Vector measures, stochastic processes and function spaces

Guillermo P. Curbera

Universidad de Sevilla

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Samara State University
Samara, Russia
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Outline

1. Motivation: the stochastic integral
2. Measures with values in (very) general spaces
3. L^0–valued measures and their L^1 spaces
4. An open problem

Stochastics processes

"A stochastic process is the mathematical abstraction of an empirical process whose development is governed by probabilistic laws."

J. L. Doob
Stochastic Processes, 1953

Stochastics processes

- $(\Omega, \mathcal{F}, \mathcal{P})$ probability space
  $(\mathcal{F}_t)_{t\geq 0}$ increasing family of σ–algebras:
  $$\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}, \quad s < t$$
  $X := \{X_t\}_{t\geq 0}$ random variables $X_t : \Omega \to \mathbb{R}$
  X_t is $\mathcal{F}_t$–measurable
- Paths (trajectories) of X are the maps given, for each $\omega \in \Omega$, by
  $$t \longmapsto X_t(\omega) \in \mathbb{R}$$
  The process is continuous, increasing, has finite variation, etc., according to the paths (or a.e. path) having these properties

The stochastic integral I

- PROBLEM: for processes $X = \{X_t\}$ and $Y = \{Y_t\}$, give meaning to the process $Y \cdot X$:

$$(Y \cdot X)_t := \int_0^t Y_s \, dX_s \qquad (INT)$$

- The difficulty is that, in general, (INT) cannot be defined **pathwise**:

$$(Y \cdot X)_t(\omega) = \int_0^t Y_s(\omega) \, dX_s(\omega) \qquad (\omega - INT)$$

i.e.: for each $\omega \in \Omega$

Two examples I

Poisson process $\{X_t\}$

- $\mathbb{N}$–valued process
- $X_t - X_s$ is independent of $\mathcal{F}_s$, for $s < t$
- $$\mathcal{P}\big(\{\omega \in \Omega : X_t(\omega) - X_s(\omega) = n\}\big) = \frac{\lambda^n (t-s)^n e^{-\lambda(t-s)}}{n!}$$
- the paths have bounded variation (a.e.), so $(\omega - INT)$ can be interpreted as a Stieltjes integral

$$(Y \cdot X)_t(\omega) = \int_0^t Y_s(\omega)\, dX_s(\omega) = \int_0^t Y_\omega(s)\, dX_\omega(s)$$

Two examples II

Brownian motion $\{X_t\}$

- $X_t - X_s$ is independent of $\mathcal{F}_s$, for $s < t$
- $$\mathcal{P}[X_t - X_s \leq A] = \frac{1}{\sqrt{2\pi(t-s)}} \int_{-\infty}^{A} \exp\left(\frac{-x^2}{2(t-s)}\right) dx$$
- a.e. path has unbounded variation on every interval $[s, t]$
- thus, the integral
$$(Y \cdot X)_t = \int_0^t Y_s \, dX_s$$
cannot be defined pathwise

The stochastic integral II

- Construction by Itô (1942) and Kunita & Watanabe (1967)
- $\{X_t\}$ continuous square integrable *martingale*:

$$EX_t^2 < \infty, \quad E(X_t|\mathcal{F}_s) = X_s,\ s < t$$

- Doob–Meyer decomposition of $X^2 := \{X_t^2\}$

$$X_t^2 = M_t + \langle X \rangle_t$$

  M_t a martingale, $\langle X \rangle_t$ an increasing process with $E\langle X \rangle_t < \infty$
- $\langle X \rangle_t$ is the *quadratic variation* of X (for Brownian motion $\langle X \rangle_t = t$)

The stochastic integral II

- Consider on $[0,\infty) \times \Omega$ the product measure $\mu_X := \langle X \rangle_t \otimes \mathcal{P}$

$$\mu_X(A) = E \int_0^\infty \chi_A(t,\omega)\, d\langle X \rangle_t(\omega)$$

for $A \in \mathcal{B}[0,\infty) \otimes \mathcal{F}$

- Set $\mathcal{L}(X) := \left\{ Y = \{Y_t\} : [Y]_t < \infty, \forall t \geq 0 \right\}$ where

$$[Y]_t := \left\| Y \cdot \chi_{[0,t]\times\Omega} \right\|_{L^2(\mu_X)} = \left(E \int_0^t Y_s^2\, d\langle X \rangle_s \right)^{1/2}$$

The stochastic integral II

- Let $\mathcal{L}_0$ be the collection of all *simple processes*:

$$Y_t(\omega) = \xi_0(\omega)\chi_{\{0\}}(t) + \sum_{i=0}^{\infty} \xi_i(\omega)\chi_{(t_i,t_{i+1}]}(t)$$

  where ξ_i is $\mathcal{F}_i$–measurable and $\sup_{i,\omega} |\xi_i(\omega)| < \infty$

- Then, the integral of $Y \in \mathcal{L}_0$ with respect to X is

$$(Y \cdot X)_t := \sum_{i=0}^{\infty} \xi_i(X_{t\wedge t_{i+1}} - X_{t\wedge t_i}) = \sum_{i=0}^{n-1} \xi_i(X_{t_{i+1}} - X_{t_i}) + \xi_n(X_t - X_{t_n})$$

- Finally, the integral is extended from $\mathcal{L}_0$ to $\mathcal{L}(X)$ by approximation (very technical)

Stochastic integration via L^0–valued measures

In the late 1970s, Metivier and Pellaumail proposed an alternative approach to the construction of the stochastic integral using integration with respect to L^0–valued measures

P. A. Meyer:
“le premier sans doute à considerer l’«horrible» espace $\mathtt{L}^0$ comme un object digne d’intérêt dans la théorie de l’intégral stochastique”

Stochastic integration via L^0–valued measures

- Let $(\Omega, \mathcal{F}, \mathcal{P}, (\mathcal{F}_t)_{t\geq 0})$
- Set $\mathcal{R} := \left\{ F \times (s,t] : s < t, F \in \mathcal{F}_s \right\}$ (predictable rectangles)
- Given a stochastic process $X = \{X_t\}$, define the measure

$$\nu_X : \mathcal{R} \longrightarrow L^0(\Omega, \mathcal{F}, \mathcal{P})$$

by

$$\nu_X\big(F \times (s,t]\big) := \chi_F \cdot (X_t - X_s) \in L^0$$

- the measure ν_X is extended (by additivity) to the ring $\mathcal{A}$ generated by $\mathcal{R}$

Stochastic integration via L^0–valued measures

Issues to be addressed:

- Under which conditions on $\{X_t\}$ the measure ν_X can be extended to the σ–algebra generated by $\mathcal{A}$?

  (Dellacherie–Meyer–Mokobodski, Bichteler) semimartingales

- An integration theory for L^0–valued measures is nedded
- Which processes $\{Y_t\}$ can be integrated with respect to ν_X?
- Do they coincide with the ones integrable (in the classical sense) with respect to $\{X_t\}$?
- What properties does the integral process $\{(Y \cdot X)_t\}$ inherit from $\{X_t\}$ and $\{Y_t\}$?

Authorship

Next we present results from joint work with

Olvido Delgado Garrido
Universidad de Sevilla
Spain

General setting

- X an F–space: complete linear metric space (no local convexity assumed)
- Examples: $L^P([0,1])$ for $0 \leq p \leq \infty$
- $\nu \colon \Sigma \to X$ a countably additive measure
- How to integrate scalar functions with respect to ν?
    - the integral of a characteristic function is (can only be)

$$\int \chi_A \, d\nu := \nu(A), \quad A \in \Sigma$$

    - next, extended to simple functions $S(\Sigma)$ by linearity

Conditions for extending the integral

- To extend the integral to bounded functions it is necessary that:

$$\text{if } (f_n) \subset S(\Sigma) \text{ with } \|f_n\|_\infty \to 0, \text{ then } \nu(f_n) := \int f_n \, d\nu \to 0 \text{ in } X$$

- Equivalently, the convex hull of the range of ν

$$co\ \nu(\Sigma) = co\{\nu(A) : A \in \Sigma\}$$

  is a bounded set in X

- Holds in: Banach spaces, locally convex spaces, locally bounded spaces,...
  Not in general: Turpin (1972, 1975)

The BMT

- Recall that X satisfies the *bounded multiplier test (BMT)*if

$$\sum x_n \text{ unconditionally convergent } \Rightarrow \sum a_n x_n \text{ converges } \forall (a_n) \in \ell^\infty$$

Holds in: Banach spaces, locally convex spaces
Not in general: Rolewicz & Ryll-Nardzewski (1967)

- The following are equivalent:
  a) X satisfies BMT
  b) Every measure $\nu : \Sigma \to X$ with bounded range has

$$co\ \nu(\Sigma) \text{ bounded}$$

Integration and L^1 space

- Following Rolewicz, Turpin, and Thomas:
- A function $f : \Omega \to \mathbb{R}$ is *integrable* with respect to ν if there exist $(\varphi_n) \subset S(\Sigma)$ such that
  a) $\varphi_n \to f$ ν–a.e.
  b) $\left(\int h\varphi_n \, d\nu \right)$ converges in X, for every $h \in L^\infty(\nu)$
- The space $L^1(\nu)$ of (classes of) integrable functions is an F–space for the *F–norm*:

$$\dot{\nu}(f) := \sup \left\{ \left\| \int g \, d\nu \right\|_X : \varphi \text{ simple }, |\varphi| \leq |f| \right\}$$

where $\|\cdot\|_X$ is the F–norm corresponding to the distance in X

The L^1 space

$L^1(\nu)$ is an order continuous F–function space.

Theorem (C. & Delgado)

Let $T : L^\infty(\lambda) \to X$ continuous and linear, X F–space, such that

- *$T(\chi_{A_n}) \to T(\chi_{\cup A_n})$ in X, for every increasing (A_n).*

Then, the measure $\nu(A) := T(\chi_A)$ satisfies

- *co $\nu(\Sigma)$ is bounded in X.*
- *$L^1(\nu)$ is the **largest** order continuous F–function space where T can be extended, still with values in X.*

Stochastic measures

A stochastic measure is $\nu\colon \Sigma \to L^0([0,1])$

- $L^0([0,1]) = \Big\{ f\colon [0,1] \to \mathbb{R} : f \text{ measurable} \Big\}$
- Topology of convergence in measure

$$f_n \to f \iff \forall \varepsilon > 0, \;\; \lim_n m\big([|f_n - f| > \varepsilon]\big) = 0$$

- L^0 is an F–space for the F–norm:

$$\|f\|_0 = \int_0^1 \frac{|f(t)|}{1 + |f(t)|}\, dt$$

- L^0 has trivial dual space

Unexpected behaviour

- Consider the measure $\nu\colon \mathcal{M}[0,1] \to L^0([0,1])$ be given by

  $$\nu(A) := m(A)\cdot\chi_{[0,1]}$$

  it has one-dimensional range, $\{a\chi_{[0,1]} : 0 \le a \le 1\}$, and bounded variation

  $$|\nu|(A) = m(A)$$

- Consider the multiplication operator $M_h\colon L^0([0,1]) \to L^0([0,1])$ given by

  $$f \longmapsto M_h(f) = f\cdot h$$

  The measure $\mu := M_h \circ \nu$ given by

  $$\mu(A) = m(A)\cdot h$$

  has also one-dimensional range, but **unbounded variation** when $h \notin L^1([0,1])$

Two big theorems I

Theorem

L^0-valued measures are bounded (have bounded range)

- Talagrand M., *Les mesures vectorielles à valeurs dans L^0 sont bornées*, Ann. Sci. École Norm. Sup. (4) **14** (1981), 445–452.
- Kalton N. J., Peck N. T. and Roberts J. W., *L_0-valued vector measures are bounded*, Proc. Amer. Math. Soc. **85** (1982), 575–582.

Two big theorems II

Theorem

L^0 satisfies the bounded multiplier test (BMT)

- B. S. Khasin, *Stability of unconditional convergence almost everywhere*, Mat. Zametki **14** (1973), 645–654.
- B. Maurey and G. Pisier, *Un théorème d'extrapolation et ses conséquences*, C. R. Acad. Sci. Paris Sér. A-B **277** (1973), A39–A42.

Consequences

- For every stochastic measure $\nu\colon \mathcal{M} \to L^0([0,1])$ the integral can be constructed and hence, the space $L^1(\nu)$
- Better conditions for integrability (with respect to L^0–valued measures) hold. The following are equivalent
  - f is integrable with respect to ν
  - The sequence $(\nu(g_n))$ is a C–sequence in $L^0([0,1])$, whenever (g_n) are disjoint simple functions with $|g_n| \le |f|$

    (x_n) is a C–sequence if $\sum a_n x_n$ converges for every $(a_n) \in c_0$
  - $\sum \big|\nu(g_n)(t)\big|^2 < \infty$, for m–a.e. $t \in [0,1]$, whenever (g_n) are disjoint simple functions with $|g_n| \le |f|$

L^1 of a stochastic measure I

- X is a *C–space* if, for any C–sequence (x_n) (i.e. $\sum a_n x_n$ converges for all $(a_n) \in c_0$), the series $\sum x_n$ is convergent in X
- $L^0([0,1])$ is a C–space (L. Schwartz, 1969)

Theorem (C. & Delgado)

Let ν be a positive stochastic measure. Then $L^1(\nu)$ is a C–space.

Note: we have not been able (yet) to remove the $\nu \geq 0$ condition

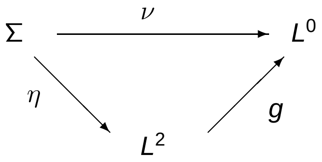
L^1 of a stochastic measure I

The proof makes use of a deep result:

Theorem (Talagrand, 1984)

L^0–valued measures have a control measure

- Proved via Maurey's factorization



- Existence of control measure: for Banach space–valued meaures
- Talagrand (2006) has solved in the negative the general question (equivalent to Maharam's problem)

L^1 of a stochastic measure II

Theorem (C. & Delgado)

Let ν be a stochastic measure. Then $L^1(\nu)$ satisfies the bounded multiplier test.

In the proof we use the following result due to Ryll-Nardzewski & Woyczyński:

Lemma

Let $(g_j)_1^m$ be measurable functions and $|b_j| \leq 1$. Then

$$m\Big(\Big[\Big|\sum_{j=1}^{m} b_j g_j(t)\Big| > 8\varepsilon\Big]\Big) \leq 8 \max_{\delta_j=\pm 1} m\Big(\Big[\Big|\sum_{j=1}^{m} \delta_j g_j(t)\Big| > \varepsilon\Big]\Big),$$

where $[|g| > r] := \{t \in [0,1] : |g(t)| > r\}$.

An important question

What spaces arise as L^1 of a stochastic measure?

Some examples:

- For $\nu(A)(x) := \int_A \frac{1}{|x+y|^\alpha}\, dy \in L^0$, we have $L^1(\nu) = L^1([0,1])$
- There exists an L^0–valued measure for which $L^1(\nu) = L^2([0,1])$
- For $\nu(A)(x) := \int_0^1 \chi_A(\frac{x+y}{2})\, dy \in L^0$, we have $L^1(\nu) = L^1_{\mathrm{loc}}(0,1)$
- For $\nu(A) := \chi_A \in L^0$, we have $L^1(\nu) = L^0([0,1])$
- We obtain function spaces which are: Hilbert, Banach, Fréchet, F–space

Related known results I

In the case of Banach space valued measures (the integration theory corresponds with that of Bartle–Dunford–Schwartz) the answer is:

Theorem (C.)

The class of spaces $L^1(\nu)$, for ν a vector measure with values in a Banach space, coincides with the class of all order continuous Banach lattices with a weak order unit.

Related known results II

In the case of F–space valued measures, the answer is:

Theorem (C. & Delgado)

The class of spaces $L^1(\nu)$, for ν a vector measure with values in an F–space, coincides with the class of all order continuous F–function spaces over a finite measure space.

- An *F–function space* over $(\Omega, \Sigma, \lambda)$, with $\lambda(\Omega) < \infty$, is an F–space X of measurable functions, satisfying:
  - (i) $f \in L^0$, $g \in X$ and $|f| \leq |g|$, imply $f \in X$ and $\|f\| \leq \|g\|$.
  - (ii) $\chi_A \in X$ for every $A \in \Sigma$.
- $L^\infty(\Omega, \Sigma, \lambda) \subset X \subset L^0(\Omega, \Sigma, \lambda)$ continuously
- The definition extends that of Banach function space

Related known results III

In the case of stochastic measures with the additional properties

- *independence*: (A_i) disjoint $\Rightarrow (\nu(A_i))$ independent random variables
- *homogeneous*: if A, B are congruent in [0,1], then $\nu(A)$ and $\nu(B)$ have the same distribution
- $\nu(A)$ are symmetric random variables

The answer is:

Theorem (Urbanik & Woyczyński)

The class of spaces $L^1(\nu)$, for ν a L^0–valued, symmetric, homogeneous, independent measure, coincides with the class of all generalized Orlicz spaces L^φ where $\varphi(0) = 0$, φ is increasing, and $\varphi(\sqrt{x})$ is concave.

Note: $L^p([0,1])$ for $0 < p \leq 2$

Still an open question

What spaces arise as L^1 of a stochastic measure?

Conjecture:
every F–function space which is an order continuous C–space satisfying the bounded multiplier test