Sobolev inequalities in function spaces: optimality and extension

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Outline

1. Extensions of Sobolev imbedding
2. An application: compactness
3. Extensions beyond r.i. spaces?
4. Refined Sobolev inequalities

Sobolev's classical inequality

Theorem (Sobolev, 1938)

Let $1 \leq p < n$*. There exist a constant* $C > 0$ *such that*

$$\|u\|_{L^q(\Omega)} \leq C \, \| \, |\nabla u| \, \|_{L^p(\Omega)}, \quad u \in \mathcal{C}_0^1(\Omega),$$

where $q := \frac{np}{n-p}$.

- $\Omega \subset R^n$ a bounded domain
- $m_n(\Omega) = 1$, m_n Lebesgue n-dimensional measure
- $\mathcal{C}_0^1(\Omega) = \{u : \Omega \to \mathbb{R}, \text{ of class } \mathcal{C}^1 \text{ with compact support}\}$

- $q := \frac{np}{n-p} > p \Rightarrow \; L^q(\Omega) \subsetneq L^p(\Omega)$

Sobolev's imbedding

- Sobolev's inequality

$$\|u\|_{L^q(\Omega)} \leq C \, \| \, |\nabla u| \, \|_{L^p(\Omega)}, \quad u \in \mathcal{C}_0^1(\Omega),$$

- Sobolev's imbedding

$$W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$$

where $W_0^{1,p}(\Omega)$ is the closure of $C_0^1(\Omega)$ for the norm

$$\|u\|_{W_0^{1,p}(\Omega)} := \|u\|_{L^p(\Omega)} + \| \, |\nabla u| \, \|_{L^p(\Omega)}$$

Refining the inequality–Extending the imbedding

- Q1: Can we find a norm $\| \cdot \|_{X(\Omega)}$, smaller than $\| \cdot \|_{L^p(\Omega)}$, such that

$$\|u\|_{L^q(\Omega)} \leq C \| \, |\nabla u| \, \|_{X(\Omega)} \ \left(\leq C \| \, |\nabla u| \, \|_{L^p(\Omega)} \right)$$

- Q1': Can we find a function space $X(\Omega)$, larger than $L^p(\Omega)$, such that

$$\left(W_0^{1,p}(\Omega) \subseteq \right) W_0^1 X(\Omega) \hookrightarrow L^q(\Omega)$$

  here $W_0^1 X(\Omega)$ is the closure of $C_0^1(\Omega)$ for the norm

$$\|u\|_{W_0^1 X(\Omega)} := \|u\|_{X(\Omega)} + \| \, |\nabla u| \, \|_{X(\Omega)}$$

- Q2: Which are the smallest (largest) of such norms (spaces)?

Reduction to a one dimensional problem

Theorem (Kerman & Pick, 2006)

Let X, Y be $\square$ spaces on [0,1]. (*for example* $X = L^q, Y = L^p$)

Then:

$$\left.\begin{array}{c} \|u\|_{X(\Omega)} \leq C\| \, |\nabla u| \, \|_{Y(\Omega)} \\ \text{for all } u \in C_0^1(\Omega) \end{array}\right\} \iff \left\{\begin{array}{c} \|Tf\|_X \leq K\|f\|_Y \\ \text{for all } f \in X \end{array}\right.$$

where T is the kernel operator associated with Sobolev's inequality

$$t \in [0,1] \longmapsto Tf(t) := \int_t^1 f(s) s^{\frac{1}{n}-1} \, ds, \qquad f : [0,1] \to \mathbb{R}$$

Adequate spaces

What are □ spaces?

Rearrangement invariant (r.i.) spaces:

- $X(\Omega)$ Banach space of (classes) of measurable functions on Ω
- with norm $\|\cdot\|_{X(\Omega)}$ compatible with the a.e. order:

$$\left.\begin{array}{r} |v| \le |u| \\ u \in X(\Omega) \end{array}\right\} \Rightarrow \left\{\begin{array}{l} v \in X(\Omega) \\ \|v\|_{X(\Omega)} \le \|u\|_{X(\Omega)} \end{array}\right.$$

- rearrangement invariant:

$$\left.\begin{array}{r} u \in X(\Omega) \\ v \sim u \end{array}\right\} \Rightarrow \left\{\begin{array}{l} v \in X(\Omega) \\ \|v\|_{X(\Omega)} = \|u\|_{X(\Omega)} \end{array}\right.$$

where $v \sim u$ if $\quad m_n(\{|v| > \lambda\}) = m_n(\{|u| > \lambda\}),\ \lambda > 0$

r.i. spaces on Ω

How to define r.i. spaces on Ω from r.i. spaces o [0,1]:

- let X be a r.i. space on [0,1]
- set $X(\Omega) := \{u : \Omega \to \mathbb{R} : u^* \in X\}$

  here u^* is the *decreasing rearrangement* of u:

  $u^* : [0,1] \to [0,\infty)$ decreasing with $u^* \sim u$
- define the norm $\|u\|_{X(\Omega)} := \|u^*\|_X$

Examples of r.i. spaces

- L^p, Orlicz spaces (change t^p by $\varphi(t)$, increasing and convex)
- spaces of functions with p-th exponential integrability

$$\int_\Omega \exp(|u|/\lambda)^p < \infty, \text{ for some } \lambda > 0$$

- Lorentz $\Lambda(\varphi)$ spaces, for φ increasing and concave

$$\|u\|_{\Lambda(\varphi)} := \int_0^1 u^*(s)\, d\varphi(s) < \infty \quad (ex.\ L^{p,1})$$

- Marcinkiewicz $M(\varphi)$ spaces, for $\varphi(t)$ quasi-concave...

$$\|u\|_{M(\varphi)} := \sup_{0<t\le 1} \frac{1}{\varphi(t)} \int_0^t u^*(s)\, ds < \infty \quad (ex.\ weak\ L^p)$$

The 1-dim associated operator

- Recall: $Tf(t) = \int_t^1 f(s) s^{\frac{1}{n}-1}\, ds$
- Given a r.i. space X

  let $[T, X]^{ri}$ be the largest r.i. space Y such that

  $$T : Y \longrightarrow X \qquad \text{continuously}$$

- This means:
  - $T : [T, X]^{ri} \longrightarrow X$ continuously
  - If Z is a r.i. space with $T : Z \to X$ continuously, then

    $$Z \subset [T, X]^{ri}$$

Refining–Extending

Consequently

- The optimal r.i. Sobolev inequality for fixed range $X(\Omega)$ is

$$\|u\|_{X(\Omega)} \leq C\| \, |\nabla u| \, \|_{[T,X]^{ri}(\Omega)} \quad \left(= C\| \, |\nabla u|^* \, \|_{[T,X]^{ri}}\right)$$

- The optimal r.i. Sobolev imbedding for fixed range $X(\Omega)$ is

$$W_0^1[T,X]^{ri}(\Omega) \hookrightarrow X(\Omega)$$

where

$$\|u\|_{W_0^1[T,X]^{ri}(\Omega)} := \|u^*\|_{[T,X]^{ri}} + \| \, |\nabla u|^* \, \|_{[T,X]^{ri}}$$

Examples I: optimal inequalities

- For $X = L^p \quad \Rightarrow \quad [T, L^p]^{ri} = L^{\frac{np}{n+p},p} \quad (n' < p < \infty)$

$$\|u\|_{L^p(\Omega)} \leq C \, \| \, |\nabla u| \, \|_{L^{\frac{np}{n+p},p}(\Omega)} \leq \| \, |\nabla u| \, \|_{L^{\frac{np}{n+p}}(\Omega)}$$

- For $X = L^\infty \quad \Rightarrow \quad [T, L^\infty]^{ri}$ is the Lorentz space $L^{n,1}$

$$\|u\|_{L^\infty(\Omega)} \leq C \, \| \, |\nabla u| \, \|_{L^{n,1}(\Omega)}$$

- For $X = L^{p,1} \quad \Rightarrow \quad [T, L^{p,1}]^{ri} = L^{\frac{n}{n+p},1}$

$$\|u\|_{L^{p,1}(\Omega)} \leq C \, \| \, |\nabla u| \, \|_{L^{\frac{n}{n+p},1}(\Omega)}$$

- For $X = L^{p,\infty} \quad \Rightarrow \quad [T, L^{p,\infty}]^{ri} = L^{\frac{n}{n+p},\infty}$

$$\|u\|_{L^{p,\infty}(\Omega)} \leq C \, \| \, |\nabla u| \, \|_{L^{\frac{n}{n+p},\infty}(\Omega)}$$

Examples II: optimal imbeddings

- For $X = L^p$: $W_0^1 L^{\frac{np}{n+p},p}(\Omega) \hookrightarrow L^p(\Omega)$
- For $X = L^\infty$: $W_0^1 L^{n,1}(\Omega) \hookrightarrow L^\infty(\Omega)$
- For $X = L^{p,1}$: $W_0^1 L^{\frac{n}{n+p},1}(\Omega) \hookrightarrow L^{p,1}(\Omega)$
- For $X = L^{p,\infty}$: $W_0^1 L^{\frac{n}{n+p},\infty}(\Omega) \hookrightarrow L^{p,\infty}(\Omega)$

All these imbeddings are **optimal**

The limiting case $p = n$

Determining $[T, X]^{ri}$ need not be simple.

The limiting case $p = n$:

- If $|\nabla u| \in L^n(\Omega)$ $\quad \Rightarrow \quad |\nabla u| \in L^p(\Omega)$, for every $1 \le p < n$,

$$\Rightarrow \quad u \in L^{\frac{np}{n-p}}(\Omega), \text{ for every } 1 \le p < n,$$

$$\Rightarrow \quad u \in L^q(\Omega), \text{ for every } 1 \le q < \infty$$

but $\quad \not\Rightarrow \quad u \in L^\infty(\Omega)$

- There exists u with $|\nabla u| \in L^n(\Omega)$ but

$$u \in \bigcap_{1 \le q < \infty} L^q(\Omega) \setminus L^\infty(\Omega)$$

The limiting case $p = n$

Theorem (Pokhozhaev 1965, Trudinger 1967)

There exist a constant $C > 0$ such that

$$\|u\|_{L_\varphi(\Omega)} \leq C \| \, |\nabla u| \, \|_{L^n(\Omega)}, \quad u \in \mathcal{C}_0^1(\Omega),$$

where $L_\varphi(\Omega)$ is the Orlicz space for $\varphi(t) := \exp(t^{n'}) - 1$

- $L_\varphi(\Omega) = \mathrm{Exp}\, L^{n'}$ is the space of all functions with n'–exponential integrability

$$\int_\Omega \exp \left| \frac{u(\omega)}{\lambda} \right|^{n'} d\omega < \infty \quad \textit{for some} \quad \lambda > 0$$

- Note: $L^\infty(\Omega) \subsetneq L_\varphi(\Omega) \subsetneq \bigcap_{1 \leq q < \infty} L^q(\Omega)$

The limiting case $p = n$

- The optimal imbedding is

$$
\begin{array}{ccc}
W_0^{1,n}(\Omega) & \longrightarrow & \mathrm{Exp}\, L^{n'} \\
\downarrow & \nearrow & \\
W_0^1[T, \mathrm{Exp}\, L^{n'}]^{ri}(\Omega) & &
\end{array}
$$

- It can be shown that

$$L^{n,1}(\log L)^{-1/n'} \subsetneq [T, \mathrm{Exp}\, L^{n'}]^{ri} \subsetneq L^{n,\infty}(\log L)^{-1/n'}$$

$[T, \mathrm{Exp}\, L^{n'}]^{ri} \neq L^{n,q}(\log L)^{-1/n'}$, for $1 \leq q \leq \infty$ (Lorentz-Zygmund)

A compactness theorem

Theorem (Rellich–Kondrachov, 1930, 1945)

Let $1 \leq p < n$. The imbedding

$$W_0^{1,p}(\Omega) \longrightarrow L^q(\Omega)$$

is compact whenever $q < \frac{np}{n-p}$

- Q: Under what conditions on the r.i. spaces X, Y we have compactness of the imbedding

$$W_0^1 Y(\Omega) \hookrightarrow X(\Omega)?$$

- Q: When is it compact the optimal imbedding

$$W_0^1 [T, X]^{ri}(\Omega) \hookrightarrow X(\Omega)?$$

A compactness theorem

Theorem

$$\left.\begin{array}{c} W^1_0 Y(\Omega) \hookrightarrow X(\Omega) \\ \textit{is compact} \end{array}\right\} \iff \left\{\begin{array}{l} T : Y \to X \\ \textit{is compact} \end{array}\right.$$

Theorem

$$\left.\begin{array}{c} W^1_0 [T, X]^{ri}(\Omega) \hookrightarrow X(\Omega) \\ \textit{is compact} \end{array}\right\} \begin{array}{c} \textit{(almost)} \\ \iff \end{array} \lim_{t \to 0^+} t^{-1/n'} \|\chi_{[0,t]}\|_X = 0$$

A compactness theorem

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Extension to Banach function spaces

- Q: Given a r.i. space X does there exists a Banach function space Z such that we have the imbedding

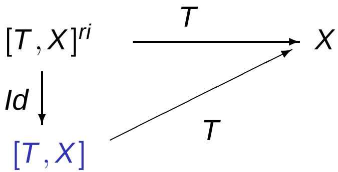
$$W_0^1 Z(\Omega) \hookrightarrow X(\Omega)$$

- Only interesting in the case $[T, X]^{ri}(\Omega) \subsetneq Z(\Omega)$

$$\begin{array}{ccc} W_0^1[T, X]^{ri}(\Omega) & \longrightarrow & X(\Omega) \\ \downarrow & \nearrow & \\ W_0^1 Z(\Omega) & & \end{array}$$

Extending from r.i. to B.f.s.

- Evidence: the associated operator T can be extended to a domain $[T,X]$ larger (in general) than $[T,X]^{ri}$



- Ex.: $[T, L^{p,1}]^{ri} = L^{\frac{np}{n+p},1}[0,1] \subsetneq L^1\big(s^{\frac{1}{n}+\frac{1}{p}-1}ds\big) = [T, L^{p,1}]$
- If $[T,X]^{ri} \subsetneq [T,X]$, then it is theoretically possible to extend Sobolev imbedding

Theorem (C. & Ricker)

$$[T,X]^{ri} = [T,X] \iff X = L^{n',1}([0,1])$$

Difficulties

Two main difficulties:

- If $[T, X]$ is not r.i., there are problems for defining $[T, X](\Omega)$ since, for $u : \Omega \to \mathbb{R}$,

$$\|u\|_{[T,X](\Omega)} = \|u^*\|_{[T,X]} = \|Tu^*\|_X$$

  does not (necessarily) give rise to a norm

- The optimal B.f.s. Sobolev's inequality

$$\|u\|_{X(\Omega)} \leq C \, \| \, |\nabla u| \, \|_{[T,X](\Omega)}$$

  may or may not hold

Answers I

Theorem (C.& Ricker)

Let X be a r.i. space over [0,1]. Then $[T,X](\Omega)$, with

$$\|u\|_{[T,X](\Omega)} := \|u^*\|_{[T,X]} = \|Tu^*\|_X,$$

is always a r.i. quasi-Banach function space.

Proof: the kernel $K(t,s) = \chi_{[t,1]}(s)s^{1/n-1}$, associated to Sobolev inequality, satisfies

$$K\left(t,\frac{s}{2}\right) \leq 2^{1/n'} \cdot K\left(\frac{t}{2},s\right), \quad (t,s) \in [0,1] \times \left[0,\frac{1}{2}\right]$$

Answers II

Theorem (C.& Ricker)

Lat X be r.i. over [0,1]. Suppose that $\lim_{t\to 0} \frac{\varphi_X(t)}{t^{1/n'}} = 0$*. Then, the optimal B.f.s. Sobolev inequality*

$$\|u\|_{X(\Omega)} \leq C \, \| \, |\nabla u| \, \|_{[T,X](\Omega)}$$

does not hold

- $\varphi_X(t) := \|\chi_{[0,t]}\|_X$ is the fundamental function of X
- the result holds, for example, for

$$X = L^{p,q}([0,1]), \text{ with } 1 \leq p < n', 1 \leq q \leq \infty$$

Answers III

Theorem (C.& Ricker)

Lat X be a r.i. over [0,1]. Suppose that the Boyd indices of X satisfy $0 < \underline{\alpha}_X \leq \overline{\alpha}_X < 1/n'$. *Then, the optimal B.f.s. Sobolev inequality*

$$\|u\|_{X(\Omega)} \leq C \, \| \, |\nabla u| \, \|_{[T,X](\Omega)}$$

does hold, but it is not a further extension of the optimal ***r.i.*** *B.f.s. Sobolev inequality*

$$\|u\|_{X(\Omega)} \leq C \, \| \, |\nabla u| \, \|_{[T,X]^{ri}(\Omega)}$$

- Boyd indices are related to the norm in X of the *dilation operators* $D_a f(t) = f(t/a)$

Answers III

Comments:

- The result is surprising since for all X with $0 < \underline{\alpha}_X \leq \overline{\alpha}_X < 1/n'$ we have

$$[T, X]^{ri} \neq [T, X]$$

- the underling reason is that $[T, X](\Omega)$ and $[T, X]^{ri}(\Omega)$ are isomorphic iff

  the cone of positive, decreasing functions of $[T, X] \subset [T, X]^{ri}$

- the result holds, for example, for

$$X = L^{p,q}([0,1]), \text{ with } n' < p < \infty, 1 \leq q \leq \infty$$

Answers III

The proof is a consequence of a more general result:

Theorem (C.& Ricker)

Lat X be a r.i. over [0,1]. Suppose that the operator

$$g \longmapsto t^{-1/n}\big(s^{1/n}Hg(s)\big)^{*}(t),$$

where H is the Hardy averaging operator, is bounded from X' to X' (the associate space of X).

Then, the optimal B.f.s. Sobolev inequality ***does hold*** *but it* ***is not*** *a further extension of the optimal r.i. Sobolev inequality*

- the proof is selfimproving:
  first is done for $L^{p,q}$, then for all X with $0 < \underline{\alpha}_X \leq \overline{\alpha}_X < 1/n'$

Refined Sobolev inequalities

Refined Sobolev inequalities:

From a result of Edmunds, Kerman & Pick (2000) we have:

$$T : [T, X] \longrightarrow X \iff \|u\|_{X(\Omega)} \leq C \left\| \frac{d}{dt} u_{\triangledown} \right\|_{[T,X]}$$

where:

$$u_{\triangledown}(t) := \int_{\{x\in\Omega : |u(x)|>u^*(t)\}} |\nabla u(x)|\, dx,$$

$u_{\triangledown}$ appears in Talenti's inequality:

$$t^{1/n'} \frac{d}{dt} u^*(t) \lesssim \frac{d}{dt} \int_{\{x\in\Omega : |u(x)|>u^*(t)\}} |\nabla u(x)|\, dx$$

Refined Sobolev inequalities

Theorem (C.& Ricker)

For X a r.i. space on [0,1]:

$$\|u\|_{X(\Omega)} \leq C \left\| t \mapsto \int_{\Omega(u,t)} |\nabla u(x)| \, \alpha(u,x)^{-1/n'} \, dx \right\|_X$$

where

$$\Omega(u,t) := \{z \in \Omega \, : \, 0 < |u(z)| \leq u^*(t)\},$$

and

$$\alpha(u,x) := \begin{cases} m_n(\{z \in \Omega : |u(z)| \geq |u(x)|\}) & \text{if } u(x) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Refined Sobolev inequalities

Are this inequalities of interest?

We compare the refined Sobolev inequalities with the optimal r.i. Sobolev inequalities:

For $X = L^\infty$:

$$\|u\|_\infty \le C \int_\Omega |\nabla u(x)|\, \alpha(u,x)^{-1/n'}\, dx \le C \int_0^1 |\nabla u|^*(t) t^{-1/n'}\, dt$$

For $X = L^1$:

$$\|u\|_1 \le C \int_\Omega |\nabla u(x)|\, \alpha(u,x)^{1/n}\, dx \le C \int_0^1 |\nabla u|^*(t)\, dt$$

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