

Extensions of the Cesàro operator on Hardy spaces on the disc

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Outline

- 1 The Cesàro operator and Hardy spaces
- 2 Extending the Cesàro operator
- 3 The optimal domain $[\mathcal{C}, H^p]$

The Cesàro operator on sequence spaces

- Let: $(a_n) \in \ell^2 \longmapsto \mathcal{C}(a_n) := \left(\frac{1}{n+1} \sum_{k=0}^n a_k \right) \in \ell^2$
- For:

$$H^2(\mathbb{D}) := \left\{ f(z) = \sum_{k=0}^{\infty} a_k z^k \in H(\mathbb{D}) : \text{ with } \sum_{k=0}^{\infty} |a_k|^2 < \infty \right\}$$

we have

$$f \in H^2(\mathbb{D}) \longmapsto \mathcal{C}(f)(z) := \sum_{n=0}^{\infty} \left(\frac{1}{n+1} \sum_{k=0}^n a_k \right) z^n \in H^2(\mathbb{D})$$

The Hardy space $H^2(\mathbb{D})$

- For $f(z) = \sum_{k=0}^{\infty} a_k z^k \in H(\mathbb{D})$, and $0 < r \leq 1$

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 dr &= \frac{1}{2\pi} \int_0^{2\pi} \left(\sum_{k=0}^{\infty} a_k r^k e^{ik\theta} \right) \overline{\left(\sum_{n=0}^{\infty} a_n r^n e^{in\theta} \right)} dr \\ &= \sum_{k=0}^{\infty} a_k \overline{a_k} r^{2k} \end{aligned}$$

- Then

$$H^2(\mathbb{D}) := \left\{ f \in H(\mathbb{D}) : \text{ with } \sup_{0 \leq r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 dr < \infty \right\}$$

Back in 1915

THE MEAN VALUE OF THE MODULUS OF AN ANALYTIC FUNCTION

By G. H. HARDY.

[Read November 12th 1914.—Received December 10th, 1914.—
Received, in revised form, February 10th, 1915.]

[Extracted from the Proceedings of the London Mathematical Society, Ser. 2, Vol. 14, Part 4.]

1. Suppose that $f(x)$ is an analytic function of the complex variable x , regular for $|x| < \rho$, and that $M(r)$ denotes, as usual, the maximum of $|f(x)|$ on the circle $|x| = r < \rho$. Then it is known that $M(r)$ possesses the following properties:—

- (i) $M(r)$ is a steadily increasing function of r ;
- (ii) $\log M(r)$ is a convex function of $\log r$, so that

$$\log M(r) \leq \frac{\log(r_3/r)}{\log(r_3/r_1)} \log M(r_1) + \frac{\log(r_2/r)}{\log(r_2/r_1)} \log M(r_2),$$

if $0 < r_1 \leq r \leq r_2 < \rho$.

Further, when $f(x)$ is an integral function, so that $\rho = \infty$, it is known that

- (iii) $M(r)$ tends to infinity with (r) , and, unless $f(x)$ is a polynomial, more rapidly than any power of r .*

It was suggested to me by Dr. H. Bohr and Prof. E. Landau, rather more than a year ago, that the property (i) is possessed also by the mean value of $|f(x)|$ on the circle $|x| = r$, i.e., by the function

$$\mu(r) = \frac{1}{2\pi} \int_0^\pi |f(re^{i\theta})| d\theta.$$

* The theorems (i) and (iii) are classical. Theorem (ii) was discovered independently by Blumenthal (*Jahresbericht der Deutschen Math.-Vereinigung*, Vol. 15, p. 97), Faber (*Math. Annalen*, Vol. 68, p. 549), and Hadamard (*Bulletin de la Soc. Math. de France*, Vol. 24, p. 186). The first statement of the theorem was due to Hadamard and the first proof to Blumenthal. The theorem is a corollary of one concerning the associated radii of convergence of a power series in two variables, due to Fabry (*Comptes Rendus*, Vol. 154, p. 1190), and Hartogs (*Math. Annalen*, Vol. 69, p. 1).

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The Hardy spaces $H^p(\mathbb{D})$

- For $1 \leq p < \infty$,

$$H^p(\mathbb{D}) := \left\{ f \in H(\mathbb{D}) : \text{ with } \sup_{0 \leq r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty \right\}$$

- $H^p(\mathbb{D})$ is a Banach space for the norm

$$\|f\|_{H^p(\mathbb{D})} := \sup_{0 \leq r < 1} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{1/p}$$

- $f \in H^p(\mathbb{D})$ has *boundary values*: $f(e^{i\theta}) := \lim_{r \rightarrow 1^-} f(re^{i\theta})$

The Cesàro operator on Hardy spaces

Theorem

For $1 \leq p < \infty$, the Cesàro operator

$$\mathcal{C} : H^p(\mathbb{D}) \longrightarrow H^p(\mathbb{D})$$

is bounded and linear.

- For $1 < p < \infty$, is classical (Hardy, M. Riesz)
- For $p = 1$, by A. Siskakis, 1987.

Extensions

- $X(\mathbb{D})$ a *Banach space of analytic functions*

(a vector subspace of $H(\mathbb{D})$ which is complete for its norm)

such that

- $\mathcal{C}: X(\mathbb{D}) \rightarrow H^p(\mathbb{D})$ continuously
- $H^p(\mathbb{D}) \subsetneq X(\mathbb{D})$
- Since, for $1 \leq q < p$ we have

$$H^p(\mathbb{D}) \subsetneq H^q(\mathbb{D}),$$

Could it happen that $\mathcal{C}: H^q(\mathbb{D}) \rightarrow H^p(\mathbb{D})$?

The answer is: NO (Aleman & Cima, 2001)

Weighted Hardy spaces

- A weight is: $\omega > 0$ a.e. on \mathbb{T} with $\log \omega \in L^1(\mathbb{T})$

An associated *outer function* is: $\psi \in H(\mathbb{D})$ with $|\psi| = \omega$ on \mathbb{T}

- For $1 \leq p < \infty$,

$$H^p(\omega) := \left\{ f \in H(\mathbb{D}) : f = \psi^{-1/p} \cdot g, \text{ for } g \in H^p(\mathbb{D}) \right\}$$

with the norm: $\|f\|_{H^p(\omega)} := \|\psi^{1/p} \cdot f\|_{H^p(\mathbb{D})}$

- $H^p(\omega) \supsetneqq H^p(\mathbb{D}) \iff \psi \text{ bounded and } \psi^{-1} \text{ unbounded}$

An extension theorem

Theorem

Let $1 \leq p < \infty$, and let ω, ψ with

- ψ bounded and ψ^{-1} unbounded
- there exist distinct points $a_1, \dots, a_m \in \mathbb{T} \setminus \{1\}$ such that

$$\psi^{-1}(z) = \mathcal{O}\left(\frac{1}{\prod_{k=1}^m |z - a_k|^p}\right), \quad |z| \rightarrow 1^-.$$

Then $\mathcal{C}: H^p(\omega) \rightarrow H^p(\mathbb{D})$ and $H^p(\omega) \supsetneq H^p(\mathbb{D})$.

Why?

$$\begin{aligned}
 \mathcal{C}(f)(z) &= \frac{1}{z} \int_0^z \frac{f(\xi)}{1-\xi} d\xi \\
 &= \frac{1}{z} \int_0^z \frac{\psi^{-1/p}(\xi) g(\xi)}{1-\xi} d\xi \\
 &= \frac{1}{z} \int_0^z \frac{\psi^{-1/p}(\xi) g(\xi) \prod_1^m (\xi - a_k)}{(1-\xi) \prod_1^m (\xi - a_k)} d\xi \\
 &= \sum_{k=0}^m \frac{A_k}{z} \int_0^z \frac{h(\xi) g(\xi)}{a_k - \xi} d\xi \\
 &= \sum_{k=0}^m \frac{A_k}{a_k} \frac{1}{(z/a_k)} \int_0^{(z/a_k)} \frac{h(a_k \eta) g(a_k \eta)}{1-\eta} d\eta \\
 &= \sum_{k=0}^m \frac{A_k}{a_k} \mathcal{C}(h(a_k \cdot) g(a_k \cdot))(z/a_k)
 \end{aligned}$$

The truth behind

Theorem

Let $1 \leq p < \infty$ and ω, ψ . TFAE:

a) $\mathcal{C}: H^p(\omega) \rightarrow H^p(\mathbb{D})$

b) The function

$$z \in \mathbb{D} \longmapsto \int_0^z \frac{\psi^{-1/p}(\xi)}{1 - \xi} d\xi$$

belongs to the space $BMOA$.

$$BMOA := \left\{ f \in H(\mathbb{D}) : f(e^{i\theta}) \in BMO(\mathbb{T}) \right\} \subsetneq \bigcap_{1 \leq p < \infty} H^p(\mathbb{D})$$

The optimal domain $[\mathcal{C}, H^p]$

- We define the *optimal domain* for \mathcal{C} with values in $H^p(\mathbb{D})$:

$$[\mathcal{C}, H^p] := \left\{ f \in H(\mathbb{D}) : \mathcal{C}(f) \in H^p(\mathbb{D}) \right\}$$

- It is a Banach space for the norm:

$$\|f\|_{[\mathcal{C}, H^p]} := \|\mathcal{C}(f)\|_{H^p(\mathbb{D})}$$

- Example: for $p = 2$, the space $[\mathcal{C}, H^2]$ is a Hilbert space

$$f(z) = \sum_0^{\infty} a_n z^n \in [\mathcal{C}, H^2] \iff ((n+1)a_n - n a_{n-1}) \in \ell^2.$$

First properties of $[\mathcal{C}, H^p]$

- Polynomials are dense in $[\mathcal{C}, H^p]$
- Point evaluations are continuous:

$$f \in [\mathcal{C}, H^p] \longmapsto f(z_0) \in \mathbb{C}$$

- $H^p(\mathbb{D}) \subsetneq [\mathcal{C}, H^p] \subsetneq H(\mathbb{D})$
- $[\mathcal{C}, H^{p_2}] \subsetneq [\mathcal{C}, H^{p_1}]$ whenever $1 \leq p_1 < p_2 < \infty$.

Functions in $[\mathcal{C}, H^p]$

- $[\mathcal{C}, H^p]$ is NOT invariant under automorphisms of \mathbb{D}

$$z \longmapsto e^{i\theta} \frac{z + a}{1 + \bar{a}z}, \quad a \in \mathbb{D}$$

- $[\mathcal{C}, H^p] = \left\{ f \in H(\mathbb{D}) : f(z) = (1 - z)g'(z) \text{ for some } g \in H^p(\mathbb{D}) \right\}$
- There exists $f \in [\mathcal{C}, H^p]$ without boundary values

Functions in $[\mathcal{C}, H^p]$

Growth characterization of $[\mathcal{C}, H^p]$:

Theorem

Let $1 < p < \infty$, then

$$f \in [\mathcal{C}, H^p] \iff \int_0^{2\pi} \left(\int_0^1 \frac{|f(re^{i\theta})|^2}{|1 - re^{i\theta}|^2} (1 - r) dr \right)^{p/2} d\theta < \infty.$$

Multipliers

Recall: $f \in H^p(\mathbb{D}) \longmapsto \varphi \cdot f \in H^p(\mathbb{D})$ continuously if and only if

$$\varphi \in H^\infty(\mathbb{D}) := \{f \in H(\mathbb{D}) : f \text{ is bounded}\}$$

Theorem

Let $1 \leq p < \infty$. Given $\varphi \in H(\mathbb{D})$, the multiplication operator

$$f \in [\mathcal{C}, H^p] \longmapsto \varphi \cdot f \in [\mathcal{C}, H^p]$$

is well defined (and hence, continuous) if and only if

$$\varphi \in H^\infty(\mathbb{D})$$

Further results on $[\mathcal{C}, H^p]$

- For $p = \infty$,

$$[\mathcal{C}, H^\infty] := \left\{ f \in H(\mathbb{D}) : \mathcal{C}(f) \in H^\infty(\mathbb{D}) \right\} \supsetneq H^\infty(\mathbb{D})$$

- Interpolation: $([\mathcal{C}, H^1], [\mathcal{C}, H^\infty])_{1-\frac{1}{p}, p} = [\mathcal{C}, H^p]$
- Multipliers: $\varphi \in H(\mathbb{D})$ satisfies

$$f \in H^p(\mathbb{D}) \mapsto \varphi \cdot f \in [\mathcal{C}, H^p] \text{ is continuous}$$

if and only if

$$\varphi \in [\mathcal{C}, BMOA] := \left\{ f \in H(\mathbb{D}) : \mathcal{C}(f) \in BMOA \right\}$$

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Thank you