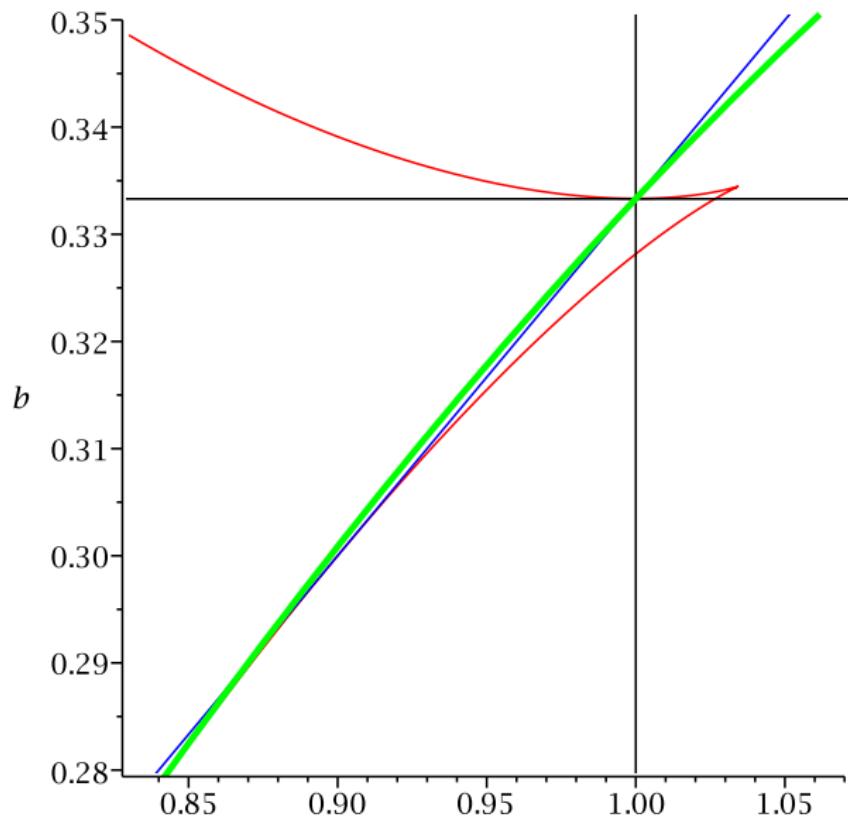
Phase space analysis: global behaviour

Contains ≈ 11 cells with 0 to 3 real roots such as $y > 1$



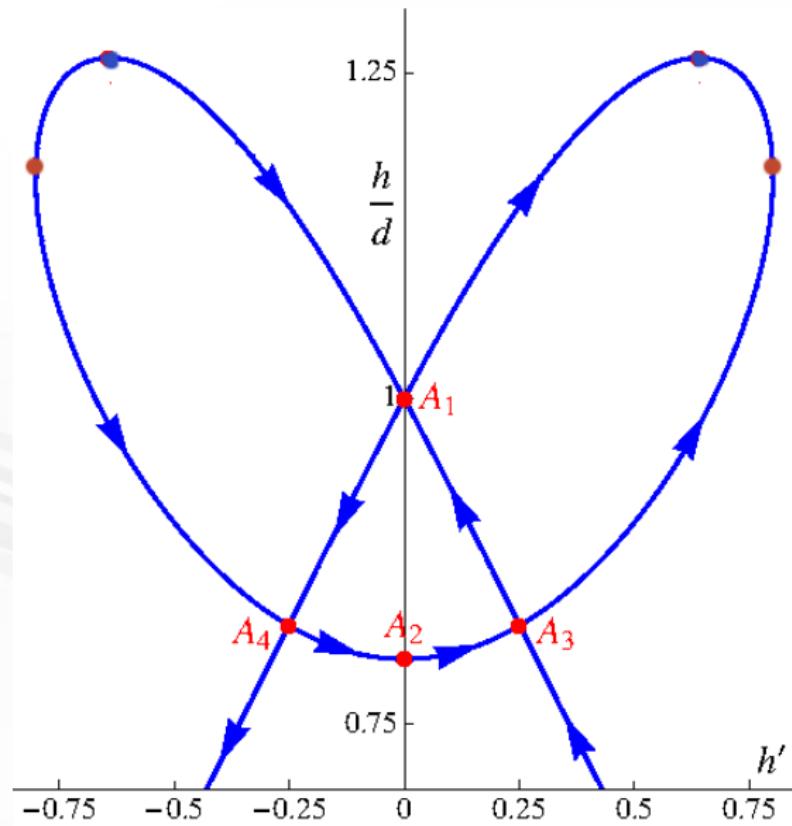
Phase space analysis: global behaviour

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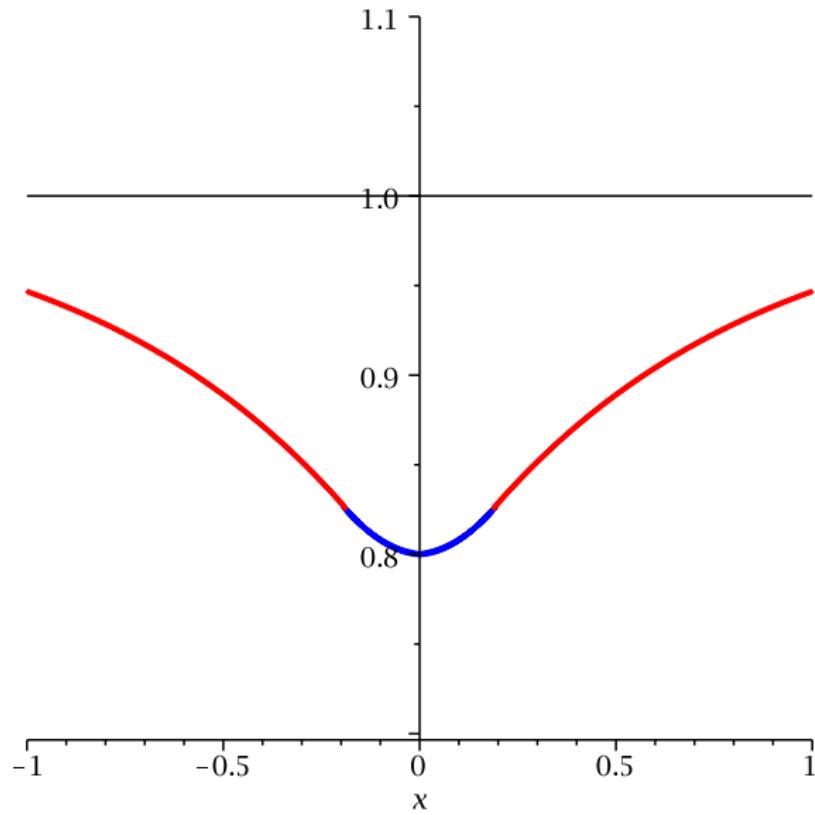
Phase-space analysis: weakly singular solitary wave

A particular example for $\text{Fr} = 0.8$, $\text{Bo} = 0.3538557 > 1/3$



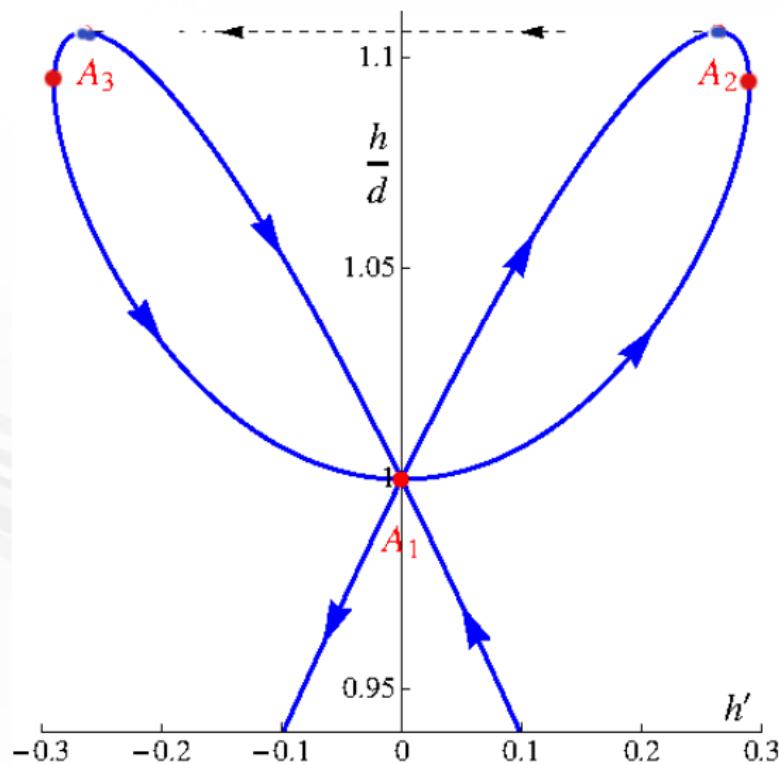
Phase-space analysis: weakly singular solitary wave

A particular example for $\text{Fr} = 0.8$, $\text{Bo} = 0.3538557 > 1/3$



Phase-space analysis: wave with algebraic decay

A particular example for $\text{Fr} = 1.0$, $\text{Bo} = 1/3$



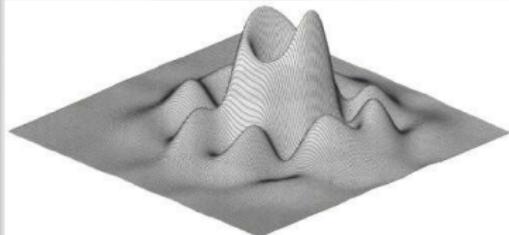
Conclusions & Perspectives

Conclusions:

- Capillary-gravity solitary waves were analyzed in shallow water regime
- Fully nonlinear & weakly dispersive model
 - Phase-space analysis using the methods of the algebraic geometry
- Only two types of regular solitary waves ($+a$, $-a$)

Perspectives:

- Analysis of periodic CG-waves
- Go to 3D !
 - Compute fully nonlinear lump-solitary waves



Thank you for your attention!



<http://www.denys-dutykh.com/>

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