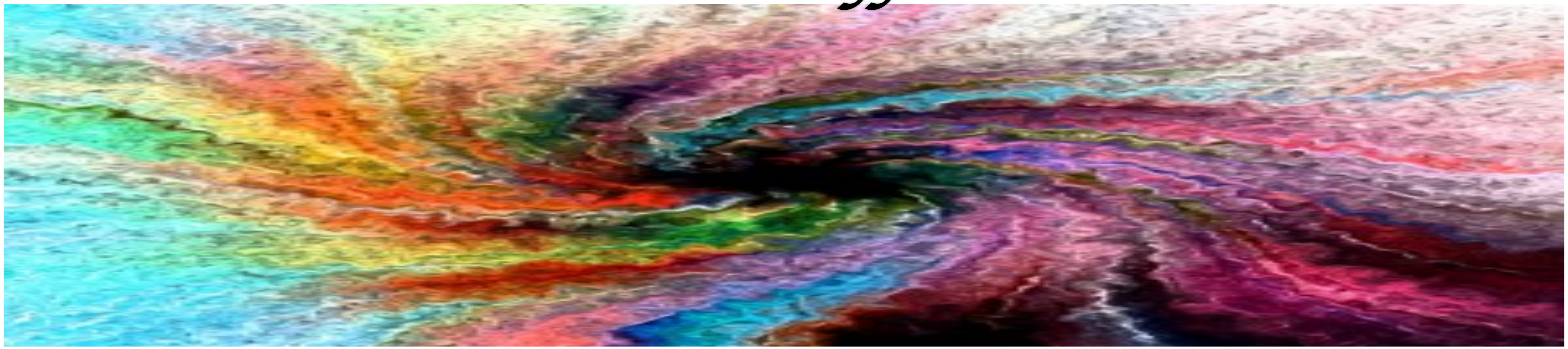
Dark Energy and Dark Matter as Curvature Effects



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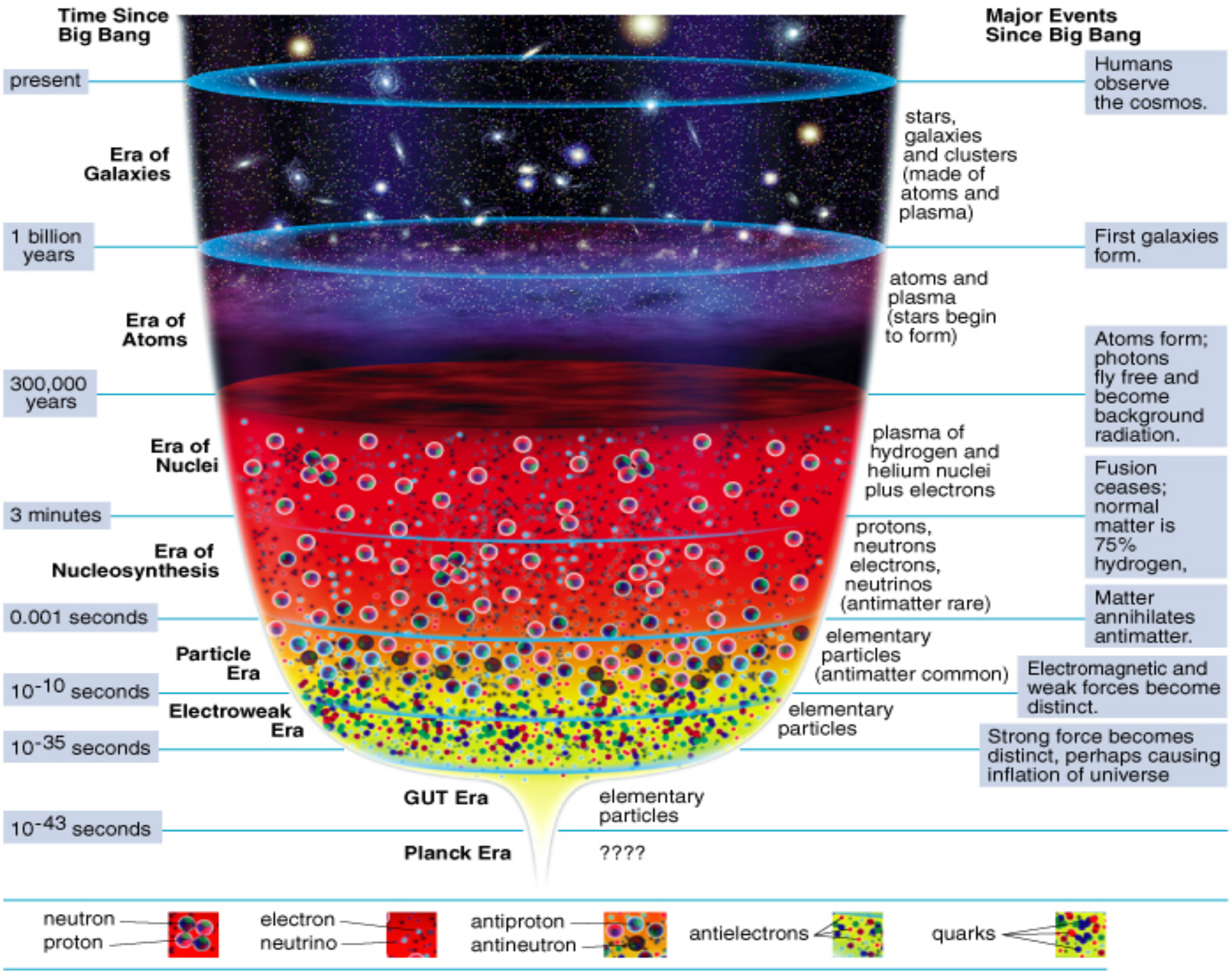
Summary

- Dark Energy and Dark matter problems
- Extending General Relativity
- Conformal transformations
- The weak field limit in f(R)-gravity
- Stellar structures and Jeans instability
- Quadrupolar gravitational radiation in f(R)-gravity
- Testing spiral galaxies
- Testing elliptical galaxies
- Modeling clusters of galaxies
- Cosmography with f(R)-gravity
- Conclusions

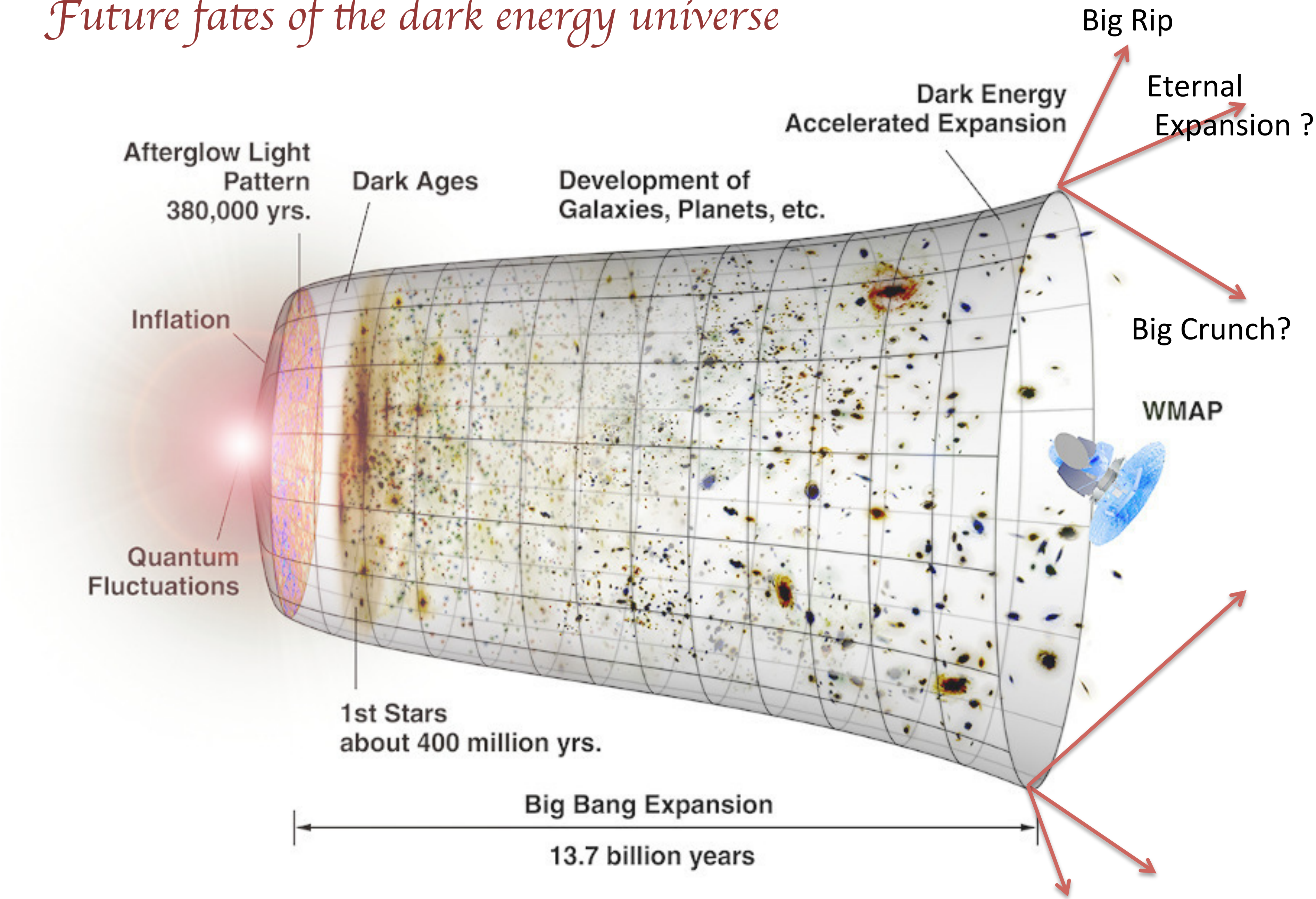
The content of the universe is, up today, absolutely unknown for its largest part. The situation is very "DARK" while the observations are extremely good!

Components of the Universe

The Observed Universe Evolution



Future fates of the dark energy universe



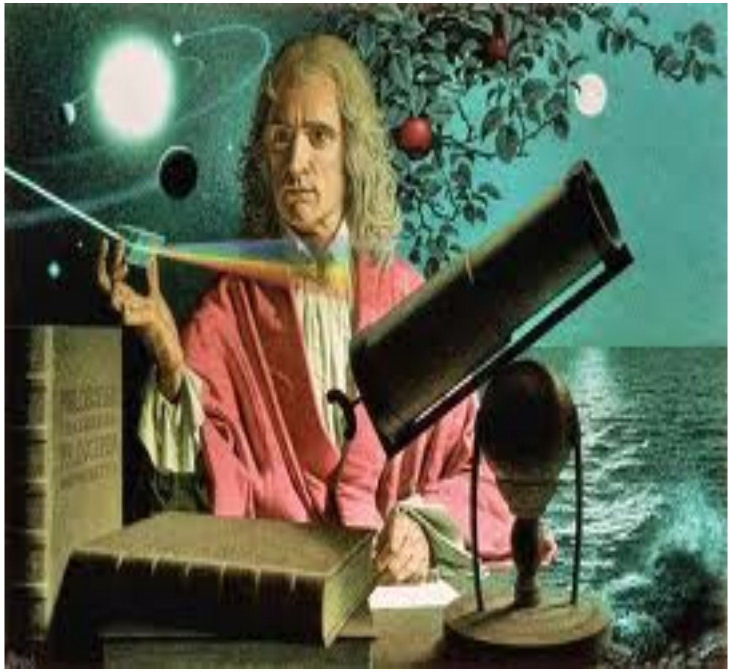
A plethora of theoretical models!!

DARK MATTER

- Neutrinos
- WIMPs
- Wimpzillas, Axions, the "particle forest".....
- MOND
- MACHOS
- Black Holes
- .....

DARK ENERGY

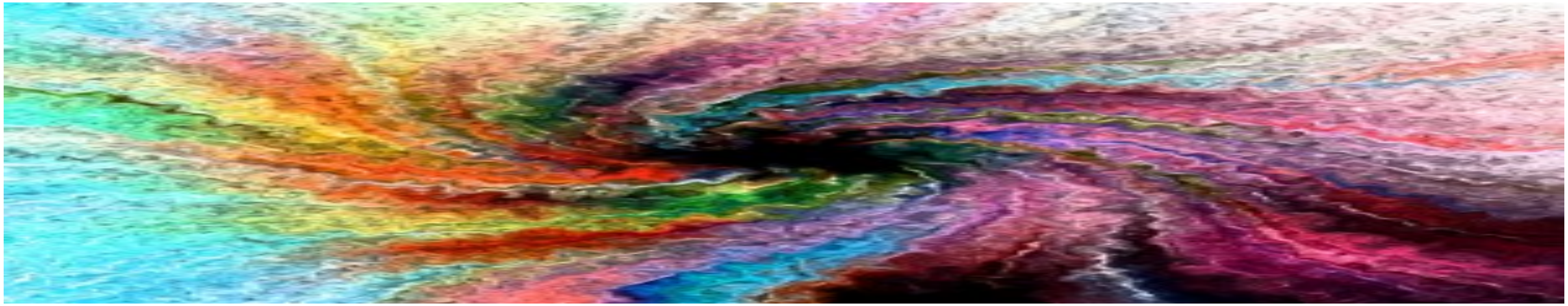
- Cosmological constant
- Scalar field Quintessence
- Phantom fields
- String-Dilaton scalar field
- Braneworlds
- Unified theories
- .....



"…there are the ones that invent OCCULT FLUIDS to understand the Laws of Nature. They will come to conclusions, but they now run out into DREAMS and CHIMERAS neglecting the true constitution of things…..

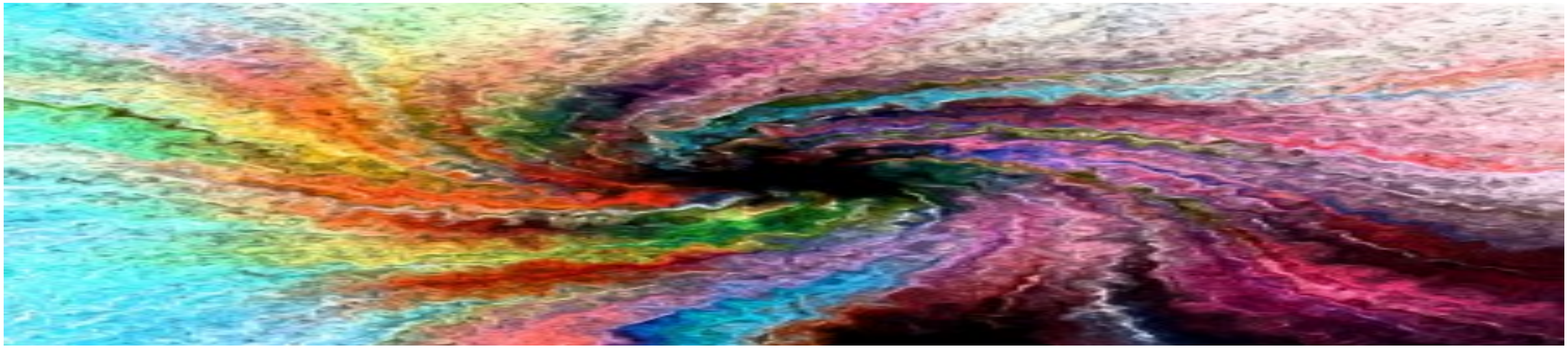
…however there are those that from the simplest observation of Nature, they reproduce New Forces (i.e. New Theories)… "

From the Preface of PRINCIPIA (II Edition) 1687 by Isaac Newton, written by Mr. Roger Cotes

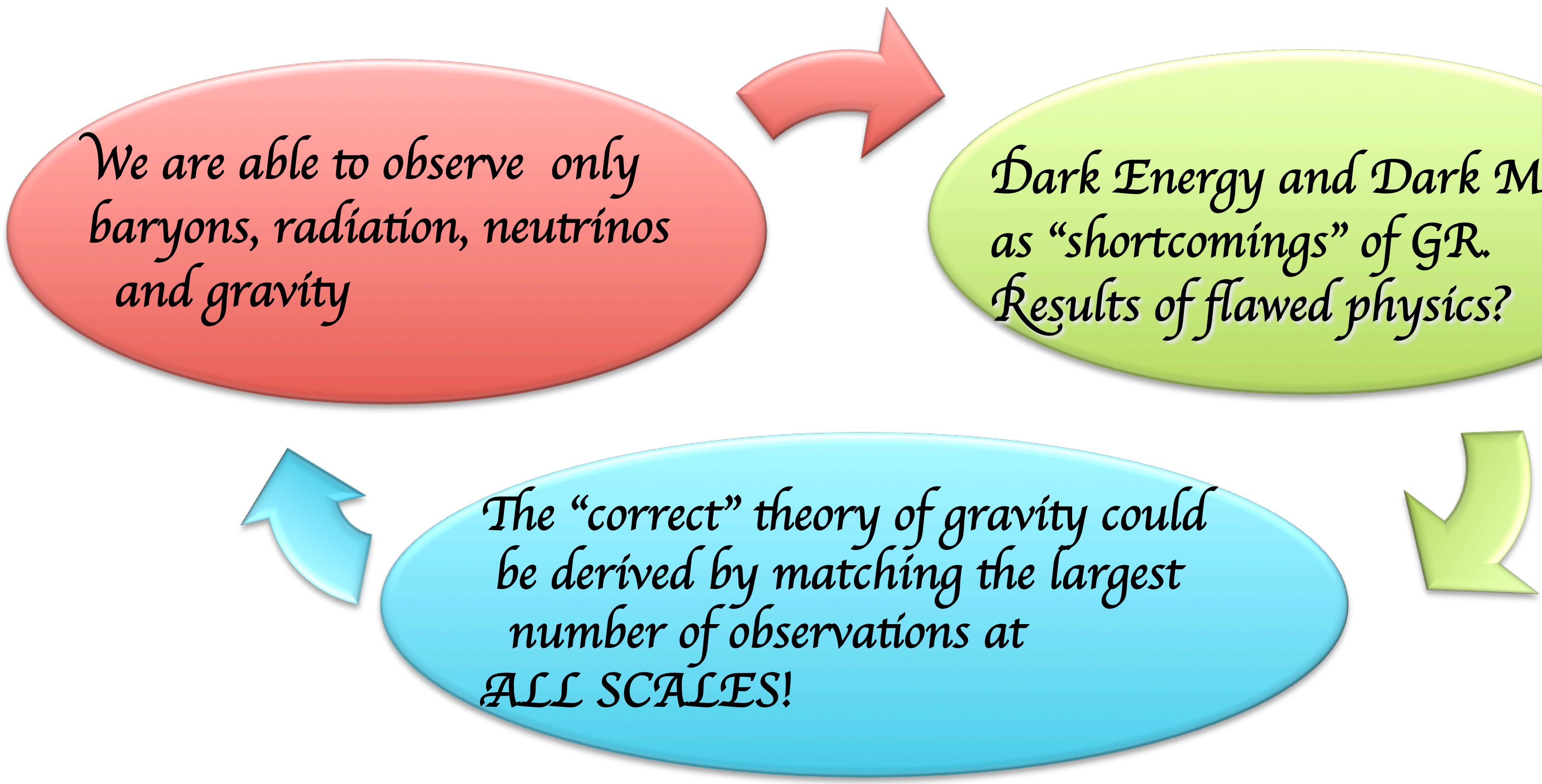


There is a fundamental issue:

Are extragalactic observations and cosmology probing the breakdown of General Relativity at large (IR) scales?



The problem could be reversed



Accelerating behaviour (DE) and dynamical phenomena (DM) as CURVATURE EFFECTS

Extending General Relativity

In order to extend General Relativity we will consider two main features:

- the geometry can couple non-minimally to matter and some scalar field;
- higher than second order derivatives of the metric may appear into dynamics

In the first case, we say that we have scalar-tensor gravity, and in the second case we have higher-order theories

A. A. Starobinsky, Phys. Lett. B91, 99 (1980).
S. Capozziello, Int. Jou. Mod. Phys. D 11, 483 (2002) .
A. De Felice, S Tsujikawa, Living Rev.Rel. 13 (2010) 3
S. Capozziello, M. De Laurentis, Phys. Rep. 509, 167 (2011).
S. Nojiri, S.D. Odintsov, Phys. Rep. 505, 59 (2011).

Extending General Relativity

A general class of higher-order-scalar-tensor theories in four dimensions is given by the action

$$\mathcal{S} = \int d^4x\,\sqrt{-g}\left[F\left(R, \Box R, \Box^2 R, \ldots, \Box^k R, \phi\right) - \frac{\epsilon}{2} g^{\mu\nu}\phi_{;\mu}\phi_{;\nu} + \mathcal{L}^{(m)}\right]$$

In the metric approach, the field equations are obtained by varying with respect to $g_{\mu\nu}$

$$\begin{aligned} G^{\mu\nu} = \frac{1}{\mathcal{G}}\Bigg[& \kappa T^{\mu\nu} + \frac{1}{2} g^{\mu\nu}(F - \mathcal{G}R) \\ & + (g^{\mu\lambda}g^{\nu\sigma} - g^{\mu\nu}g^{\lambda\sigma})\mathcal{G}_{;\lambda\sigma} \\ & + \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{i}(g^{\mu\nu}g^{\lambda\sigma} + g^{\mu\lambda}g^{\nu\sigma})(\Box^{j-i})_{;\sigma} \\ & \times \left(\Box^{i-j}\frac{\partial F}{\partial \Box^i R}\right)_{;\lambda} - g^{\mu\nu}g^{\lambda\sigma} \\ & \times \left((\Box^{j-1}R)_{;\sigma}\Box^{i-j}\frac{\partial F}{\partial \Box^i R}\right)_{;\lambda}\Bigg], \end{aligned}$$

- $G^{\mu\nu}$ is the Einstein tensor

and

$$\mathcal{G} \equiv \sum_{j=0}^{n} \Box^j \left(\frac{\partial F}{\partial \Box^j R}\right)$$

Extending General Relativity

- The simplest extension of GR is achieved assuming $F = f(R)$, $\varepsilon = 0$, in the action
- The standard Hilbert–Einstein action is recovered for $f(R) = R$

By varying with respect to $g_{\mu\nu}$, we get

$$f'(R)R_{\mu\nu} - \frac{f(R)}{2}g_{\mu\nu} = \nabla_\mu\nabla_\nu f'(R) - g_{\mu\nu}\Box f'(R)$$

and, after some manipulations

$$G_{\mu\nu} = \frac{1}{f'(R)}\left\{\nabla_\mu\nabla_\nu f'(R) - g_{\mu\nu}\Box f'(R) + g_{\mu\nu}\frac{[f(R) - f'(R)R]}{2}\right\}$$

where the gravitational contribution due to higher-order terms can be reinterpreted as a "curvature" stress-energy tensor related to the form of $f(R)$.

Such a tensor disappears for $f(R)=R$

Extending General Relativity

Considering also the standard perfect-fluid matter contribution, we have

$$G_{\alpha\beta} = \frac{1}{f'(R)}\left\{\frac{1}{2}g_{\alpha\beta}\left[f(R) - Rf'(R)\right] + f'(R)_{;\alpha\beta} - g_{\alpha\beta}\Box f'(R)\right\} + \frac{\kappa T^{(m)}_{\alpha\beta}}{f'(R)} = T^{(\mathrm{curv})}_{\alpha\beta} + \frac{T^{(m)}_{\alpha\beta}}{f'(R)}$$

In the case of GR, identically vanishes while the standard, minimal coupling is recovered for the matter contribution

is an effective stress-energy tensor constructed by the extra curvature terms

The peculiar behavior of $f(R) = R$ is due to the particular form of the Lagrangian itself which, even though it is a second-order Lagrangian, can be non-covariantly rewritten as the sum of a first-order Lagrangian plus a pure divergence term.

Extending General Relativity

From the general action it is possible to obtain another interesting case by choosing

$$F = F(\phi)R - V(\phi), \qquad \epsilon = -1$$

In this case, we get

$$\mathcal{S} = \int V(\phi)\left[F(\phi)R + \frac{1}{2}g^{\mu\nu}\phi_{;\mu}\phi_{;\nu} - V(\phi)\right]$$

The variation with respect to $g_{\mu\nu}$ gives the second-order field equations

$$F(\phi)G_{\mu\nu} = F(\phi)\left[R_{\mu\nu} - \frac{1}{2}R_{\mu\nu}\right] = -\frac{1}{2}T^{\phi}_{\mu\nu} - g_{\mu\nu}\Box_g F(\phi) + F(\phi)_{;\mu\nu}$$

The energy-momentum tensor related to the scalar field is

$$T^{\phi}_{\mu\nu} = \phi_{;\mu}\phi_{;\nu} - \frac{1}{2}g_{\mu\nu}\phi_{;\alpha}\phi^{\alpha}_{;} + g_{\mu\nu}V(\phi)$$

The variation with respect to ϕ provides the Klein-Gordon equation, i.e. the field equation for the scalar field:

$$\Box_g\phi - RF_{\phi}(\phi) + V_{\phi}(\phi) = 0$$

This last equation is equivalent to the Bianchi contracted identity

Conformal transformations

Conformal transformations are mathematical tools that are very useful in Extended Theories of Gravity in order to disentangle the further gravitational degrees of freedom coming from extended actions.

The idea is to perform a conformal rescaling of the space-time metric.

Conformal transformations

Let the pair $\{M, g_{\mu\nu}\}$ be a space-time, with M a smooth manifold of dimension $n \geq 2$ and $g_{\mu\nu}$ a Lorentzian or Riemannian metric on M.

The point-dependent rescaling of the metric tensor

$$g_{\mu\nu} \longrightarrow \tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$$

is a nowhere vanishing, regular function, is called a Weyl or conformal transformation

- Due to this metric rescaling, the lengths of spacelike and timelike intervals and the norms of spacelike and timelike vectors are changed, while null vectors and null intervals of the metric $g_{\mu\nu}$ remain null in the rescaled metric
- The light cones are left unchanged by the transformation
- A vector that is timelike, spacelike, or null with respect to the metric $g_{\mu\nu}$ has the same character with respect to rescaling metric and vice versa

Conformal transformations

In general, tensorial quantities are not invariant under conformal transformations, neither are the tensorial equations describing geometry and physics

In fact, the Christoffel symbols are

$$\tilde{\Gamma}^{\sigma}_{\lambda\mu} = \Gamma^{\sigma}_{\lambda\mu} + g^{\nu\sigma}\left(\frac{\partial\omega}{\partial x^{\lambda}}g_{\mu\nu} + \frac{\partial\omega}{\partial x^{\mu}}g_{\lambda\nu} - \frac{\partial\omega}{\partial x^{\nu}}g_{\lambda\mu}\right)$$

the Ricci tensor

$$\tilde{R}_{\alpha\beta} = R_{\alpha\beta} - 2\omega_{;\alpha;\beta} + 2\omega_{;\alpha}\omega_{;\beta} - g_{\alpha\beta}\Box\omega - 2g_{\alpha\beta}\omega_{;\gamma}\omega^{;\gamma}$$

The Ricci scalar $\tilde{R} = e^{-2\omega}\left(R - 6\Box\omega - 6\omega_{;\gamma}\omega^{;\gamma}\right)$

The only tensor that is invariant under conformal transformations is the Weyl tensor

$$\tilde{C}^{\alpha}_{\beta\gamma\delta} = C^{\alpha}_{\beta\gamma\delta}$$

Conformal transformations

Performing the conformal transformation on f(R) field equations we get

$$\tilde{G}_{\alpha\beta} = \frac{1}{f'(R)}\left\{\frac{1}{2}g_{\alpha\beta}\left[f(R) - Rf'(R)\right] + f'(R)_{;\alpha\beta} - g_{\alpha\beta}\Box f'(R)\right\} + 2\left(\omega_{;\alpha;\beta} + g_{\alpha\beta}\Box\omega - \omega_{;\alpha}\omega_{;\beta} + \frac{1}{2}g_{\alpha\beta}\omega_{;\gamma}\omega^{;\gamma}\right)$$

We can then choose the conformal factor to be $\omega = \frac{1}{2}\ln|f'(R)|$

Rescaling ω in such a way that kϕ = ω, and k =√ 1/6, we obtain the Lagrangian equivalence

$$\sqrt{-g}\,f(R) = \sqrt{-\tilde{g}}\left(-\frac{1}{2}\tilde{R} + \frac{1}{2}\tilde{\phi}_{;\alpha}\tilde{\phi}^{\alpha}_{;} - \tilde{V}\right)$$

and the Einstein equations in standard form $\tilde{G}_{\alpha\beta} = \phi_{;\alpha}\phi_{;\beta} - \frac{1}{2}\tilde{g}_{\alpha\beta}\phi_{;\gamma}\phi^{;\gamma} + \tilde{g}_{\alpha\beta}V(\phi)$

With the potential

$$V(\phi) = \frac{e^{-4k\phi}}{2}\left[\mathcal{P}(\phi) - \mathcal{N}\left(e^{2k\phi}\right)e^{2k\phi}\right] = \frac{1}{2}\frac{f(R) - Rf'(R)}{f'(R)^2}.$$

Here N is the inverse function of P'(ϕ) and $\mathcal{P}(\phi) = \int \exp(2k\phi)\mathrm{d}\mathcal{N}$

However, the problem is completely solved if P'(ϕ) can be analytically inverted

In summary, a fourth-order theory is conformally equivalent to the standard second-order Einstein theory plus a scalar field

Conformal transformations

This procedure can be extended to more general theories. If the theory is assumed to be higher than fourth order, we may have Lagrangian densities of the form

$$\mathcal{L} = \mathcal{L}(R, \Box R, \ldots, \Box^k R)$$

Every $\Box$ operator introduces two further terms of derivation into the field equations.

For example a theory like

$$\mathcal{L} = R \Box R,$$

is a sixth-order theory and the above approach can be pursued by considering a conformal factor of the form

$$\omega = \frac{1}{2} \ln \left| \frac{\partial \mathcal{L}}{\partial R} + \Box \frac{\partial \mathcal{L}}{\partial \Box R} \right|$$

Conformal transformations

In general, increasing of two orders of derivation in the field equations (i.e., for every term $\Box R$), corresponds to adding a scalar field in the conformally transformed frame

A sixth-order theory can be reduced to an Einstein theory with two minimally coupled scalar fields; a 2n-order theory can be, in principle, reduced to an Einstein theory plus $(n-1)$-scalar fields

S. Gottlober, H-J Schmidt, and A A Starobinsky, Class. Quantum Grav. 7, 893 (1990)

Conformal transformations work at three levels:

(i) on the Lagrangian of the given theory;
(ii) on the field equations;
(iii) on the solutions.

They allow to classify gravitational degrees of freedom and reduce any higher-order theory to Einstein plus scalar fields

DM and DE vs ETGs

- *How we can use ETGs to address the problem of Self-gravitating structures ?*

- *Do we really need DM and DE?*

- *How gravity behaves at various infrared scales?*

The weak field limit in f(R)-gravity

We can deal with the Newtonian and the post-Newtonian limit of $f\,(R)$ gravity adopting the spherical symmetry

The solution of field equations can be obtained considering the general spherically symmetric metric:

$$ds^2 = g_{\sigma\tau} dx^\sigma dx^\tau$$

$$= g_{00}(x^0, r) dx^{0 2} - g_{rr}(x^0, r) dr^2 - r^2 d\Omega,$$

In order to develop the Newtonian limit, let us consider the perturbed metric With respect to a Minkowskian backgroun $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

The metric entries can be developed as:

$$\begin{cases} g_{tt}(t,r) \simeq 1 + g_{tt}^{(2)}(t,r) + g_{tt}^{(4)}(t,r), \\ g_{rr}(t,r) \simeq -1 + g_{rr}^{(2)}(t,r), \\ g_{\theta\theta}(t,r) = -r^2, \\ g_{\phi\phi}(t,r) = -r^2 \sin^2\theta, \end{cases}$$

The weak field limit in f(R)-gravity

We assume, analytic Taylor expandable $f(R)$ functions with respect to a certain value $R = R_0$:

$$f(R) = \sum_n \frac{f^n(R_0)}{n!}(R - R_0)^n \simeq f_0 + f_1 R + f_2 R^2 + f_3 R^3 + \ldots .$$

In order to obtain the weak field approximation, one has to insert expansions into field equations and expand the system up to the orders O(0), O(2) e O(4).

If we consider the O(2) - order approximation, the field equations in vacuum, results to be

$$\begin{cases} f_1 r R^{(2)} - 2 f_1 g^{(2)}_{tt,r} + 8 f_2 R^{(2)}_{,r} - f_1 r g^{(2)}_{tt,rr} + 4 f_2 r R^{(2)} = 0\,, \\ f_1 r R^{(2)} - 2 f_1 g^{(2)}_{rr,r} + 8 f_2 R^{(2)}_{,r} - f_1 r g^{(2)}_{tt,rr} = 0\,, \\ 2 f_1 g^{(2)}_{rr} - r \\ \times \left[f_1 r R^{(2)} - f_1 g^{(2)}_{tt,r} - f_1 g^{(2)}_{rr,r} + 4 f_2 R^{(2)}_{,r} + 4 f_2 r R^{(2)}_{,rr} \right] = 0\,, \\ f_1 r R^{(2)} + 6 f_2 \left[2 R^{(2)}_{,r} + r R^{(2)}_{,rr} \right] = 0\,, \\ 2 g^{(2)}_{rr} + r \left[2 g^{(2)}_{tt,r} - r R^{(2)} + 2 g^{(2)}_{rr,r} + r g^{(2)}_{tt,rr} \right] = 0\,. \end{cases} \qquad (33)$$

It is evident that the trace equation provides a differential equation with respect to the Ricci scalar which allows to solve exactly the system at O(2) - order

The weak field limit in f(R)-gravity

Finally, one gets the general solution:

Where $\xi \doteq \frac{f_1}{6f_2}$

$$\begin{cases} g_{tt}^{(2)} = \delta_0 - \frac{Y}{f_1 r} - \frac{\delta_1(t)e^{-r\sqrt{-\xi}}}{3\xi r} + \frac{\delta_2(t)e^{r\sqrt{-\xi}}}{6(-\xi)^{3/2}r} \\ g_{rr}^{(2)} = -\frac{Y}{f_1 r} + \frac{\delta_1(t)[r\sqrt{-\xi}+1]e^{-r\sqrt{-\xi}}}{3\xi r} \\ -\frac{\delta_2(t)[\xi r+\sqrt{-\xi}]e^{r\sqrt{-\xi}}}{6\xi^2 r} \\ R^{(2)} = \frac{\delta_1(t)e^{-r\sqrt{-\xi}}}{r} - \frac{\delta_2(t)\sqrt{-\xi}e^{r\sqrt{-\xi}}}{2\xi r} \end{cases}$$

For limit $f(R) \rightarrow R$, in the case of a point-like source of mass M, we recover the standard Schwarzschild solution

The two arbitrary functions of time $\delta_1(t)$ and $\delta_2(t)$ have respectively the dimensions of length^{-1} and length^{-2} .

They are completely arbitrary since the differential equation system contains only spatial derivatives and can be fixed to constant values.

The weak field limit in f(R)-gravity

In order to match at infinity the Minkowskian prescription for the metric, one can discard the Yukawa growing mode in and then we have:

$$\begin{cases} ds^2 = \left[1 - \dfrac{2GM}{f_1 r} - \dfrac{\delta_1(t) e^{-r\sqrt{-\xi}}}{3\xi r}\right] dt^2 \\ -\left[1 + \dfrac{2GM}{f_1 r} - \dfrac{\delta_1(t)(r\sqrt{-\xi}+1) e^{-r\sqrt{-\xi}}}{3\xi r}\right] dr^2 - r^2 d\Omega, \\ R = \dfrac{\delta_1(t) e^{-r\sqrt{-\xi}}}{r}. \end{cases}$$

In particular, since $g_{tt} = 1+2\Phi_{grav} = 1+ g(2)_{tt}$, the gravitational potential of $f(R)$-gravity, analytic in the Ricci scalar R, is

$$\Phi_{\text{grav}} = -\left(\frac{GM}{f_1 r} + \frac{\delta_1(t) e^{-r\sqrt{-\xi}}}{6\xi r}\right)$$

This general result means that the standard Newton potential is achieved only in the particular case $f(R) = R$ while it is not so for any analytic $f(R)$ models

The parameters $f_{1,2}$ and the function δ_1 represent the deviations with respect the standard Newton potential

S. Capozziello, M. De Laurentis Ann. Phys. 524, 545 (2012)

The weak field limit in f(R)-gravity

We note that the ξ parameter can be related to an effective mass being

$$m^2 = (3\xi)^{-1} = -\frac{f_1}{3f_2}$$

and can be interpreted also as an effective length $\mathcal{L}$

$$\Phi(r) = -\frac{GM}{(1+\delta)r}\left(1+\delta e^{-\frac{r}{L}}\right)$$

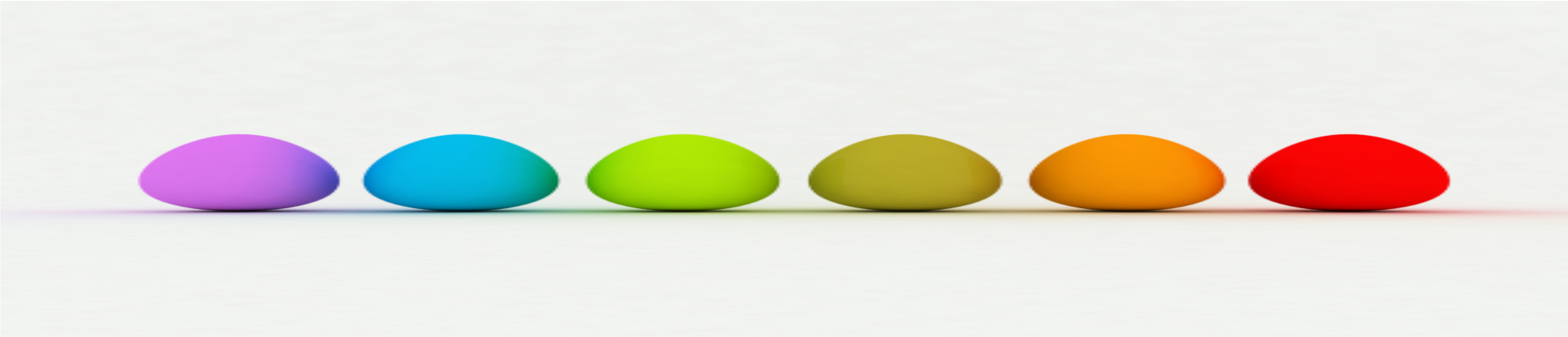
The second term is a modification of the gravity including a scale length

If $\delta = 0$ the Newtonian potential and the standard gravitational coupling are recovered.

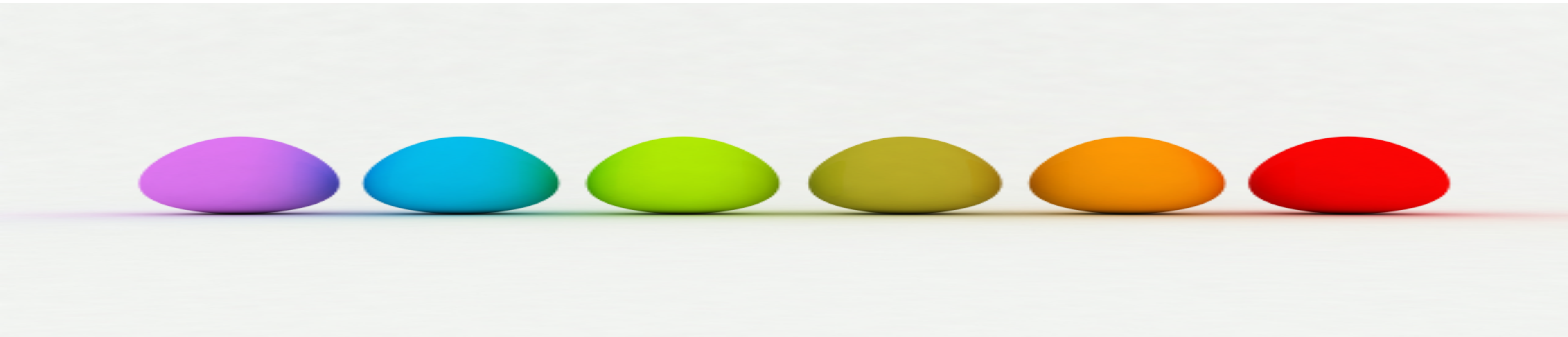
Assuming $1+\delta = f_1$, δ is related to $\delta_1(t)$ through

$$\delta_1 = -\frac{6GM}{L^2}\left(\frac{\delta}{1+\delta}\right)$$

Under this assumption, the scale length $\mathcal{L}$ could naturally arise and reproduce several phenomena that range from Solar System to cosmological scales.



Understanding on which scales the modifications to General Relativity are working or what is the weight of corrections to gravitational potential is a crucial point that could confirm or rule out these extended approaches to gravitational interaction.



Stellar structures and Jeans instability

It is usually assumed that the dynamics of stellar objects is completely determined by the Newton law of gravity

Considering potential corrections in strong field regimes could be another way to check the viability of Extended Theories of Gravity

In particular, stellar systems are an ideal laboratory to look for signatures of possible modifications of standard law of gravity

Some observed stellar systems are incompatible with the standard models of stellar structure : these are peculiar objects, as star in instability strips, protostars or anomalous neutron stars (the so-called "magnetars" with masses larger than their expected Volkoff mass) that could admit dynamics in agreement with modified gravity and not consistent with standard General Relativity

Stellar structures and Jeans instability

Field equations at O (2)-order, that is at the Newtonian level, are

$$R_{tt}^{(2)} - \frac{R^{(2)}}{2} - f''(0) \triangle R^{(2)} = \mathcal{X} T_{tt}^{(0)}$$

$$-3f''(0) \triangle R^{(2)} - R^{(2)} = \mathcal{X} T^{(0)},$$

$f^n(R)=f^n(R^{(2)}+O(4))=f^n(0)+f^{n+1}(0)R^{(2)}+\ldots$

The energy-momentum tensor for a perfect fluid is

$$T_{\mu\nu} = (\epsilon + p)u_\mu u_\nu - p g_{\mu\nu}.$$

The pressure contribution is negligible in the field equations of Newtonian approximation

$$\triangle \Phi + \frac{R^{(2)}}{2} + f''(0) \triangle R^{(2)} = -\mathcal{X}\rho$$

modified Poisson equation

$$3f''(0) \triangle R^{(2)} + R^{(2)} = -\mathcal{X}\rho,$$

S. Capozziello, M. De Laurentis Ann. Phys. 524, 545 (2012)

For $f'(R) = 0$ we have the standard Poisson equation $\triangle\Phi = -4\pi G\rho$

From the Bianchi identity we have $T^{\mu\nu}{}_{;\mu} = 0 \rightarrow \frac{\partial p}{\partial x^k} = -\frac{1}{2}(p+\epsilon)\frac{\partial \ln g_{tt}}{\partial x^k}.$

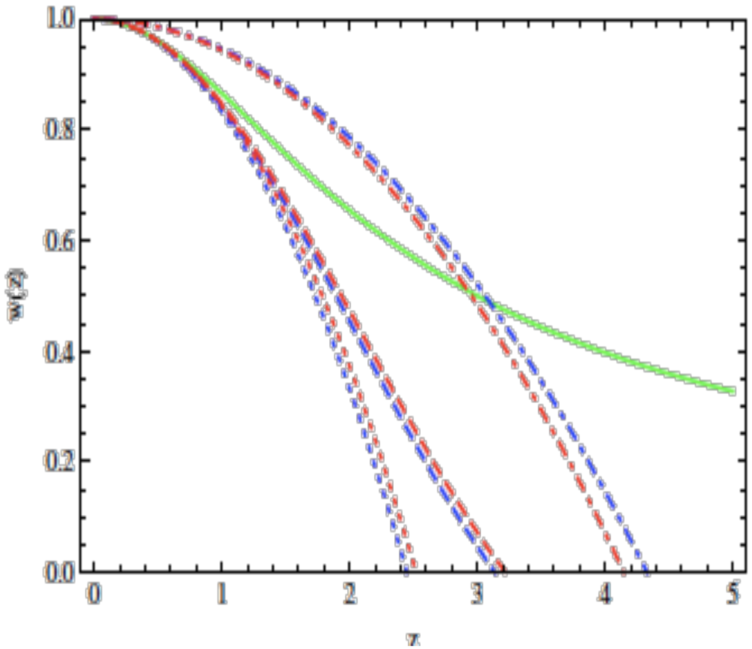
Stellar structures and Jeans instability

Let us suppose that matter satisfies a polytropic equation $p = K^{\gamma}\rho^{\gamma}$

We obtain an integro-differential equation for the gravitational potential, that is

$$\frac{d^2w(z)}{dz^2} + \frac{2}{z}\frac{dw(z)}{dz} + w(z)^n = \frac{m\xi_0}{8}\frac{1}{z}\int_0^{\xi/\xi_0} dz'\, z' \left\{ e^{-m\xi_0|z-z'|} - e^{-m\xi_0|z+z'|} \right\} w(z')^n$$

Lané-Emden equation in f(R)-gravity



We find the radial profiles of the gravitational potential by solving for some values of n (polytropic index)

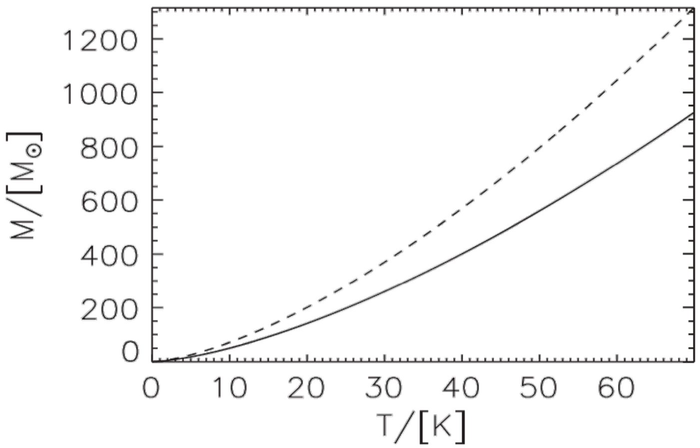
New solutions are physically relevant and could explain exotic systems out of Main Sequence (magnetars, variable stars).

S. Capozziello, M. De Laurentis, A. Stabile, S.D. Odintsov, PRD 83, 064004, (2011)

Stellar structures and Jeans instability

We have also compared the behavior with the temperature of the Jeans mass for various types of interstellar molecular clouds

$$\tilde{M}_J = 6\sqrt{\frac{6}{(3+\sqrt{21})^3}}M_J$$



In our model the limit (in unit of mass) to start the collapse of an interstellar cloud is lower than the classical one advantaging the structure formation.

S. Capozziello, M. De Laurentis, I. De Martino, M. Formisano, S.D. Odintsov
Phys.Rev. D85 (2012) 044022

Subject	T (K)	n (10^8 m^{-3})	μ	M_J ($M_\odot$)	$\tilde{M}_J$ ($M_\odot$)
Diffuse hydrogen clouds	50	5.0	1	795.13	559.68
Diffuse molecular clouds	30	50	2	82.63	58.16
Giant molecular clouds	15	1.0	2	206.58	145.41
Bok globules	10	100	2	11.24	7.91

Addressing stellar systems by this approach could be extremely important to test observationally Extended Theories of Gravity. Also gravitational waves could be a test-bed for these theories.

Quadrupolar gravitational radiation in f(R)-gravity

We calculate the Minkowskian limit for a class of analytic f(R)-Lagrangian

$$f(R) = \sum_n \frac{f^n(R_0)}{n!}(R-R_0)^n \simeq f_0 + f_0' R + \frac{1}{2} f_0'' R^2 + \cdots$$

Field equations at the first order of approximation in term of the perturbation , become:

$$f_0'\left[R^{(1)}_{\mu\nu} - \frac{R^{(1)}}{2}\eta_{\mu\nu}\right] - f_0''\left[R^{(1)}_{,\mu\nu} - \eta_{\mu\nu}\Box R^{(1)}\right] = \frac{\mathcal{X}}{2} T^{(0)}_{\mu\nu}$$

The explicit expressions of the Ricci tensor and scalar, at the first order in the metric perturbation, read

$$\begin{cases} R^{(1)}_{\mu\nu} = h^{\sigma}_{(\mu,\nu)\sigma} - \frac{1}{2}\Box h_{\mu\nu} - \frac{1}{2} h_{,\mu\nu} \\ R^{(1)} = h_{\sigma\tau}^{,\sigma\tau} - \Box h \end{cases}$$

S. Capozziello, M. De Laurentis, Phys. Rep. 509, 167 (2011)

M. De Laurentis, S. Capozziello, Astroparticle Physics 35 , 257 (2011)

Quadrupolar gravitational radiation in f(R)-gravity

If we assume that the source is localized in a finite region as a consequence outside this region $T_{\mu\nu} = 0$, and then we have that $R^{(1)}_{\mu\nu} = \Box h_{\mu\nu} = 0$

With this assumption we can calculate the energy momentum tensor of gravitational field in f(R)-gravity adopting the definition given in Landau and Lifshitz (1962)

$$t^{\lambda}_{\alpha} = f'\left\{\left[\frac{\partial R}{\partial g_{\rho\sigma,\lambda}} - \frac{1}{\sqrt{-g}}\partial_{\xi}\left(\sqrt{-g}\frac{\partial R}{\partial g_{\rho\sigma,\lambda\xi}}\right)\right]g_{\rho\sigma,\alpha} + \frac{\partial R}{\partial g_{\rho\sigma,\lambda\xi}}g_{\rho\sigma,\xi\alpha}\right\} - f''R_{,\xi}\frac{\partial R}{\partial g_{\rho\sigma,\lambda\xi}}g_{\rho\sigma,\alpha} - \delta^{\lambda}_{\alpha}\, f$$

The energy momentum tensor consists of a sum of GR contribution plus a term coming from f(R)-gravity :

$$t^{\lambda}_{\alpha} = f'_0 t^{\lambda}_{\alpha}\,{}_{|\mathrm{GR}} + f''_0 t^{\lambda}_{\alpha}\,{}_{|f(R)}$$

M. De Laurentis, S. Capozziello, Astroparticle Physics 35 , 257 (2011)

Quadrupolar gravitational radiation in f(R)-gravity

...in term of the perturbation h is

$$t^{\lambda}_{\alpha} \sim f'_0 t^{\lambda}_{\alpha|\mathrm{GR}} + f''_0 \Big\{ (h^{\rho\sigma}_{,\rho\sigma} - \Box h) \Big[h^{\lambda\xi}_{,\xi\alpha} - h^{,\lambda}_{\alpha} - + \frac{1}{2}\delta^{\lambda}_{\alpha}(h^{\rho\sigma}_{,\rho\sigma} - \Box h) \Big]$$
$$- h^{\rho\sigma}_{,\rho\sigma\xi} h^{\lambda\xi}_{,\alpha} + h^{\rho\sigma}_{,\rho\sigma}{}^{\lambda} h_{,\alpha} + h^{\lambda\xi}_{,\alpha} \Box h_{,\xi} - \Box h^{,\lambda} h_{,\alpha} \Big\}.$$

In the weak field limit, the source $h_{\mu\nu}$ is written as function of time $t' = t - r$, and plane wave approximation

the energy momentum tensor assumes the form:

$$t^{\lambda}_{\alpha} = \underbrace{f'_0 k^{\lambda} k_{\alpha} \left(\dot{h}^{\rho\sigma} \dot{h}_{\rho\sigma} \right)}_{GR} - \underbrace{\frac{1}{2} f''_0 \delta^{\lambda}_{\alpha} \left(k_{\rho} k_{\sigma} \ddot{h}^{\rho\sigma} \right)^2}_{f(R)}$$

M. De Laurentis, S. Capozziello, Astroparticle Physics 35 , 257 (2011)
De Laurentis M., De Martino I., 2013, MNRAS., doi:10.1093/mnras/stt216

Quadrupolar gravitational radiation in f(R)-gravity

In order to calculate the radiated energy of a gravitational waves sources, we consider the average energy flux dE/dt away from the systems and the momenta of the mass-energy distribution

Finally the result is

$$\underbrace{\left\langle \frac{dE}{dt} \right\rangle}_{(total)} = \frac{G}{60} \left\langle \underbrace{f'_0 \left(\dddot{Q}^{ij} \dddot{Q}_{ij} \right)}_{GR} - \underbrace{f''_0 \left(\ddddot{Q}^{ij} \ddddot{Q}_{ij} \right)}_{f(R)} \right\rangle$$

for $f''_0 = 0$ and $f'_0 = 4/3$

$$\underbrace{\left\langle \frac{dE}{dt} \right\rangle}_{(GR)} = \frac{G}{45} \left\langle \dddot{Q}^{ij} \dddot{Q}_{ij} \right\rangle$$

The massive mode contribution is evident.

$$\underbrace{\left\langle \frac{dE}{dt} \right\rangle}_{(total)} = \frac{G f'_0}{60} \left\langle \left(\dddot{Q}^{ij} \dddot{Q}_{ij} \right) - \frac{1}{m^2} \left(\ddddot{Q}^{ij} \ddddot{Q}_{ij} \right) \right\rangle$$

This means that this further term affects both the total energy release and the waveform.

This could represent a further signature to investigate such theories in the GW strong-field regime.

Application to the binary systems

Assuming Keplerian motion and the orbit in the (x; y)-plane

the quadrupole matrix is

$$Q_{ij} = \mu r^2 \begin{pmatrix} \cos^2\psi & \sin\psi\cos\psi \\ \sin\psi\cos\psi & \sin^2\psi \end{pmatrix}_{ij}$$

the time average of the radiated power

$$\left\langle \frac{dE}{dt} \right\rangle = \frac{1}{T}\int_0^T dt \frac{dE(\psi)}{dt} = \frac{1}{T}\int_0^{2\pi} \frac{d\psi}{\dot{\psi}} \frac{dE(\psi)}{dt}$$

Where $\dot{\psi} = \left(\frac{Gm_c}{a^3}\right)^{\frac{1}{2}} \left(1-\epsilon^2\right)^{-\frac{3}{2}} \left(1+\epsilon\cos\psi\right)^2$

The time derivative of the orbital period

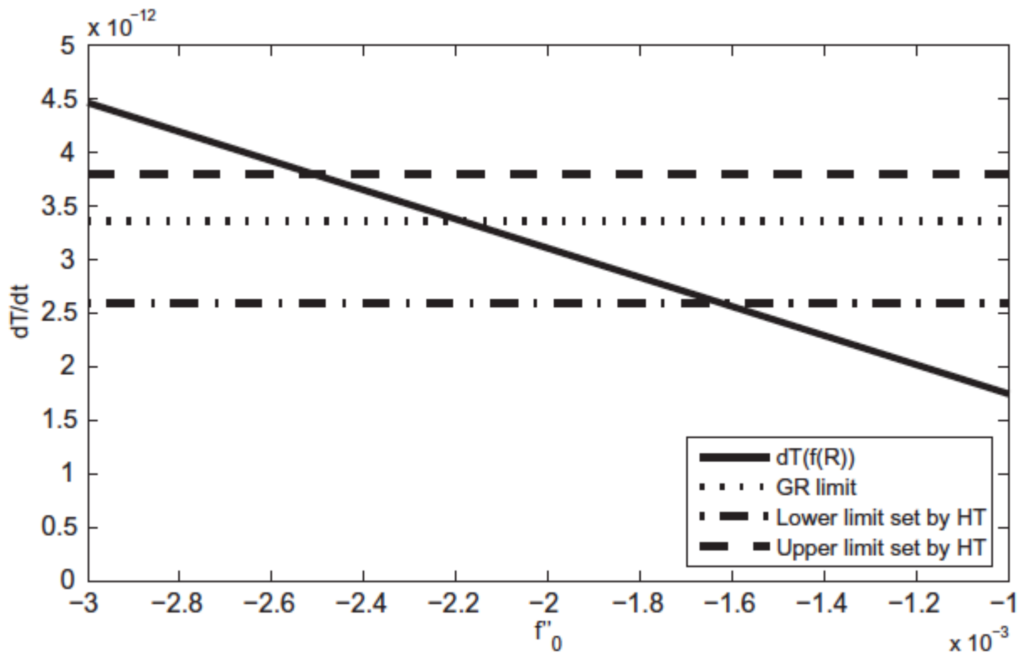
$$\dot{T}_b = -\frac{3}{20}\left(\frac{T}{2\pi}\right)^{-\frac{5}{3}} \frac{\mu G^{\frac{5}{3}}(m_c+m_p)^{\frac{2}{3}}}{c^5(1-n^2)^{\frac{7}{2}}} \times$$
$$\times \left[f'_0 \left(37\epsilon^4 + 292\epsilon^2 + 96\right) - \frac{f''_0 \pi^2 T^{-1}}{2(1+n^2)^3} \times \right.$$
$$\left. \times \left(891\epsilon^8 + 28016\epsilon^6 + 82736\epsilon^4 + 43520\epsilon^2 + 3072\right) \right]$$

We will go on to constrain the f (R) theories estimating f''$_0$ from the comparison between the theoretical predictions of dT$_b$ and the observed one.

Application to the binary systems: The PSR 1913 + 16 case

Using the values for the specific example of PSR 1913 + 16 to numerically evaluate the above equations

PSR 1913 + 16	Chacteristic features
Pulsar mass	$m = 1.39 M_\odot$
Companion mass	$M = 1.44 M_\odot$
Inclination angle	$\sin i = 0.81$
Orbit semimajor axis	$a = 8.67 \times 10^{10}$ cm
Eccentricity	$\epsilon = 0.617155$
Gravitational constant	$G = 6.67 \times 10^{-8}$ dyn cm^2 g^{-2}
Speed of light	$c = 2.99 \times 10^{10}$ cm s^{-1}

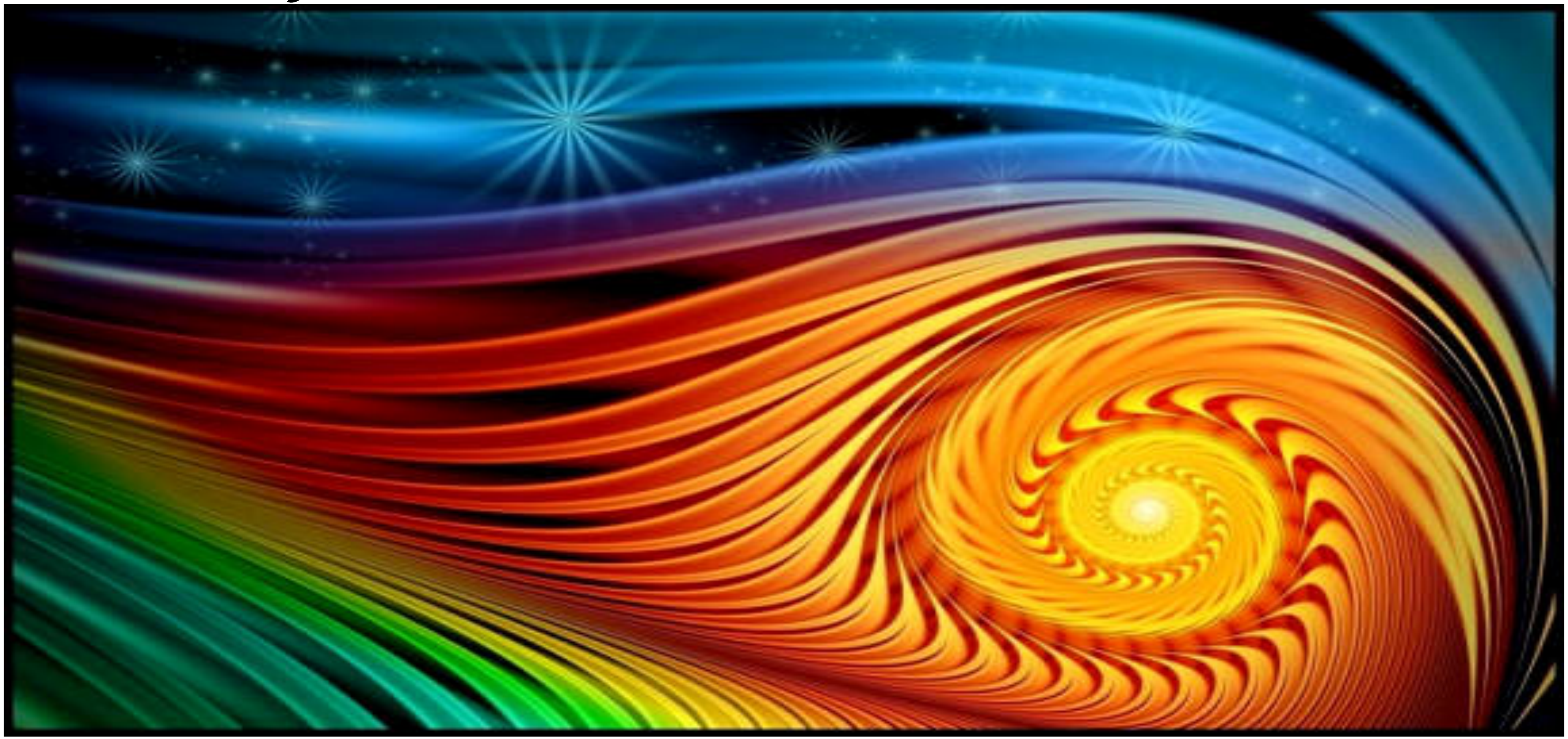


Orbital decay rate for PSR 1913 + 16 in $f(R)$-gravity. Upper limit set by Taylor et al. in dashed line. GR limit 3.36×10^{-12} in dotted line and the lower limit set by Taylor et al. in dashdot line. Solid line is $dT_{f(R)}$

A class of $f(R)$ agrees with data!

Extended Theories of Gravity can also impact on the estimate of DM properties on galactic scales

Modified gravity could be a possible way to solve the cusp/core and similar problems of the DM scenario without asking for huge amounts of DM



Testing spiral galaxies

Yukawa-like corrections are a general feature, in the framework of $f(R)$-gravity

This equation $\Rightarrow$ $$\Phi(r) = -\frac{GM}{(1+\delta)r}\left(1+\delta e^{-\frac{r}{L}}\right)$$ is the starting point for the computation of the rotation curve of an extended system.

Considering a general expression derived for a generic potential giving rise to a separable force

$$F_p(\mu, r) = \frac{GM_{\odot}}{r_s^2} f_{\mu}(\mu) f_r(\eta)$$

With $\mu = M/M_{\odot}$, $\eta = r/r_s$ and $(M_{\odot}, r_s)$ the Solar mass and a characteristic length of the problem

In our case, $f_{\mu} = 1$ and:

$$f_r(\eta) = \left(1+\frac{\eta}{\eta_L}\right)\frac{\exp\left(-\eta/\eta_L\right)}{(1+\delta)\eta^2}$$

With $\eta_L = L/r_s$

Testing spiral galaxies

Using cylindrical coordinates (R, θ, z) and the corresponding dimensionless variables (η, θ, ξ) (with $\xi = z/rs$), the total force then reads:

$$F(\mathbf{r}) \quad = \frac{G\rho_0 r_s}{1+\delta} \int_0^{\infty} \eta' d\eta' \int_{-\infty}^{\infty} d\zeta' \int_0^{\pi} f_r(\Delta)\tilde{\rho}(\eta', \theta', \zeta') d\theta'$$

With $\tilde{\rho} = \rho/\rho_0$, ρ_0 a reference density, we have

$$\Delta = \left[\eta^2 + \eta'^2 - 2\eta\eta' \cos(\theta - \theta') + (\zeta - \zeta')^2\right]^{1/2}$$

For obtaining axisymmetric systems, one can set $\tilde{\rho} = \tilde{\rho}(\eta, \xi)$.

S. Capozziello, M. De Laurentis Ann. Phys. 524, 545 (2012)

Testing spiral galaxies

The systems we are considering here are spiral galaxies which will be modeled as the sum of an infinitesimally thin disc and a spherical halo, and then the scaling radius r_s will be the disc scale length R_d

Under these assumptions, the rotation curve may be obtained as:

$$v_c^2(R) = \frac{G\rho_0 R_d^2 \eta}{1+\delta} \int_0^\infty \eta' d\eta' \int_{-\infty}^{\infty} \tilde{\rho}(\eta',\zeta') d\zeta' \int_0^\pi f_r(\Delta_0) d\theta'$$

With $\Delta_0 = \Delta(\theta = \zeta = 0) = \left[\eta^2 + \eta'^2 - 2\eta\eta' \cos\theta' + \zeta'^2\right]^{1/2}$

It is evident that the total rotation curve may be split in the sum of the standard Newtonian term and a corrective one disappearing for $L \rightarrow \infty$, i.e. when ETGs have no deviations from GR at galactic scales.

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Testing spiral galaxies

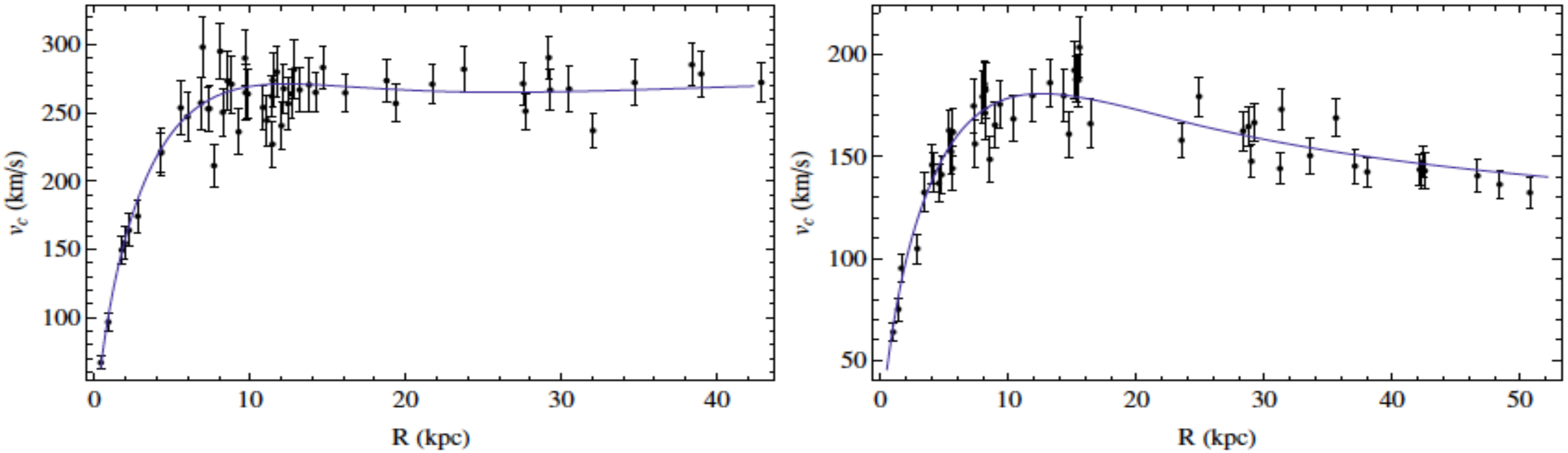
The total rotation curve is:

$$v_c^2(R, M_d, \mathbf{p}_i)$$
$$= v_{dN}^2(R, M_d) + v_{hN}^2(R, \mathbf{p}_i) + v_{dY}^2(R, M_d) + v_{hY}^2(R, \mathbf{p}_i)$$

Md *is the disc mass,* d *and* h *denote disc and halo related quantities, while* N *and* Y *refer to the Newtonian and Yukawa-like contributions*

One may model a spiral galaxy as the sum of a hick disc and a spherical halo Without DM contribution.

Testing spiral galaxies



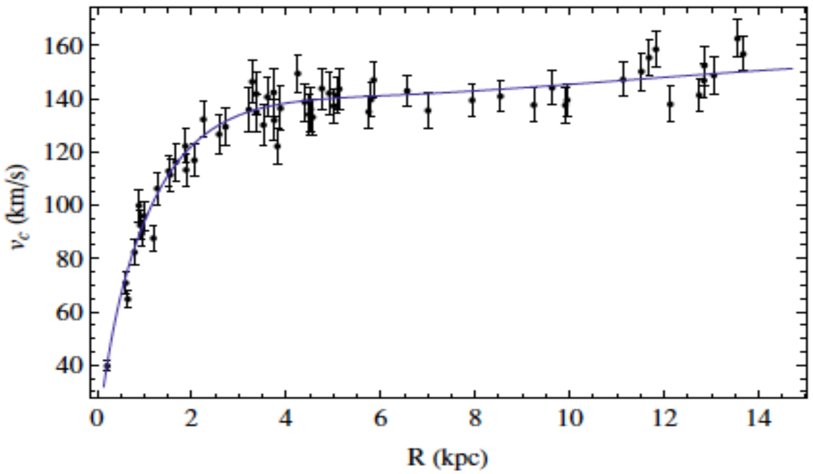


Figure 4 Examples of simulated rotation curves with superimposed theoretical curves. From left to right, model parameters are $(\log M_d, \log M_{\rm vir}, c, f_{DM}, \log \eta_L) =$ (11.15, 12.90, 10.24, 0.47, 0.36),(10.90, 11.76, 14.77, 0.45, −0.92), (10.04, 12.10, 13.76, 0.54, 1.11), while the simulation parameters are set as discussed in the text. Note that, depending on how the model parameters are set, it is possible to get rotation curves which are flat, decreasing or increasing in the outer region.

Testing elliptical galaxies

The modified potential can be tested also for elliptical galaxies checking whether it is able to provide a reasonable match to their kinematics.

Such self-gravitating systems are very different with respect to spirals so addressing both classes of objects under the same standard could be a fundamental step versus DM

One may construct equilibrium models based on the solution of the radial Jeans equation to interpret the kinematics of planetary nebulae

We use the inner long slit data and the extended planetary nebulae kinematics for three galaxies which have published dynamical analyses within DM halo framework

They are:
NGC 3379 , (DL +09) , NGC 4494 N +09 , NGC 4374 (N + 11).

Testing elliptical galaxies

It is shown the circular velocity of the modified potential as a function of the potential parameters L and δ for NGC 4494 and NGC 4374.

From a theoretical point of view, δ is a free parameter that can assume positive and negative values. Comparing results for spirals and ellipticals, it is clear that the morphology of these two classes of systems strictly depends on the sign and the value of δ.

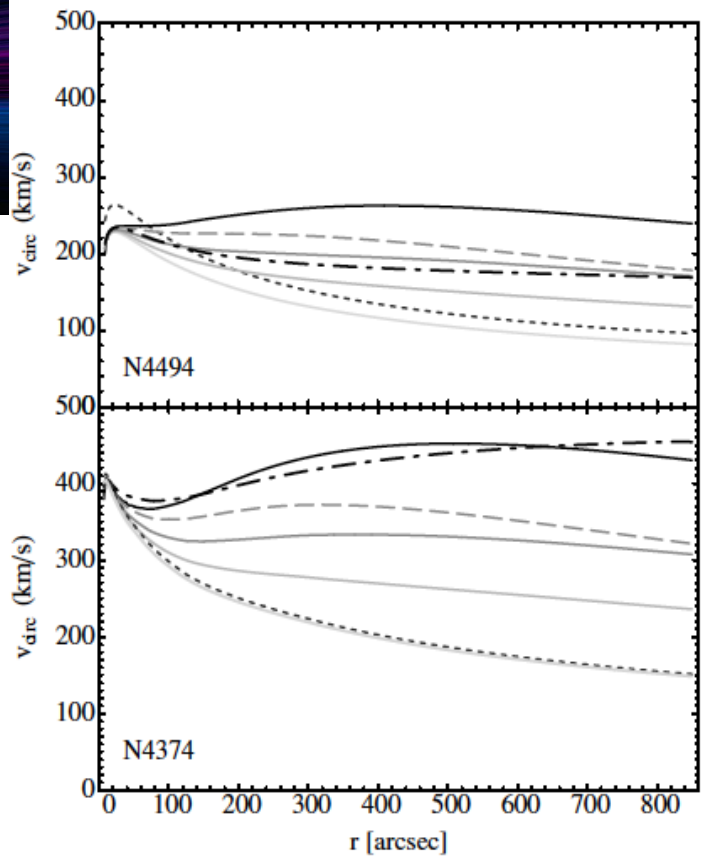


Figure 6 Circular velocity produced by the modified potential for the two galaxies N4494 (top) and N4374 (bottom). In both cases the $M/\mathscr{L}_*$ has been fixed to some fiducial value (as expected from stellar population models and Kroupa 2001 IMF): $M/\mathscr{L}_* = 4.3\Upsilon_{\odot,B}$ for NGC 4494 and $M/\mathscr{L}_* = 5.5\Upsilon_{\odot,V}$ for NGC 4374. The potential parameters adopted are: $L = 250''$ and δ=0, -0.65, -0.8, -0.9 (lighter to darker solid lines) and $L = 180''$ and δ=-0.8 (dashed lines). The dotted line is a case with positive coefficient of the Yukawa-like term and $L = 5000''$ which illustrates that positive δ cannot produce flat circular velocity curves. Finally some reference Navarro-Frenk-White (NFW) models are shown as dot-dashed lines [108].

Testing elliptical galaxies

The problem of fitting a modified potential (which is formally selfconsistent implies the same kind of degeneracies between the anisotropy parameter, $\beta = 1-\sigma^{2}{}_{\theta}/\sigma 2r$ (where σ_{θ} and σ_r are the azimuthal and radial dispersion components in spherical coordinates), and the non-Newtonian part of the potential (characterized by two parameters like typical dark haloes) in a similar way of the classical mass-anisotropy degeneracy

These degeneracies can be alleviated via higher-order Jeans equations including in the dynamical models both the dispersion $(\sigma\, p)$ and the kurtosis (κ) profiles of the tracers

Under spherical assumption, nonrotation and β = const (corresponding to the family of distribution functions $f(\mathcal{E},\mathcal{L}) = f_0\mathcal{L}^{-2\beta}$, the 2-nd and 4-th moment radial equations can be compactly written as:

$$s(r) = r^{-2\beta}\int_r^{\infty} x^{2\beta} H(x)\,dx$$

Where $s(r) = \{\rho\,\sigma^2{}_r;\rho\, v^4{}_r\}$, β is the anisotropy parameter, and

$$H(r) = \left\{\rho\frac{d\Phi}{dr};3\rho\frac{d\Phi}{dr}\overline{v_r^2}\right\}$$

respectively for the dispersion and kurtosis equations

Testing elliptical galaxies

The overall match of the model curves with data is remarkably good and it is comparable with models obtained with DM modeling (gray lines)

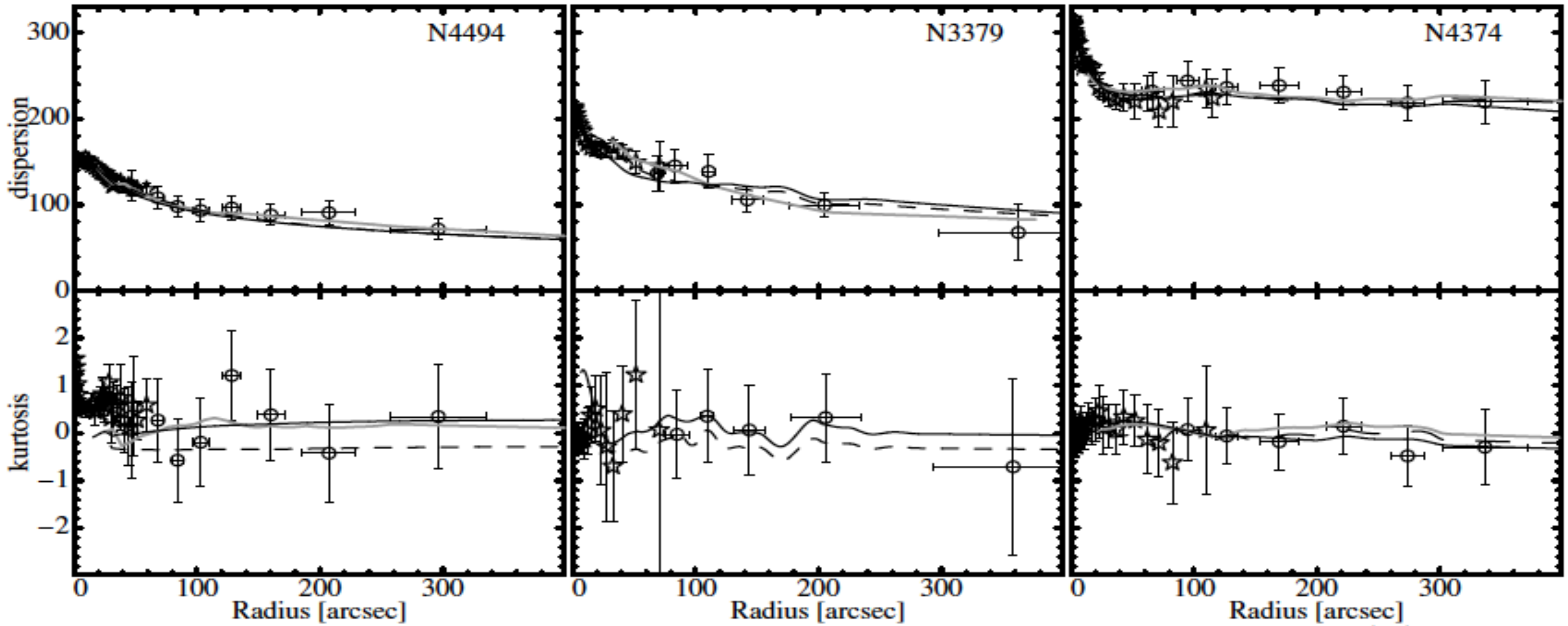


Figure 7 Dispersion in $km s$ (top) and kurtosis (bottom) fit of the galaxy sample for the different $f(R)$ parameter sets: the anisotropic solution (solid lines) is compared with the isotropic case (dashed line – for NGC 4374 and NGC 4494 this is almost indistinguishable from the anisotropic case). From the left, NGC 4494, NGC 3379 and NGC 4374 are shown with DM models as gray lines from N+09, DL+09 (no kurtosis is provided), and N+11 respectively [108].

Testing elliptical galaxies

The marginalized confidence contours of the main two potential parameters for the three galaxies

there seems to be a possible increasing trend of δ with the orbital anisotropy

Table 4 Model parameters for the $f(R)$ potential.

Galaxy	Mag (band)	R_{eff}	$M/L_\star$	L	δ	β	χ^2/dof
NGC3379	-19.8(B)	2.2	6 (7)	6	-0.75	0.5(<0.8)	14/25
NGC4374	-21.3(V)	3.4	6 (6)	24	-0.88	0.01(0.01)	14/39
NGC4494	-20.5(B)	6.1	3 (4)	20	-0.79	0.5(0.5)	18/43

Notes – Galaxy ID, total magnitude, effective radius and model parameters for the unified solution. DM–based estimates for $M/\mathscr{L}_\star$ and β (NGC 3379: DL+09; NGC 4374: N+11; NGC 4494: N+09) are shown in parentheses for comparison. $M/\mathscr{L}_\star$ are in solar units, R_{eff} and L in kpc. Typical errors on $M/\mathscr{L}_\star$ are of the order of $0.2M/\mathscr{L}_\odot$ and on β of 0.2 (see also Fig. 8). The small χ^2 values are mainly due to the large data error bars.

such a function could be related to second order effects connected to anisotropies and non-homogeneities which could trigger the formation and the evolution of stellar systems

This results can have interesting implications on the capability of the theory of making predictions on the internal structure of the gravitating systems after their spherical collapse. However, this possibility has to be confirmed on larger galaxy samples

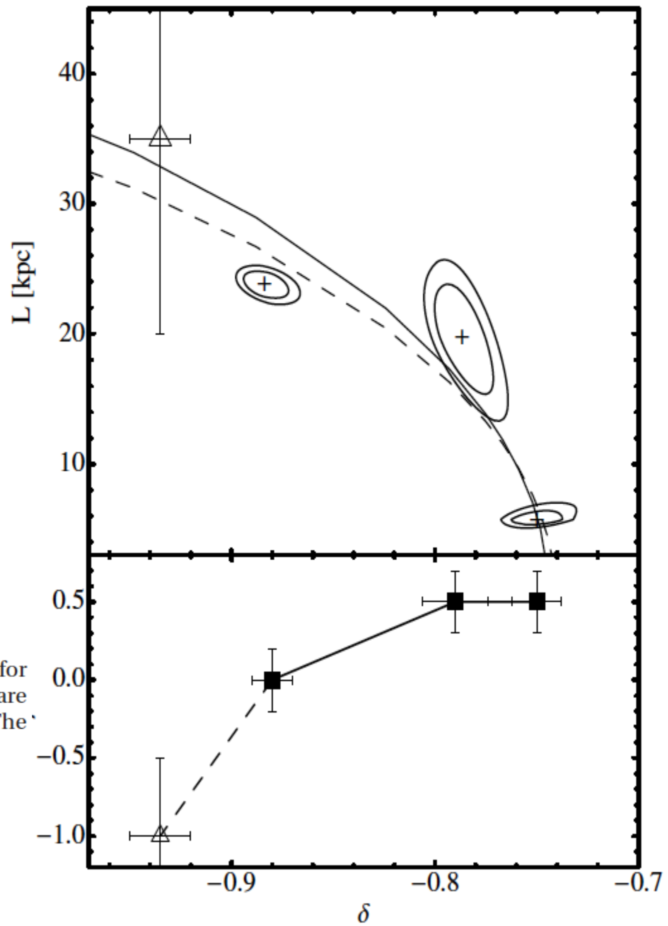


Figure 8 Top: 1- and 2-σ confidence levels in the $\delta - L$ space marginalized over $M/\mathscr{L}_\star$ and β (see also Table 4). Spiral galaxy results from [105] are shown as empty triangle with error bars. Solid (dashed) curve shows the tentative best-fit to the data including (excluding) the spiral galaxies and assuming a $L \propto \sqrt{\delta/(1+\delta)}$. Bottom: the anisotropy and the δ parameters turn out to be correlated for the elliptical sample (full squares). This correlation seems to include also the spiral sample cumulatively shown as the empty triangle (here we have assumed $\beta = -1.0 \pm 0.5$ as a fiducial value for spiral galaxies to draw a semi-quantitative trend across galaxy types) [108].

Modeling clusters of galaxies

A fundamental issue is related to clusters and superclusters of galaxies.

Such structures, essentially, rule the large scale structure, and are the intermediate step between galaxies and cosmology.

As the galaxies, they appear DM dominated but the distribution of DM component seems clustered and organized in a very different way with respect to galaxies. It seems that DM is ruled by the scale and also its fundamental nature could depend on the scale

Our goal is to reconstruct the mass profile of clusters without DM adopting the same strategy as above where DM effects are figured out by corrections to the Newton potential

Modeling clusters of galaxies

Standard Cluster Model: spherical mass distribution in hydrostatic equilibrium

- *Boltzmann equation:*

$$-\frac{d\Phi}{dr} = \frac{kT(r)}{\mu m_p r}\left[\frac{d\ln\rho_{gas}(r)}{d\ln r} + \frac{d\ln T(r)}{d\ln r}\right]$$

- *Newton classical approach:*

$$\begin{cases} \phi(r) = -\frac{GM}{r} \\ \rho_{cl,EC}(r) = \rho_{dark} + \rho_{gas}(r) + \rho_{gal}(r) + \rho_{CDgal}(r) \end{cases}$$

- *f(R) approach:*

$$\begin{cases} \phi(r) = -\frac{3GM}{4a_1 r}\left(1 + \frac{1}{3}e^{-\frac{r}{L}}\right) \\ \rho_{cl,EC}(r) = \rho_{gas}(r) + \rho_{gal}(r) + \rho_{CDgal}(r) \end{cases}$$

- *Rearranging the Boltzmann equation:*

$$\phi_N(r) = -\frac{3GM}{4a_1 r}$$

$$\phi_C(r) = -\frac{GM}{4a_1}\frac{e^{-\frac{r}{L}}}{r}$$

$$\begin{cases} M_{bar,th}(r) = \frac{4a_1}{3}\left[-\frac{kT(r)}{\mu m_p G}r\left(\frac{d\ln\rho_{gas}(r)}{d\ln r} + \frac{d\ln T(r)}{d\ln r}\right)\right] - \frac{4a_1}{3G}r^2\frac{d\Phi_C}{dr}(r) \\ M_{bar,obs}(r) = M_{gas}(r) + M_{gal}(r) + M_{CDgal}(r) \end{cases}$$

Modeling clusters of galaxies

Fitting mass Profile with data:

- *Sample: 12 clusters from Chandra (Vikhlinin 2005, 2006)*
- *Temperature profile from spectroscopy*
- *Gas density: modified beta-model*

$$n_p n_e = n_0^2 \cdot \frac{(r/r_c)^{-\alpha}}{(1+r^2/r_c^2)^{3\beta-\alpha/2}} \cdot \frac{1}{(1+r^\gamma/r_s^\gamma)^{\epsilon/\gamma}} + \frac{n_{02}^2}{(1+r^2/r_{c2}^2)^{3\beta_2}}$$

- *Galaxy density:*

$$\rho_{gal}(r) = \begin{cases} \rho_{gal,1} \cdot \left[1+\left(\frac{r}{R_c}\right)^2\right]^{-\frac{3}{2}} & r < R_c \\ \rho_{gal,2} \cdot \left[1+\left(\frac{r}{R_c}\right)^2\right]^{-\frac{2.6}{2}} & r > R_c \end{cases} \qquad \rho_{CDgal} = \frac{\rho_{0,J}}{\left(\frac{r}{r_c}\right)^2 \left(1+\frac{r}{r_c}\right)^2}$$

Table 1. Column 1: Cluster name. Column2: Richness. Column 2: cluster total mass. Column 3: gas mass. Column 4: galaxy mass. Column 5: cD-galaxy mass. All mass values are estimated at $r = r_{max}$. Column 6: ratio of total galaxy mass to gas mass. Column 7: minimum radius. Column 8: maximum radius.

name	R	$M_{cl,N}$ ($M_\odot$)	M_{gas} ($M_\odot$)	M_{gal} ($M_\odot$)	M_{cDgal} ($M_\odot$)	$\frac{gal}{gas}$	r_{min} (kpc)	r_{max} (kpc)
A133	0	$4.35874 \cdot 10^{14}$	$2.73866 \cdot 10^{13}$	$5.20269 \cdot 10^{12}$	$1.10568 \cdot 10^{12}$	0.23	86	1060
A262	0	$4.45081 \cdot 10^{13}$	$2.76659 \cdot 10^{12}$	$1.71305 \cdot 10^{11}$	$5.16382 \cdot 10^{12}$	0.25	61	316
A383	2	$2.79785 \cdot 10^{14}$	$2.82467 \cdot 10^{13}$	$5.88048 \cdot 10^{12}$	$1.09217 \cdot 10^{12}$	0.25	52	751
A478	2	$8.51832 \cdot 10^{14}$	$1.05583 \cdot 10^{14}$	$2.15567 \cdot 10^{13}$	$1.67513 \cdot 10^{12}$	0.22	59	1580
A907	1	$4.87657 \cdot 10^{14}$	$6.38070 \cdot 10^{13}$	$1.34129 \cdot 10^{13}$	$1.66533 \cdot 10^{12}$	0.24	563	1226
A1413	3	$1.09598 \cdot 10^{15}$	$9.32466 \cdot 10^{13}$	$2.30728 \cdot 10^{13}$	$1.67345 \cdot 10^{12}$	0.26	57	1506
A1795	2	$1.24313 \cdot 10^{14}$	$1.00530 \cdot 10^{13}$	$4.23211 \cdot 10^{12}$	$1.93957 \cdot 10^{12}$	0.11	79	1151
A1991	1	$1.24313 \cdot 10^{14}$	$1.00530 \cdot 10^{13}$	$1.24608 \cdot 10^{12}$	$1.08241 \cdot 10^{12}$	0.23	55	618
A2029	2	$8.92392 \cdot 10^{14}$	$1.24129 \cdot 10^{14}$	$3.21543 \cdot 10^{13}$	$1.11921 \cdot 10^{12}$	0.27	62	1771
A2390	1	$2.09710 \cdot 10^{15}$	$2.15726 \cdot 10^{14}$	$4.91580 \cdot 10^{13}$	$1.12141 \cdot 10^{12}$	0.23	83	1984
MKW4	-	$4.69503 \cdot 10^{13}$	$2.83207 \cdot 10^{12}$	$1.71153 \cdot 10^{11}$	$5.29855 \cdot 10^{11}$	0.25	60	434
RXJ1159	-	$8.97997 \cdot 10^{13}$	$4.33256 \cdot 10^{12}$	$7.34414 \cdot 10^{11}$	$5.38799 \cdot 10^{11}$	0.29	64	568

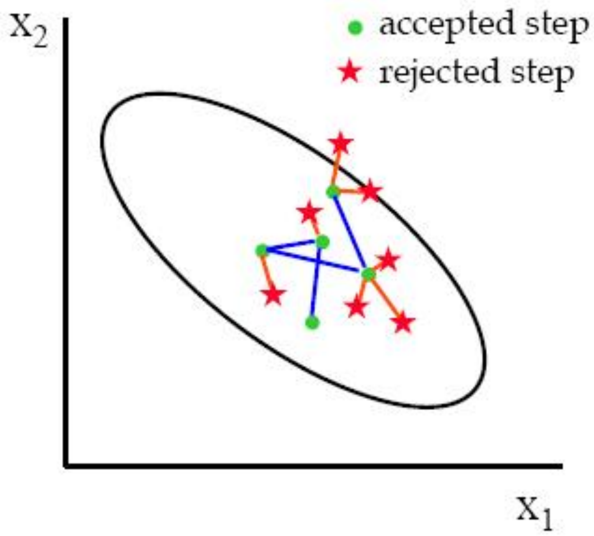
Modeling clusters of galaxies

-Minimization of chi-square:

$$\chi^2 = \frac{1}{N - n_p - 1} \cdot \sum_{i=1}^{N} \frac{(M_{bar,obs} - M_{bar,theo})^2}{M_{bar,theo}}$$

-Markov Chain Monte Carlo:

$$\alpha(\mathbf{p}, \mathbf{p}') = min\left\{1, \frac{L(\mathbf{d}|\mathbf{p}')P(\mathbf{p}')q(\mathbf{p}', \mathbf{p})}{L(\mathbf{d}|\mathbf{p})P(\mathbf{p})q(\mathbf{p}, \mathbf{p}')}\right\}$$



Reject min < 1:
- new point out of prior
- new point with greater chi-square

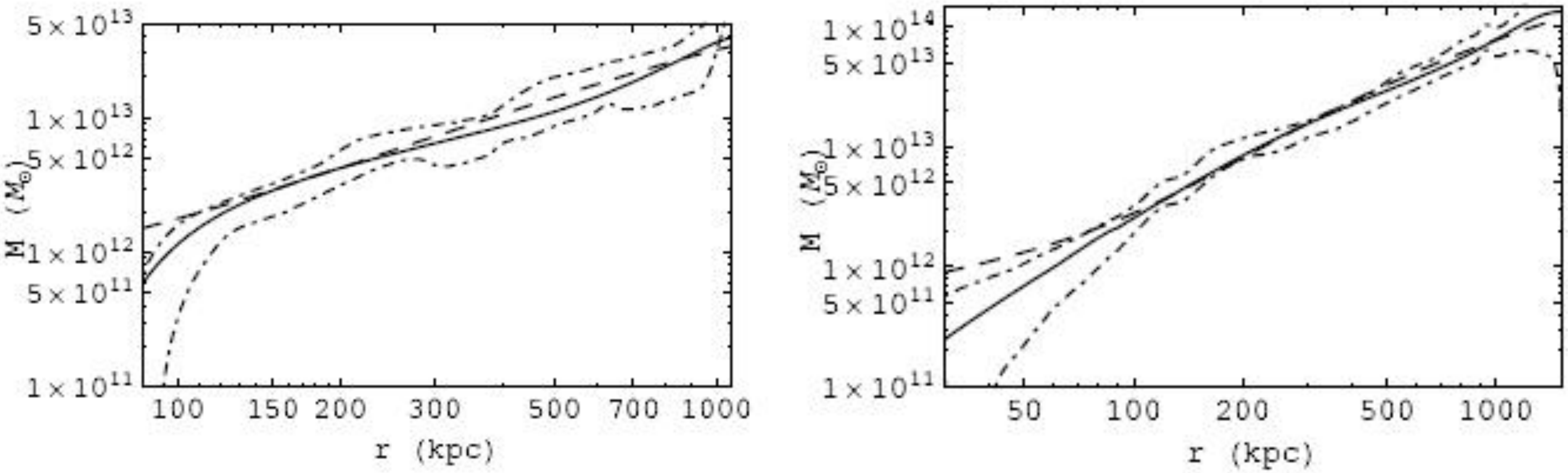
Accept min = 1: new point in prior and less chi-square

Sample of accepted points → Sampling from underlying probability distribution

- Power spectrum test convergence:

Discrete power spectrum from samples → Convergence = flat spectrum

Modeling clusters of galaxies



- Differences between theoretical and observed fit less than 5%
- Typical scale in [100; 150] kpc range where is a turning-point:
  - Break in the hydrostatic equilibrium
  - Limits in the expansion series of f(R): $R - R_0 << \frac{a_1}{a_2}$ in the range [19;200] kpc

  Proper gravitational scale (as for galaxies, see Capozziello et al MNRAS 2007)

  - Similar issues in Metric-Skew-Tensor-Gravity (Brownstein, 2006): we have better and more detailed approach

Modeling clusters of galaxies

Results

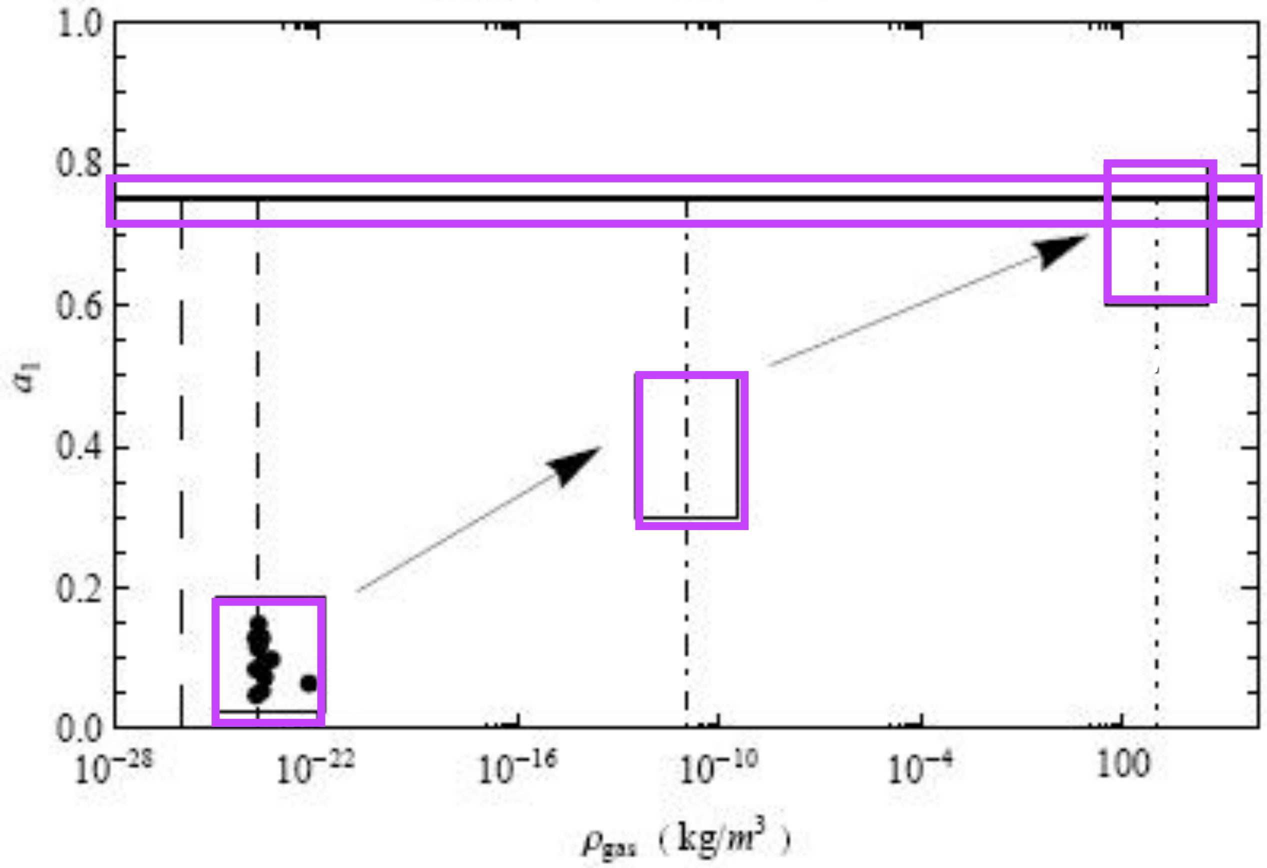
name	a_1	$[a_1 - 1\sigma,\ a_1 + 1\sigma]$	a_2 (kpc^2)	$[a_2 - 1\sigma,\ a_2 + 1\sigma]$ (kpc^2)	L (kpc)	$[L - 1\sigma,\ L + 1\sigma]$ (kpc)
A133	0.085	[0.078, 0.091]	$-4.98 \cdot 10^3$	$[-2.38 \cdot 10^4,\ -1.38 \cdot 10^3]$	591.78	[323.34, 1259.50]
A262	0.065	[0.061, 0.071]	-10.63	$[-57.65,\ -3.17]$	31.40	[17.28, 71.10]
A383	0.099	[0.093, 0.108]	$-9.01 \cdot 10^2$	$[-4.10 \cdot 10^3,\ -3.14 \cdot 10^2]$	234.13	[142.10, 478.06]
A478	0.117	[0.114, 0.122]	$-4.61 \cdot 10^3$	$[-1.01 \cdot 10^4,\ -2.51 \cdot 10^3]$	484.83	[363.29, 707.73]
A907	0.129	[0.125, 0.136]	$-5.77 \cdot 10^3$	$[-1.54 \cdot 10^4,\ -2.83 \cdot 10^3]$	517.30	[368.84, 825.00]
A1413	0.115	[0.110, 0.119]	$-9.45 \cdot 10^4$	$[-4.26 \cdot 10^5,\ -3.46 \cdot 10^4]$	2224.57	[1365.40, 4681.21]
A1795	0.093	[0.084, 0.103]	$-1.54 \cdot 10^3$	$[-1.01 \cdot 10^4,\ -2.49 \cdot 10^2]$	315.44	[133.31, 769.17]
A1991	0.074	[0.072, 0.081]	-50.69	$[-3.42 \cdot 10^2,\ -13]$	64.00	[32.63, 159.40]
A2029	0.129	[0.123, 0.134]	$-2.10 \cdot 10^4$	$[-7.95 \cdot 10^4,\ -8.44 \cdot 10^3]$	988.85	[637.71, 1890.07]
A2390	0.149	[0.146, 0.152]	$-1.40 \cdot 10^6$	$[-5.71 \cdot 10^6,\ -4.46 \cdot 10^5]$	7490.80	[4245.74, 15715.60]
MKW4	0.054	[0.049, 0.060]	-23.63	$[-1.15 \cdot 10^2,\ -8.13]$	51.31	[30.44, 110.68]
RXJ1159	0.048	[0.047, 0.052]	-18.33	$[-1.35 \cdot 10^2,\ -4.18]$	47.72	[22.86, 125.96]

Modeling clusters of galaxies

Results: expectations

- First derivative, a_1 : very well constrained → It scales with the system size

- Newtonian limit: $$\phi(r) = -\frac{3GM}{4a_1 r}\left(1+\frac{1}{3}e^{-\frac{r}{L}}\right)$$ → $a_1 \rightarrow 3/4$



Modeling clusters of galaxies

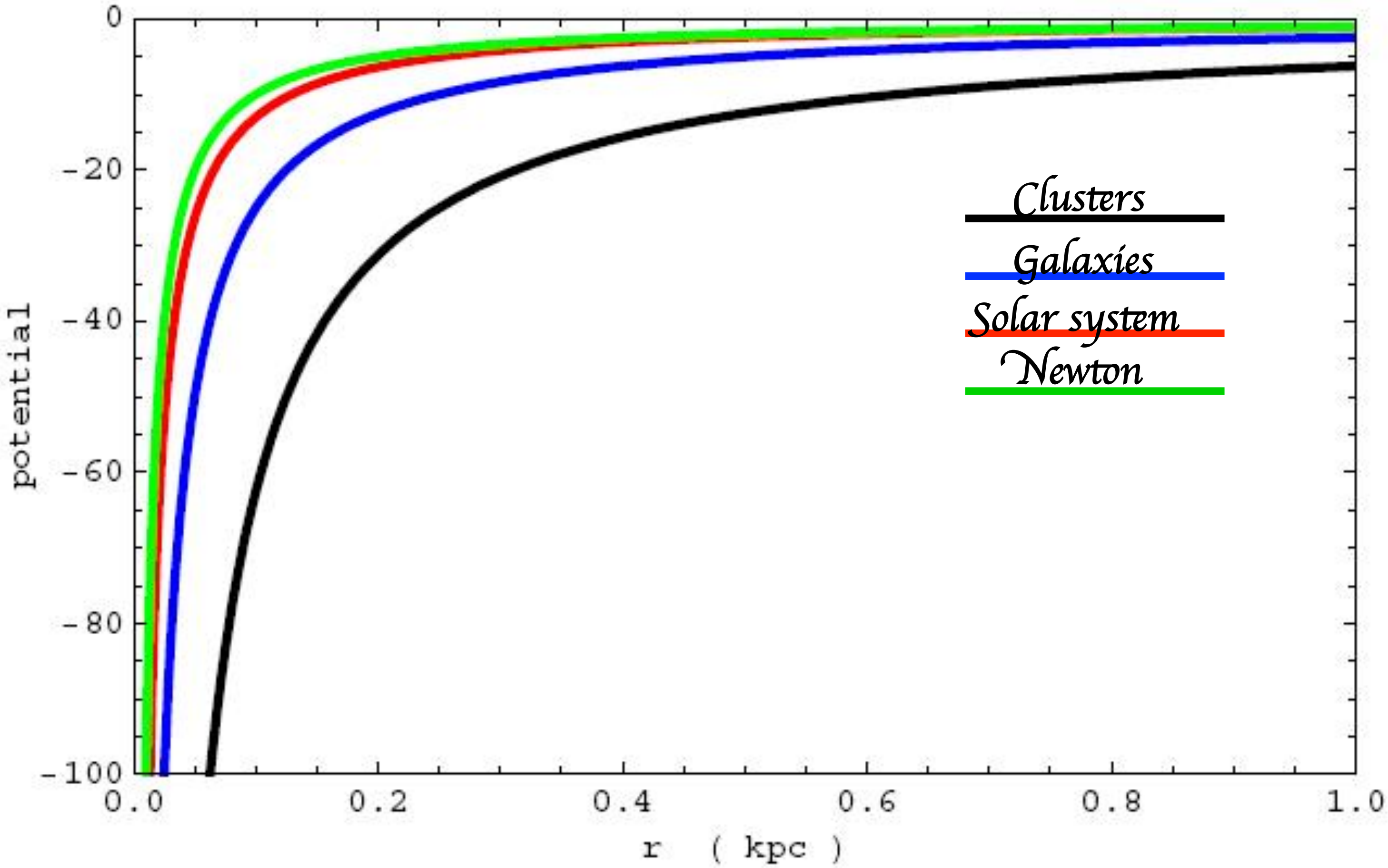
Point like potential:

Cluster of Galaxies: $a_1 = 0.16$ - $L = 1000$ kpc

Galaxies: $a_1 = 0.4$ - $L = 100$ kpc

Solar System: $a_1 = 0.75$ - $L = 1$ kpc

Newton Limit: $a_1 = 0.75$ - $L = 0$ kpc



Modeling clusters of galaxies

name	a_1	$[a_1 - 1\sigma,\ a_1 + 1\sigma]$	a_2 (kpc^2)	$[a_2 - 1\sigma,\ a_2 + 1\sigma]$ (kpc^2)	L (kpc)	$[L - 1\sigma,\ L + 1\sigma]$ (kpc)
A133	0.085	[0.078, 0.091]	$-4.98 \cdot 10^3$	$[-2.38 \cdot 10^4,\ -1.38 \cdot 10^3]$	591.78	[323.34, 1259.50]
A262	0.065	[0.061, 0.071]	-10.63	$[-57.65,\ -3.17]$	31.40	[17.28, 71.10]
A383	0.099	[0.093, 0.108]	$-9.01 \cdot 10^2$	$[-4.10 \cdot 10^3,\ -3.14 \cdot 10^2]$	234.13	[142.10, 478.06]
A478	0.117	[0.114, 0.122]	$-4.61 \cdot 10^3$	$[-1.01 \cdot 10^4,\ -2.51 \cdot 10^3]$	484.83	[363.29, 707.73]
A907	0.129	[0.125, 0.136]	$-5.77 \cdot 10^3$	$[-1.54 \cdot 10^4,\ -2.83 \cdot 10^3]$	517.30	[368.84, 825.00]
A1413	0.115	[0.110, 0.119]	$-9.45 \cdot 10^4$	$[-4.26 \cdot 10^5,\ -3.46 \cdot 10^4]$	2224.57	[1365.40, 4681.21]
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RXJ1159	0.048	[0.047, 0.052]	-18.33	$[-1.35 \cdot 10^2,\ -4.18]$	47.72	[22.86, 125.96]

Modeling clusters of galaxies

- Gravitational length: $L \equiv L(a_1, a_2) = \left(-\frac{6a_2}{a_1}\right)^{1/2}$

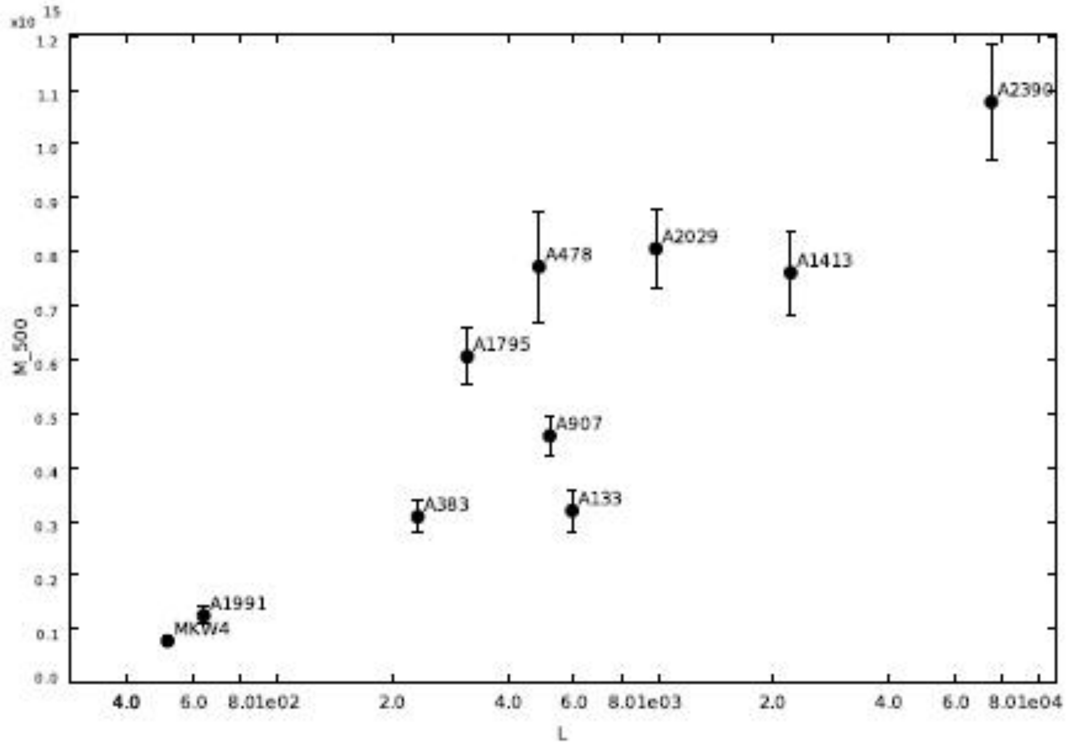
Strong characterization of Gravitational potential

-Mean length:

$$< L >_\rho \; = \; 318 \text{ kpc} \qquad < a_2 >_\rho = -3.40 \cdot 10^4$$
$$< L >_M \; = \; 2738 \text{ kpc} \qquad < a_2 >_M = -4.15 \cdot 10^5$$

- Strongly related to virial mass (the same for gas mass):

- Strongly related to average temperature:



Cosmography

GR based models vs f(R) gravity

↓

Agreement with Data...

} *How can we discriminate?*

- *No a priori dynamical model = Model Independent Approach;*
- *Robertson – Walker metric;*
- *Expansion series of the scale factor with respect to cosmic time:*

$$\frac{a(t)}{a(t_0)} = 1 + H_0(t-t_0) - \frac{q_0}{2}H_0^2(t-t_0)^2 + \frac{j_0}{3!}H_0^3(t-t_0)^3 + \frac{s_0}{4!}H_0^4(t-t_0)^4 + \frac{l_0}{5!}H_0^5(t-t_0)^5 + O[(t-t_0)^6$$

$$q(t) = -\frac{1}{a}\frac{d^2a}{dt^2}\frac{1}{H^2} \qquad j(t) = \frac{1}{a}\frac{d^3a}{dt^3}\frac{1}{H^3} \qquad s(t) = \frac{1}{a}\frac{d^4a}{dt^4}\frac{1}{H^4} \qquad l(t) = \frac{1}{a}\frac{d^5a}{dt^5}\frac{1}{H^5}$$

Deceleration *Jerk* *Snap* *Lerk*

Expansion up to fifth order :
- *error on $d_L(z)$ less than 10% up to z = 1*
- *error on $\mu(z)$ less than 3% up to z = 2*

Cosmography with f(R)- gravity

- Definition: $H(t) = \frac{1}{a}\frac{da}{dt}, \; q(t) = -\frac{1}{a}\frac{d^2a}{dt^2}\frac{1}{H^2}, \; j(t) = \frac{1}{a}\frac{d^3a}{dt^3}\frac{1}{H^3}, \; s(t) = \frac{1}{a}\frac{d^4a}{dt^4}\frac{1}{H^4}, \; l(t) = \frac{1}{a}\frac{d^5a}{dt^5}\frac{1}{H^5}$

- Derivatives of H(t):

$$\dot{H} = -H^2(1+q)$$

$$\ddot{H} = H^3(j+3q+2)$$

$$d^3H/dt^3 = H^4\left[s-4j-3q(q+4)-6\right]$$

$$d^4H/dt^4 = H^5\left[l-5s+10(q+2)j+30(q+2)q+24\right]$$

- Derivatives of scalar curvature:

$$R = -6(\dot{H}+2H^2)$$

$$R_0 = -6H_0^2(1-q_0)$$

$$\dot{R}_0 = -6H_0^3(j_0-q_0-2)$$

$$\ddot{R}_0 = -6H_0^4\left(s_0+q_0^2+8q_0+6\right)$$

$$d^3R_0/dt^3 = -6H_0^5\left[l_0-s_0+2(q_0+4)j_0-6(3q_0+8)q_0-24\right]$$

Cosmography with f(R)-gravity

- 1st Friedmann eq. :

$$H_0^2 = \frac{H_0^2\Omega_M}{f'(R_0)} + \frac{f(R_0) - R_0 f'(R_0) - 6H_0\dot{R}_0 f''(R_0)}{6f'(R_0)},$$

- 2nd Friedmann eq. :

$$-\dot{H}_0 = \frac{3H_0^2\Omega_M}{2f'(R_0)} + \frac{\dot{R}_0^2 f'''(R_0) + (\ddot{R}_0 - H_0\dot{R}_0)f''(R_0)}{2f'(R_0)}$$

- Derivative of 2nd Friedmann eq. :

$$\ddot{H} = \frac{\dot{R}^2 f'''(R) + \left(\ddot{R} - H\dot{R}\right) f''(R) + 3H_0^2\Omega_M a^{-3}}{2\left[\dot{R} f''(R)\right]^{-1}[f'(R)]^2} - \frac{\dot{R}^3 f^{(iv)}(R) + \left(3\dot{R}\ddot{R} - H\dot{R}^2\right) f'''(R)}{2f'(R)} - \frac{\left(d^3R/dt^3 - H\ddot{R} + \dot{H}\dot{R}\right) f''(R) - 9H_0^2\Omega_M H a^{-3}}{2f'(R)}$$

- Constraint from gravitational constant:

$$H^2 = \frac{8\pi G}{3f'(R)}[\rho_m + \rho_{\rm curv} f'(R)] \quad \Rightarrow \quad G_{\rm eff}(z=0) = G \rightarrow f'(R_0) = 1.$$

Cosmography with f(R)-gravity

- *Final solutions:*

$$\frac{f(R_0)}{6H_0^2} = -\frac{\mathcal{P}_0(q_0, j_0, s_0, l_0)\Omega_M + \mathcal{Q}_0(q_0, j_0, s_0, l_0)}{\mathcal{R}(q_0, j_0, s_0, l_0)}$$

$$f'(R_0) = 1$$

$$\frac{f''(R_0)}{(6H_0^2)^{-1}} = -\frac{\mathcal{P}_2(q_0, j_0, s_0)\Omega_M + \mathcal{Q}_2(q_0, j_0, s_0)}{\mathcal{R}(q_0, j_0, s_0, l_0)}$$

$$\frac{f'''(R_0)}{(6H_0^2)^{-2}} = -\frac{\mathcal{P}_3(q_0, j_0, s_0, l_0)\Omega_M + \mathcal{Q}_3(q_0, j_0, s_0, l_0)}{(j_0 - q_0 - 2)\mathcal{R}(q_0, j_0, s_0, l_0)}$$

- *Taylor expansion f(R) in series of R up to third order (higher not necessary)*

- *Linear equations in f(R) and derivatives*

- Ω_M *is model dependent:*

$$\Omega_M = 0.041$$
$$\Omega_M = 0.250.$$

f(R) derivatives and CPL models

"Precision cosmology" → Values of cosmographic parameters?

Cosmographic parameters → Dark energy parameters = equivalent f(R)

CPL approach:
(Chevallier, Polarski, Linder)

$$w = w_0 + w_a(1-a) = w_0 + w_a z(1+z)^{-1}$$

Cosmographic parameters:

$$q_0 = \frac{1}{2} + \frac{3}{2}(1-\Omega_M)w_0$$

$$j_0 = 1 + \frac{3}{2}(1-\Omega_M)\left[3w_0(1+w_0)+w_a\right]$$

$$\begin{aligned} s_0 &= -\frac{7}{2} - \frac{33}{4}(1-\Omega_M)w_a - \frac{9}{4}(1-\Omega_M)\left[9+(7-\Omega_M)w_a\right]w_0 + \\ &- \frac{9}{4}(1-\Omega_M)(16-3\Omega_M)w_0^2 - \frac{27}{4}(1-\Omega_M)(3-\Omega_M)w_0^3 \end{aligned}$$

$$\begin{aligned} l_0 &= \frac{35}{2} + \frac{1-\Omega_M}{4}\left[213+(7-\Omega_M)w_a\right]w_a + \frac{(1-\Omega_M)}{4}\left[489+9(82-21\Omega_M)w_a\right]w_0 + \\ &+ \frac{9}{2}(1-\Omega_M)\left[67-21\Omega_M+\frac{3}{2}(23-11\Omega_M)w_a\right]w_0^2 + \frac{27}{4}(1-\Omega_M)(47-24\Omega_M)w_0^3 + \\ &+ \frac{81}{2}(1-\Omega_M)(3-2\Omega_M)w_0^4 \end{aligned}$$

CPL Cosmography and f(R): the ΛCDM Model

ΛCDM model: $(w_0, w_a) = (-1, 0)$

$$q_0 = \frac{1}{2} - \frac{3}{2}\Omega_M; \quad j_0 = 1; \quad s_0 = 1 - \frac{9}{2}\Omega_M; \quad l_0 = 1 + 3\Omega_M + \frac{27}{2}\Omega_M^2$$

$$f(R_0) = R_0 + 2\Lambda, \qquad f''(R_0) = f'''(R_0) = 0,$$

ΛCDM fits well many data ⟹ cosmographic values strictly depend on Ω_M

$$q_0 = q_0^\Lambda \times (1 + \varepsilon_q), \qquad j_0 = j_0^\Lambda \times (1 + \varepsilon_j),$$
$$s_0 = s_0^\Lambda \times (1 + \varepsilon_s), \qquad l_0 = l_0^\Lambda \times (1 + \varepsilon_l),$$

$$\eta_{20} = f''(R_0)/f(R_0) \times H_0^4$$

$$\eta_{30} = f'''(R_0)/f(R_0) \times H_0^6$$

$$\eta_{20} = \frac{64 - 6\Omega_M(9\Omega_M + 8)}{[3(9\Omega_M + 74)\Omega_M - 556]\,\Omega_M^2 + 16} \times \frac{\varepsilon}{27}$$

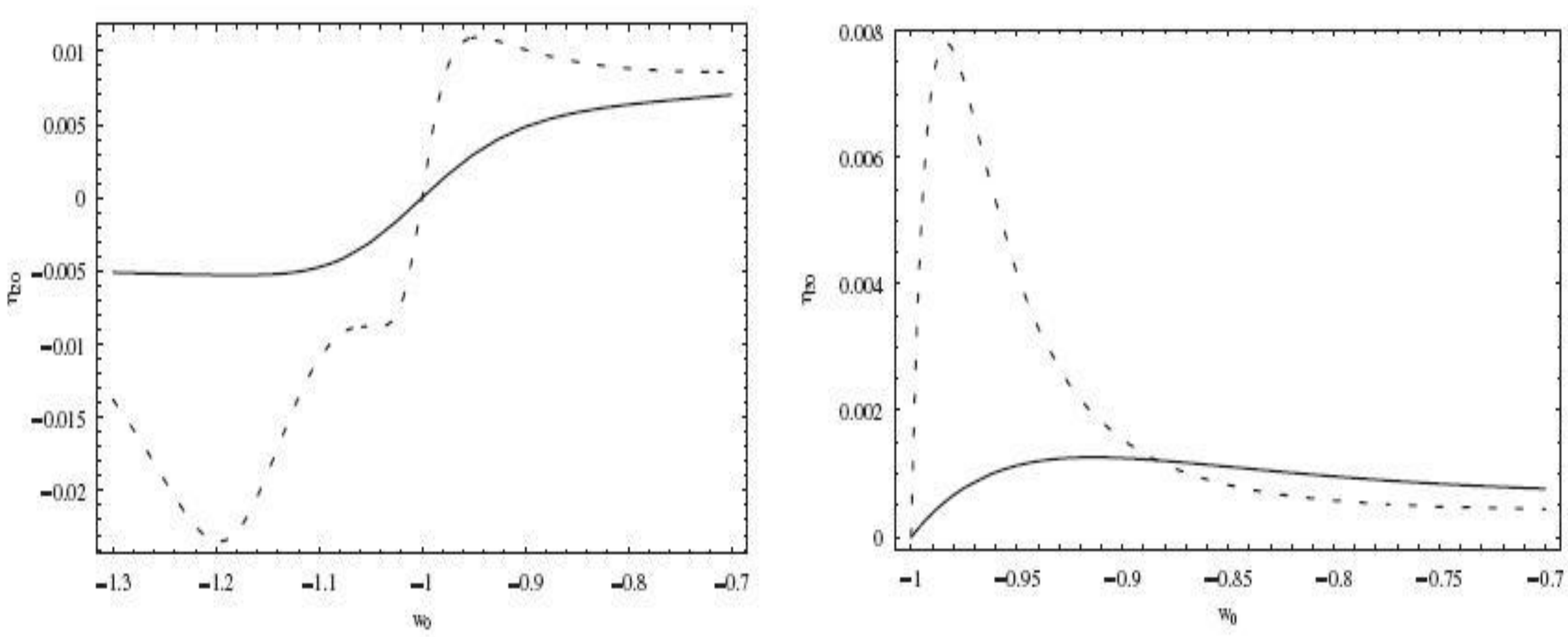
$$\eta_{30} = \frac{6\,[(81\Omega_M - 110)\Omega_M + 40]\,\Omega_M + 16}{[3(9\Omega_M + 74)\Omega_M - 556]\,\Omega_M^2 + 16} \times \frac{\varepsilon}{243\Omega_M^2}$$

$$\begin{cases} \eta_{20} \simeq 0.15 \times \varepsilon & \text{for} \quad \Omega_M = 0.041 \\ \eta_{20} \simeq -0.12 \times \varepsilon & \text{for} \quad \Omega_M = 0.250 \end{cases}$$

$$\begin{cases} \eta_{30} \simeq 4 \times \varepsilon & \text{for} \quad \Omega_M = 0.041 \\ \eta_{30} \simeq -0.18 \times \varepsilon & \text{for} \quad \Omega_M = 0.250 \end{cases}$$

CPL Cosmography and f(R): constant EoS case

- Constant EoS: $w_a = 0$
- Beware of divergences in the f(R) derivatives
- Small deviations from GR
- Large deviations for baryonic dominated universe



Constraining f(R) models by Cosmography

- Procedure:
  1. Estimate (q(0), j(0), s(0), l(0)) observationally
  2. Compute f(Ro), f '(Ro), f " (Ro), f '''(Ro)
  3. Solve for f(R) parameters from derivatives
  4. Constraint f(R) models

- e.g. Double Power-Law: $f(R) = R(1 + \alpha R^n + \beta R^{-m})$

$$\begin{cases} f(R_0) = R_0(1 + \alpha R_0^n + \beta R_0^{-m}) \\ f'(R_0) = 1 + \alpha(n+1)R_0^n - \beta(m-1)R_0^{-m} \\ f''(R_0) = \alpha n(n+1)R_0^{n-1} + \beta m(m-1)R_0^{-(1+m)} \\ f'''(R_0) = \alpha n(n+1)(n-1)R_0^{n-2} \\ \qquad - \beta m(m+1)(m-1)R_0^{-(2+m)}. \end{cases}$$

$$\begin{cases} \alpha = \frac{1-m}{n+m}\left(1 - \frac{\phi_0}{R_0}\right)R_0^{-n} \\ \beta = -\frac{1+n}{n+m}\left(1 - \frac{\phi_0}{R_0}\right)R_0^{m}, \end{cases}$$

$$\begin{cases} \alpha = \frac{\phi_2 R_0^{1-n}[1+m+(\phi_3/\phi_2)R_0]}{n(n+1)(n+m)} \\ \beta = \frac{\phi_2 R_0^{1+n}[1-n+(\phi_3/\phi_2)R_0]}{m(1-m)(n+m)}. \end{cases}$$

$$\begin{cases} \frac{n(n+1)(1-m)(1-\phi_0/R_0)}{\phi_2 R_0[1+m+(\phi_3/\phi_2)R_0]} = 1 \\ \frac{m(n+1)(m-1)(1-\phi_0/R_0)}{\phi_2 R_0[1-n+(\phi_3/\phi_2)R_0]} = 1. \end{cases}$$

$$\begin{cases} m = -[1 - n + (\phi_3/\phi_2)R_0] \\ n = \frac{1}{2}\left[1 + \frac{\phi_3}{\phi_2}R_0 \pm \frac{\sqrt{\mathcal{N}(\phi_0, \phi_2, \phi_3)}}{\phi_2 R_0(1 + \phi_0/R_0)}\right] \end{cases}$$

Constraining f(R) models by Cosmography

- Cosmographic parameters from SNeIa:
  What we have to expect from data

$$q_0 = -0.90 \pm 0.65, \qquad j_0 = 2.7 \pm 6.7,$$
$$s_0 = 36.5 \pm 52.9, \qquad l_0 = 142.7 \pm 320.$$

- Fisher information matrix method:

$$F_{ij} = \left\langle \frac{\partial^2 L}{\partial \theta_i \partial \theta_j} \right\rangle$$

- FM ingredients :

$$\begin{cases} \chi^2(H_0, \mathbf{p}) = \sum_{n=1}^{\mathcal{N}_{SNeIa}} \left[\frac{\mu_{obs}(z_i) - \mu_{th}(z_n, H_0, \mathbf{p})}{\sigma_i(z_i)} \right]^2 \\ d_L(z) = \mathcal{D}_L^0 z + \mathcal{D}_L^1 \, z^2 + \mathcal{D}_L^2 \, z^3 + \mathcal{D}_L^3 \, z^4 + \mathcal{D}_L^4 \, z^5 \\ \sigma(z) = \sqrt{\sigma_{\rm sys}^2 + \left(\frac{z}{z_{\rm max}}\right)^2 \sigma_m^2} \end{cases}$$

- Estimating error on g:

$$\sigma_g^2 = \left| \frac{\partial g}{\partial \Omega_M} \right|^2 \sigma_M^2 + \sum_{i=1}^{i=4} \left| \frac{\partial g}{\partial p_i} \right|^2 \sigma_{p_i}^2 + \sum_{i \neq j} 2 \frac{\partial g}{\partial p_i} \frac{\partial g}{\partial p_j} C_{ij}$$

- Survey: Davis (2007)

$\sigma_M/\Omega_M = 10\%$; $\sigma_{sys} = 0.15$

$N_{SNeIa} = 2000$; $\sigma_m = 0.33$

$z_{max} = 1.7$

→ $\sigma_1 = 0.38$, $\sigma_2 = 5.4$, $\sigma_3 = 28.1$, $\sigma_4 = 74.0$

→ $\sigma_{20} = 0.04$, $\sigma_{30} = 0.04$

- Snap like survey:

$\sigma_M/\Omega_M = 1\%$; $\sigma_{sys} = 0.15$

$N_{SNeIa} = 2000$; $\sigma_m = 0.02$

$z_{max} = 1.7$

→ $\sigma_1 = 0.08$, $\sigma_2 = 1.0$, $\sigma_3 = 4.8$, $\sigma_4 = 13.7$

→ $\sigma_{20} = 0.007$, $\sigma_{30} = 0.008$

- Ideal PanSTARRS survey:

$\sigma_M/\Omega_M = 0.1\%$; $\sigma_{sys} = 0.15$

$N_{SNeIa} = 60000$; $\sigma_m = 0.02$

$z_{max} = 1.7$

→ $\sigma_1 = 0.02$, $\sigma_2 = 0.2$, $\sigma_3 = 0.9$, $\sigma_4 = 2.7$

→ $\sigma_{20} = 0.0015$, $\sigma_{30} = 0.0016$

Conclusions (DE)

- *Extended Gravity seems a viable approach to describe the Dark Side of the Universe. It is based on a straightforward generalization of Einstein Gravity and does not account for exotic fluids.*
- *Following Starobinsky, R can be considered a "geometric" scalar field!).*
- *Comfortable results are obtained by matching the theory with data (SNeIa, Radio-galaxies, Age of the Universe, CMBR).*
- *Transient dust-like Friedman solutions evolving in de Sitter- like expansion (DE) at late times are particularly interesting (debated issue).*
- *Generic quintessential and DE models can be easily "mimicked" by f(R) through an inverse scattering procedure. Cosmography.*
- *A comprehensive cosmological model from early to late epochs should be achieved by f(R). LSS issues have to be carefully addressed.*

Conclusions (DM)

- Rotation curves of galaxies can be naturally reproduced, without huge amounts of DM, thanks to the corrections to the Newton potential, which come out in the low energy limit.
- The baryonic Tully- Fisher relation has a natural explanation in the framework of f(R) theories.
- Effective haloes of elliptical galaxies are reproduced by the same mechanism..
- Good evidences also for galaxy clusters

Furthermore…..

- Orbital period for PSR 1913 + 16 and other binary systems in agreement With f(R)-gravity (probe for massive GWs?).
- Exotic stellar structures could be compatible with f(R)..
- Search for EXPERIMENTUM CRUCIS

Perspectives:

DE & DM as curvature effects

- Matching other DE models
- Jordan Frame and Einstein Frame
- Systematic studies of rotation curves for other galaxies
- Galaxy cluster dynamics (virial theorem, SZE, etc.)
- Luminosity profiles of galaxies in f(R).
- Faber-Jackson & Tully-Fisher, Bullet Cluster

Weak Fields, GW, Further results

- Systematic studies of PPN formalism
- Relativistic Experimental Tests in f(R)
- Gravitational waves and lensing
- Birkhoff 's Theorem in f (R)-gravity
- f(R) with torsion

WORK in PROGRESS! (suggestions are welcome!)