# Solid extensions of the Cesàro operator on the Hardy space $H^2(\mathbb{D})$ and the case $p \neq 2$.

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# Outline

# The Cesàro operator on $\ell^p$

- In 1915 Hardy proved:

$$\sum_{n=0}^{\infty} |a_n|^2 < \infty \Longrightarrow \sum_{n=0}^{\infty} \left| \frac{1}{n+1} \sum_{k=0}^{n} a_k \right|^2 < \infty.$$

In 1920 he extended the result for $1 < p < \infty$:

$$\sum_{n=0}^{\infty} \left| \frac{1}{n+1} \sum_{k=0}^{n} a_k \right|^p \leq \left( \frac{p}{p-1} \right)^p \sum_{n=0}^{\infty} |a_n|^p.$$

- This is the boundedness $\ell^p \to \ell^p$ of the Cesáro operator $\mathcal{C}$ acting on sequences:

$$a = (a_n)_0^\infty \longmapsto \mathcal{C}(a) := \left( \frac{1}{n+1} \sum_{k=0}^{n} a_k \right)_{n=0}^{\infty}.$$

# Extension I

- The Cesáro operator $\mathcal{C}\colon \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ is a Frechét space isomorphism.
- The largest sequence space to which $\mathcal{C}$ can be extended, still with values in $\ell^p$, is:

$$[\mathcal{C}, \ell^p] := \Big\{ a = (a_n)_0^\infty \in \mathbb{C}^{\mathbb{N}} : \mathcal{C}(a) = \Big( \frac{1}{n+1} \sum_{k=0}^{n} a_k \Big)_{n=0}^{\infty} \in \ell^p \Big\},$$

which is a Banach space for the norm

$$\|a\|_{[\mathcal{C},\ell^p]} := \|\mathcal{C}(a)\|_{\ell^p} = \left( \sum_{n=0}^{\infty} \left| \frac{1}{n+1} \sum_{k=0}^{n} a_k \right|^p \right)^{1/p}.$$

- $[\mathcal{C}, \ell^p] = \mathcal{C}^{-1}(\ell^p)$ is isomorphically isometric to $\ell^p$ via $\mathcal{C}$, but

$$\ell^p \subsetneq [\mathcal{C}, \ell^p].$$

# Extension II

- A special subspace is the solid core (for the coordinate-wise order) of $[\mathcal{C}, \ell^p]$:

$$ces_p := \left\{ (a_n) : \Big( \frac{1}{n+1} \sum_{k=0}^{n} |a_k| \Big)_{n=0}^{\infty} \in \ell^p \right\}, 1 < p < \infty,$$

which is a Banach space for the norm

$$\|a\|_{ces_p} := \|\mathcal{C}(|a|)\|_{\ell^p} = \left( \sum_{n=0}^{\infty} \Big( \frac{1}{n+1} \sum_{k=0}^{n} |a_k| \Big)^p \right)^{1/p}.$$

- Properties:
    - $\ell^p \subsetneq ces_p \subsetneq [\mathcal{C}, \ell^p]$.
    - $ces_p$ contains sequences with arbitrarily large terms.
    - Control on the partial sums of $a \in ces_p$:

$$\lim_{n\to\infty} (n+1)^{-1/p} \sum_{k=0}^{n} |a_k| = 0.$$

# The space $ces_p$

- The dual of $ces_p$ (Jagers, 1974). Isomorphic identification, $1/p + 1/q = 1$:

$$ces_p' = d(q) := \Big\{ b = (b_n) : (\tilde{b}_n) \in \ell^q \Big\}, \quad \text{for} \quad \tilde{b}_n := \sup_{k \geq n} |b_k|,$$

  and norm $\|b\|_{d(q)} := \|\tilde{b}\|_{\ell^q}$. ($\tilde{b}$ is the *least decreasing majorant*.)

- Identification of $\mathcal{C}(ces_p)$ for $\mathcal{C} \colon ces_p \to \ell^p$:

$$\mathcal{C}(ces_p) = \Big\langle (a_n) \in \ell^p : a_n \geq 0, (a_n(n+1)) \text{ increasing} \Big\rangle.$$

# The space $ces_p$

- A *"striking property"* of the Cesàro operator on $ces_p$:

Theorem (G. Bennett, *Factorizing the Classical Inequalities*, Memoirs AMS, 1996)

$$a \in ces_p \iff \mathcal{C}(|a|) \in ces_p.$$

- Recently it has been shown that for many other sequence spaces $\mathbb{X}$ it holds that

$$\mathcal{C}\,\mathcal{C}(|a|) \in \mathbb{X} \Rightarrow \mathcal{C}(|a|) \in \mathbb{X}.$$

# The Cesàro operator on Hardy spaces

- RECALL: the Cesáro operator acting on analytic functions:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \longmapsto \mathcal{C}(f)(z) := \sum_{n=0}^{\infty} \Big( \frac{1}{n+1} \sum_{k=0}^{n} a_k \Big) z^n$$

a Frechét space isomorphism $\mathcal{C} \colon \mathcal{H}(\mathbb{D}) \to \mathcal{H}(\mathbb{D})$.

**Theorem**

*For $1 \leq p < \infty$, the Cesàro operator $\mathcal{C}$ is bounded*

$$\mathcal{C} \colon \mathcal{H}^p(\mathbb{D}) \longrightarrow \mathcal{H}^p(\mathbb{D}).$$

# Extensions

- $\mathbb{X}(\mathbb{D})$ a *Banach space of analytic functions (BSAF)* such that:
  - $\mathcal{H}^p(\mathbb{D}) \subsetneq \mathbb{X}(\mathbb{D})$,
  - $\mathcal{C} : \mathbb{X}(\mathbb{D}) \to \mathcal{H}^p(\mathbb{D})$ continuously.

- We define the *optimal domain* for $\mathcal{C}$ with values in $\mathcal{H}^p(\mathbb{D})$:

$$[\mathcal{C}, \mathcal{H}^p] := \Big\{ f \in \mathcal{H}(\mathbb{D}) : \mathcal{C}(f) \in \mathcal{H}^p(\mathbb{D}) \Big\}.$$

It is a Banach space for the norm:

$$\|f\|_{[\mathcal{C},\mathcal{H}^p]} := \|\,\mathcal{C}(f)\|_{\mathcal{H}^p(\mathbb{D})}.$$

# Optimal extension of $\mathcal{C}$

**Theorem**

*Let* $1 \leq p < \infty$ *and* $\mathbb{X}(\mathbb{D})$ *a BSAF. Then, TFAE:*

(a) $\mathcal{C} \colon \mathbb{X}(\mathbb{D}) \to \mathcal{H}^p(\mathbb{D})$ *continuously.*

(b) $\mathbb{X}(\mathbb{D}) \subset [\mathcal{C}, \mathcal{H}^p]$.

Consequently, the optimal (BSAF) extension of the Cesàro operator is

$$\mathcal{C} \colon [\mathcal{C}, \mathcal{H}^p] \longrightarrow \mathcal{H}^p(\mathbb{D}) \quad \text{(optimally)}.$$

# Solid BSAFs

- Let $f \in \mathcal{H}^2(\mathbb{D})$ and $g \in \mathcal{H}(\mathbb{D})$:
  - (a) If $|g(z)| \le |f(z)|$ for $z \in \mathbb{D}$, then $g \in \mathcal{H}^2(\mathbb{D})$.
  - (b) If $|\hat{g}(n)| \le |\hat{f}(n)|$ for $n \ge 0$, then $g \in \mathcal{H}^2(\mathbb{D})$ .
- Let $f \in [\mathcal{C}, \mathcal{H}^2]$ and $g \in \mathcal{H}(\mathbb{D})$:
  - (a) If $|g(z)| \le |f(z)|$ for $z \in \mathbb{D}$, then $g \in [\mathcal{C}, \mathcal{H}^2]$,

$$\text{since } f \in [\mathcal{C}, \mathcal{H}^2] \iff \int_0^{2\pi} \int_0^1 \frac{|f(re^{i\theta})|^2}{|1 - re^{i\theta}|^2}(1-r)\, dr d\theta < \infty.$$

  - (b) However, since $\sum_0^\infty a_n z^n \in [\mathcal{C}, \mathcal{H}^2] \iff (\frac{1}{n+1}\sum_{k=0}^n a_k) \in \ell^2$:

$$\frac{1}{1+z} = \sum_{n=0}^\infty (-1)^n z^n \in [\mathcal{C}, \mathcal{H}^2] \quad \text{but} \quad \frac{1}{1-z} = \sum_{n=0}^\infty z^n \notin [\mathcal{C}, \mathcal{H}^2].$$

# Solid extension of the Cesàro operator

- A BSAF $\mathbb{X}(\mathbb{D})$ is solid for the coefficient-wise order if

$$\sum_{n=0}^{\infty} a_n z^n \in \mathbb{X}(\mathbb{D}) \quad \& \quad |b_n| \leq |a_n| \quad \Longrightarrow \sum_{n=0}^{\infty} b_n z^n \in \mathbb{X}(\mathbb{D}).$$

- Examples:
  - $\mathcal{H}^2(\mathbb{D})$ is solid (since $\ell^2$ is solid).
  - The optimal domain $[\mathcal{C}, \mathcal{H}^2]$ is not solid.
  - The smallest solid space containing $\mathcal{H}^\infty(\mathbb{D})$ is $\mathcal{H}^2(\mathbb{D})$ (Kisliakov, 1981).

# The optimal solid domain

- The largest solid BSAF to which the Cesàro operator

$$\mathcal{C}\colon \mathcal{H}^2(\mathbb{D}) \longrightarrow \mathcal{H}^2(\mathbb{D})$$

can be extended –still with values in $\mathcal{H}^2(\mathbb{D})$–
is the optimal solid domain

$$\mathcal{H}(ces_2) := \left\{ \sum_{n=0}^{\infty} a_n z^n \in \mathcal{H}(\mathbb{D}) : (a_n) \in ces_2 \right\},$$

where

$$ces_2 := \left\{ (a_n) : \left( \frac{1}{n+1} \sum_{k=0}^{n} |a_k| \right)_{n=0}^{\infty} \in \ell^2 \right\}.$$

A BSAF for the norm:

$$\|f\|_{\mathcal{H}(ces_2)} = \|a\|_{ces_2} = \left( \sum_{n=0}^{\infty} \left( \frac{1}{n+1} \sum_{k=0}^{n} |a_k| \right)^2 \right)^{1/2}.$$

# Functions in $\mathcal{H}(ces_2)$

**Theorem**

*A function $f \in \mathcal{H}(ces_2)$ if and only it is a (complex) linear combination of functions of the form*

$$(1-z)\sum_{n=0}^{\infty} a_n z^n,$$

*where $(a_n)$ is non-negative, increasing, with $\left(\frac{a_n}{n+1}\right) \in \ell^2$.*

- Proof based on identifying the range $\mathcal{C}(ces_2)$ of the operator

$$\mathcal{C}\colon ces_2 \to \ell^2,$$

for which we use Bennett's result:

$$a \in ces_2 \iff \mathcal{C}(|a|) \in ces_2.$$

# The space $\mathcal{H}(ces_2)$

It is a 'good' space:

- Point evaluations on $\mathcal{H}(ces_2)$ are continuous (as $\mathcal{H}(ces_2) \subseteq [\mathcal{C}, \mathcal{H}^2]$ continuously).
- $\mathcal{H}(ces_2)$ is reflexive.
- Polynomial are dense in $\mathcal{H}(ces_2)$.
- Functions in $\mathcal{H}(ces_2)$ are the sum, in $\mathcal{H}(ces_2)$, of their Taylor series.
- $\{z^n : n \geq 0\}$ form an unconditional, boundedly complete and shrinking basis for $\mathcal{H}(ces_2)$.

# The stability of $\mathcal{H}(ces_2)$

- Consider $\mathcal{C}\colon \mathcal{H}(ces_2) \to \mathcal{H}(ces_2)$ rather than $\mathcal{C}\colon \mathcal{H}^2(\mathbb{D}) \to \mathcal{H}^2(\mathbb{D})$.

  Look for the optimal solid extension domain.

**Theorem**

*The largest solid BSAF which $\mathcal{C}$ maps continuously into $\mathcal{H}(ces_2)$ is $\mathcal{H}(ces_2)$ itself.*

- Even though the target space $\mathcal{H}(ces_2)$ is substantially larger than $\mathcal{H}^2(\mathbb{D})$, it turns out that no further solid extension occurs.

- Proof based (again!) on Bennett's result:

$$a \in ces_2 \iff \mathcal{C}(|a|) \in ces_2.$$

# Multipliers

- A function $\varphi \in \mathcal{H}(\mathbb{D})$ is a multiplier on a BSAF $\mathbb{X}(\mathbb{D})$ if

$$f \in \mathbb{X}(\mathbb{D}) \longmapsto M_\varphi(f) := \varphi{\cdot}f \in \mathbb{X}(\mathbb{D}) \quad \text{boundedly.}$$

  Denote $\mathcal{M}(\mathbb{X}(\mathbb{D}))$ for the space of multipliers.

- Examples:
  - $\mathcal{M}(\ell^p) = \ell^\infty$:

$$(x_n) \in \ell^p \longmapsto (\varphi_n{\cdot}x_n) \in \ell^p \iff (\varphi_n) \in \ell^\infty.$$

  - $\mathcal{M}(ces_p) = \ell^\infty$:

$$(x_n) \in ces_p \longmapsto (\varphi_n{\cdot}x_n) \in ces_p \iff (\varphi_n) \in \ell^\infty.$$

  - $\mathcal{M}(\mathcal{H}^p(\mathbb{D})) = \mathcal{H}^\infty(\mathbb{D})$:

$$f \in \mathcal{H}^p(\mathbb{D}) \longmapsto \varphi{\cdot}f \in \mathcal{H}^p(\mathbb{D}) \iff \varphi \in \mathcal{H}^\infty(\mathbb{D}).$$

# Multipliers for optimal domains I

**Theorem**

*Let $1 \leq p < \infty$. Given $\varphi \in \mathcal{H}(\mathbb{D})$, the multiplication operator*

$$f \in [\mathcal{C}, \mathcal{H}^p] \longmapsto \varphi \cdot f \in [\mathcal{C}, \mathcal{H}^p]$$

*is bounded if and only if*

$$\varphi \in \mathcal{H}^\infty(\mathbb{D}) \quad \textit{with} \quad \|M_\varphi\|_{op} = \|\varphi\|_\infty.$$

*That is, $\mathcal{M}([\mathcal{C}, \mathcal{H}^p]) = \mathcal{H}^\infty(\mathbb{D})$.*

Proof: The growth characterization of functions in $[\mathcal{C}, \mathcal{H}^p]$ applied to $\varphi^n$, assuming $\|M_\varphi\|_{op} = 1$.

# Multipliers for optimal domains II

**Theorem**

*Given $\varphi \in \mathcal{H}(\mathbb{D})$, the multiplication operator*

$$f \in \mathcal{H}(ces_2) \longmapsto \varphi \cdot f \in \mathcal{H}(ces_2)$$

*is bounded if and only if*

$$\varphi(z) = \sum_{n=0}^{\infty} b_n z^n, \quad \text{with} \quad \|M_\varphi\|_{op} = \sum_{n=0}^{\infty} |b_n| < \infty.$$

The proof is done via convolution of sequences in $ces_2$.

# Idea of the proof

Multiplication of functions in $\mathcal{H}(\mathbb{D})$ corresponds to convolution of their sequences of Taylor coefficients:

$$\Big(\sum_{n=0}^{\infty} a_n z^n\Big)\Big(\sum_{m=0}^{\infty} b_m z^m\Big) = \sum_{k=0}^{\infty}\Big(\sum_{j=0}^{k} a_j b_{k-j}\Big) z^k.$$

Thus, $\varphi(z) = \sum_0^\infty b_n z^n \in \mathcal{M}(\mathcal{H}(ces_2))$ when $b = (b_n)_0^\infty$ defines a bounded convolution operator $T_b$ on $ces_2$:

$$ces_2 \ni a = (a_n)_0^\infty \longmapsto T_b(a) := a * b = \Big(\sum_{j=0}^{k} a_j b_{k-j}\Big)_{k=0}^{\infty} \in ces_2.$$

$\|M_\varphi\|_{op} = \|T_b\|_{op}$ due to the isometric isomorphism $\mathcal{H}(ces_2) \equiv ces_2$.

# Idea of the proof

### Theorem

*Let $1 < p < \infty$. Given $b = (b_n)_0^\infty$, the associated convolution operator on sequences*

$$a = (a_n)_0^\infty \longmapsto T_b(a) := a * b = \Big( \sum_{j=0}^{k} a_j b_{k-j} \Big)_{k=0}^{\infty},$$

*is bounded $T_b \colon ces_p \to ces_p$ if and only if*

$$b \in \ell^1, \quad \text{and in this case} \quad \|T_b\|_{ces_p \to ces_p} = \|b\|_{\ell^1}.$$

- The multipliers for $\mathcal{H}(ces_2)$ are a solid BSAF.
- This is not so for the multipliers of $\mathcal{H}^2(\mathbb{D})$ and $[\mathcal{C}, \mathcal{H}^2]$ (which is $\mathcal{H}^\infty$).

# Spectra

A classical result:

Theorem (Brown, Halmos, Shields, 1965)

*Consider the Cesàro operator* $\mathcal{C}\colon \ell^2 \to \ell^2$. *Then,* $\|\mathcal{C}\| = 2$, *and the spectrum is*

$$\sigma(\mathcal{C}, \ell^2) = \big\{ z \in \mathbb{C} : |z-1| \le 1 \big\}.$$

Theorem (Hardy, 1925; Rhoades, 1971)

*Consider the Cesàro operator* $\mathcal{C}\colon \ell^p \to \ell^p$. *Then,* $\|\mathcal{C}\|_{\ell^p \to \ell^p} = \frac{p}{p-1}$, *and the spectrum is*

$$\sigma(\mathcal{C}, \ell^p) = \Big\{ z \in \mathbb{C} : \Big| z - \frac{p}{2(p-1)} \Big| \le \frac{p}{2(p-1)} \Big\}.$$

# Spectra

**Theorem (Siskakis, 1987)**

*Consider the Cesàro operator* $\mathcal{C}\colon \mathcal{H}^p(\mathbb{D}) \to \mathcal{H}^p(\mathbb{D})$. *Then, the spectrum is*

$$\sigma(\mathcal{C}, \mathcal{H}^p) = \left\{ z \in \mathbb{C} : \left| z - \frac{p}{2} \right| \leq \frac{p}{2} \right\},$$

*and*

(a) *For* $2 \leq p < \infty$, *we have* $\|\mathcal{C}\|_{\mathcal{H}^p \to \mathcal{H}^p} = p$.

(b) *For* $1 \leq p < 2$, *we have* $p \leq \|\mathcal{C}\|_{\mathcal{H}^p \to \mathcal{H}^p} \leq 2$.

The proof uses semigroups of weighted composition operators on $\mathcal{H}^p(\mathbb{D})$.

# The case of $\mathcal{H}(ces_2)$

### Theorem

*Consider the Cesàro operator* $\mathcal{C}\colon \mathcal{H}(ces_2) \to \mathcal{H}(ces_2)$. *Then, the spectrum is*

$$\sigma(\mathcal{C}, \mathcal{H}(ces_2)) = \{z \in \mathbb{C} : |1 - z| \le 1\},$$

*and* $\|\mathcal{C}\|_{\mathcal{H}(ces_2)\to\mathcal{H}(ces_2)} = 2$.

### Theorem

*Let* $1 < p < \infty$. *Consider the Cesàro operator* $\mathcal{C}\colon ces_p \to ces_p$. *Then, the spectrum is*

$$\sigma(\mathcal{C}, ces_p) = \left\{z \in \mathbb{C} : \left|z - \frac{p}{2(p-1)}\right| \le \frac{p}{2(p-1)}\right\},$$

*and* $\|\mathcal{C}\|_{ces_p\to ces_p} = \frac{p}{p-1}$.

# Proof for $p = 2$

- By studying the adjoint operator $\mathcal{C}': ces_2' \to ces_2'$ we deduce
  - $\|C\|_{ces_2 \to ces_2} = 2$.
  - $\overline{\mathcal{U}} = \{z \in \mathbb{C} : |1 - z| \leq 1\} \subseteq \sigma(\mathcal{C}) \subset \{z : |z| \leq 2\}$.
- We write: $\boxed{(\mathcal{C} - \lambda I)^{-1} = D_\lambda - \frac{1}{\lambda^2} E_\lambda}$.
  - $D_\lambda$ is a diagonal operator:

$$|d_{nn}| = \left| \frac{n+1}{1 - (n+1)\lambda} \right| \leq \frac{1}{\operatorname{dist}(\lambda, \mathcal{U})}.$$

For $\lambda \notin \overline{\mathcal{U}}$ we have $D_\lambda : ces_2 \to ces_2$ bounded.

# Proof for $p = 2$

Recall: $\boxed{(\mathcal{C} - \lambda I)^{-1} = D_\lambda - \frac{1}{\lambda^2} E_\lambda}$.

- $E_\lambda = (e_{nm})$ is a lower triangular matrix:

$$e_{nm} = \frac{1}{(n+1)\prod_{k=m}^{n}(1 - \frac{1}{(k+1)\lambda})}, \quad 0 \leq m < n.$$

- For $\Re(\lambda) \leq 0$ we have:

$$|e_{nm}| \leq \frac{1}{n+1}.$$

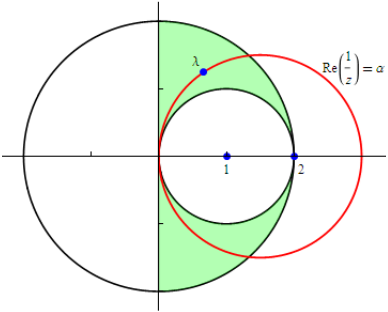
Thus, $|E_\lambda| \leq \mathcal{C}$, and so $E_\lambda\colon ces_2 \to ces_2$ is bounded.

# Proof for $p = 2$

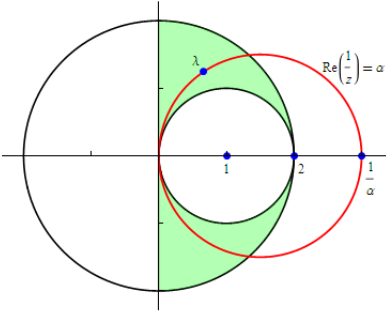
The remaining region for $\lambda$ can be written as:

$$\Big\{ z : |z| \leq 2 \Big\} \cap \Big( \bigcup_{0<\alpha<\frac{1}{2}} \Big\{ z : \Re\Big(\frac{1}{z}\Big) = \alpha \Big\} \Big).$$

where $\Re(\frac{1}{z}) = \alpha$ is a circle centered at $(\frac{1}{2\alpha}, 0)$ with radius $\frac{1}{2\alpha}$.



# Proof for $p = 2$




Then, for $a \in ces_2$:

$$\left|\left(E_\lambda(a)\right)_n\right| = \left|\frac{1}{n+1}\sum_{m=0}^{n-1}\frac{a_m}{\prod_{k=m}^{n}\left(1-\frac{1}{(k+1)\lambda}\right)}\right| \le \left(E_{\frac{1}{\alpha}}(|a|)\right)_n.$$

Consequently:

$$\frac{1}{\alpha} > 2 \Longrightarrow E_{\frac{1}{\alpha}} \text{ bounded on } ces_2 \Longrightarrow E_\lambda \text{ bounded on } ces_2.$$

# The case $p \neq 2$. Functions

- For a BSAF $\mathbb{X}(\mathbb{D})$:
  - $\mathbb{X}(\mathbb{D})_s$ the solid core of $\mathbb{X}(\mathbb{D})$.
  - $\mathbb{X}(\mathbb{D})_{uc}$ the unconditionally converging Taylor series.
  - $\mathbb{X}(\mathbb{D})_s = \mathbb{X}(\mathbb{D})_{uc}$.
- For $1 < p \leq 2$ and $\frac{1}{p} + \frac{1}{q} = 1$:

$$\begin{array}{ccccccc} \mathcal{H}([\mathcal{C}, \ell^p]) & \subset & [\mathcal{C}, \mathcal{H}^q] & \subset & [\mathcal{C}, \mathcal{H}^p] & \subset & \mathcal{H}([\mathcal{C}, \ell^q]) \\ \bigcup & & \bigcup & & \bigcup & & \bigcup \\ \mathcal{H}(ces_p) & \subset & [\mathcal{C}, \mathcal{H}^q]_s & \subset & [\mathcal{C}, \mathcal{H}^p]_s & \subset & \mathcal{H}(ces_q) \end{array}$$

- For $p = 2$ we have $\mathcal{H}([\mathcal{C}, \ell^2]) = [\mathcal{C}, \mathcal{H}^2]$ and $[\mathcal{C}, \mathcal{H}^2]_s = \mathcal{H}(ces_2)$.

# The case $p \neq 2$. Coefficients

- For $1 < r < \infty$, let:
  - $N^r = \{(a_n) : |a| \leq \hat{f}, \text{ for some } f \in \mathcal{H}^r(\mathbb{D})\}$.
  - $B_r = \{(a_n) : a = \hat{f}, \text{ for some } f \in [\mathcal{C}, \mathcal{H}^r]_{uc}\}$.
  - $K_r = \{(a_n) : \mathcal{C}(|a|) \leq \hat{f}, \text{ for some } f \in \mathcal{H}^r(\mathbb{D})\}$.
- For $1 < p \leq 2$ and $\frac{1}{p} + \frac{1}{q} = 1$:

$$\begin{array}{ccccccc}
\ell^p & \subset & N^q & \subset & N^p & \subset & \ell^q \\
\cap & & \cap & & \cap & & \cap \\
ces_p & \subset B_q \subset & K_q & \subset B_p \subset & K_p & \subset & ces_q
\end{array}$$

- For $p = 2$ we have $\ell^2 = N^2$ and $ces_2 = B_2 = K_2$.
- The study of $N^p$ is related to the upper majorant property in $L^p(\mathbb{T})$ of Hardy and Littlewood.

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Thank you