A Banach function space X for which all operators from l^p to X are compact

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Outline

1. Vector measures
2. Operators
3. The construction

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Vector measure

Let (Ω, Σ) be a measurable space and X a Banach space.
A (countably additive) vector measure is

$$\nu \colon \Sigma \to X$$

satisfying, for any disjoint family $(A_i)_1^\infty$ of Σ-measurable sets, that

$$\nu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \nu(A_i),$$

with convergence in the topology of X.

Examples

- For $1 \leq p < \infty$:

$$A \in \mathcal{M}([0,1]) \mapsto \nu_p(A) := \chi_A \in L^p([0,1]).$$

- For (r_n) the Rademacher functions:

$$A \in \mathcal{M}([0,1]) \mapsto \nu(A) := \left(\int_A r_n(t)\, dt \right)_{n=1}^{\infty} \in \ell^2,$$

where

$$r_n(t) := \text{sign}(\sin(2^n \pi t), \quad n \in \mathbb{N}.$$

The space of vector measures

- Fix (Ω, Σ) a measurable space and X a Banach space.
  The collection of all vector measures $\nu\colon \Sigma \to X$ is a linear space denoted by
  $$ca(\Sigma, X).$$

- It becomes a Banach space when endowed with the norm of the (total) semivariation:
  $$\|\nu\| := \sup\Big\{|x^*\nu|(\Omega) : x^* \in X^*, \|x^*\| \leq 1\Big\},$$
  where
  - X^* is the topological dual space of X.
  - $x^*\nu$ is the scalar measure $A \in \Sigma \mapsto x^*\nu(A) \in \mathbb{R}$.
  - $|x^*\nu|$ is the variation of $x^*\nu$.

The space of vector measures

- The semivariation of ν is a sub-additive set function:

$$A \in \Sigma \longmapsto \|\nu\|(A) := \sup\left\{|x^*\nu|(A) : x^* \in X^*, \|x^*\| \leq 1\right\}.$$

- An equivalent expression for the semivariation is

$$\sup_{B\in\Sigma, B\subset A} \|\nu(B)\|_X \leq \|\nu\|(A) \leq 2 \sup_{B\in\Sigma, B\subset A} \|\nu(B)\|_X.$$

- Countably additive vector measures are bounded:

$$\nu \in ca(\Sigma, X) \Rightarrow \|\nu\| < \infty.$$

The range of a vector measure

- The range of a vector measure $\nu\colon \Sigma \to X$ is

$$rg(\nu) := \Big\{\nu(A) : A \in \Sigma\Big\} \subset X \quad \text{(it is a bounded set in } X\text{).}$$

- Bartle-Dunford-Schwartz (1955): $rg(\nu)$ is relatively weakly compact in X:

1. There exists μ with $\|\nu\| << \mu = |x_0^*\nu|$, $x_0^* \in X^*$ (Rybakov, 1970).
2. The ν-integration operator T

$$S = \sum_{i=1}^{n} a_i \chi_{A_i} \longmapsto T(S) := \sum_{i=1}^{n} a_i \nu(A_i) \in X,$$

   can be extended to $T\colon L^\infty(\mu) \to X$.

3. $T\colon L^\infty(\mu) \to X$ is weak*-to-weak continuous.

Measures with compact range

- A relevant subspace of $ca(\Sigma, X)$ is

$$cca(\Sigma, X) := \Big\{\nu \in ca(\Sigma, X) : \ \nu \text{ has relatively compact range in } X\Big\}.$$

$cca(\Sigma, X)$ is a closed subspace of $ca(\Sigma, X)$ for the norm of the semivariation.

- The situation of $cca(\Sigma, X) \subseteq ca(\Sigma, X)$ is a question of interest:

$$ca(\Sigma, X) = cca(\Sigma, X) \iff \mathcal{L}(\ell^\infty, X) = \mathcal{K}(\ell^\infty, X),$$

every bounded linear operator $T\colon \ell^\infty \to X$ is compact.

A result of Lech Drewnowski

Theorem (1990, 1998)

Suppose that the σ-algebra Σ admits a nonzero, atomless, finite, positive measure. TFAE:

(a) $ca(\Sigma, X) \supset \ell^\infty$.

(b) $ca(\Sigma, X) \supset c_0$.

(c) $cca(\Sigma, X) \supset c_0$.

(d) $\mathcal{L}(\ell^2, X) \neq \mathcal{K}(\ell^2, X)$.

- (d): There exists a bounded, linear and non-compact operator

$$T\colon \ell^2 \to X.$$

A problem

Let X be a Banach space.

Is it true that

$$\mathcal{L}(\ell^2, X) = \mathcal{K}(\ell^2, X) \Rightarrow \mathcal{L}(\ell^\infty, X) = \mathcal{K}(\ell^\infty, X)?$$

A definition by Hans Jarchow

- For $1 \leq p \leq \infty$, denote by

$$\mathcal{K}_p$$

  the collection of all Banach spaces X satisfying

$$\mathcal{W}(L^p([0,1]), X) = \mathcal{K}(L^p([0,1]), X).$$

- Here:
  - $\mathcal{L}(Y, X)$ are the bounded linear operators $T\colon\ Y \to X$.
  - $\mathcal{W}(Y, X)$ are the weakly compact linear operators $T\colon\ Y \to X$.
  - $\mathcal{K}(Y, X)$ are the compact linear operators $T\colon\ Y \to X$.
- In general

$$\mathcal{K}(Y, X) \subseteq \mathcal{W}(Y, X) \subseteq \mathcal{L}(Y, X).$$

Some facts

- For $p = 1$:

$$\mathcal{W}(L^1, X) = \mathcal{K}(L^1, X) \iff X \text{ is a Schur space}$$

(compact and weakly compact subsets of X coincide).

- For $1 < p < \infty$:

$$X \in \mathcal{K}_p \iff \mathcal{L}(L^p, X) = \mathcal{K}(L^p, X).$$

(due to reflexivity of L^p).

- For $p = \infty$:

$$X \in \mathcal{K}_\infty \iff \mathcal{L}(\ell^\infty, X) = \mathcal{K}(\ell^\infty, X).$$

(since $\ell^\infty \approx L^\infty([0, 1])$ and facts on operators).

Results by Hans Jarchow

Theorem (1991)

(a) *For* $1 < r < p < 2$ *we have*

$$\mathcal{K}_1 \subsetneq \mathcal{K}_r \subsetneq \mathcal{K}_p \subsetneq \mathcal{K}_2.$$

(b) *For* $2 < q < \infty$ *we have*

$$\mathcal{K}_2 = \mathcal{K}_q.$$

(c) *For* $q = \infty$ *we have*

$$\mathcal{K}_1 \subsetneq \mathcal{K}_\infty \subseteq \mathcal{K}_2.$$

For $1 < r < p < 2 < q < \infty$

$$\mathcal{K}_r \subsetneq \mathcal{K}_p$$

$$\subsetneq \qquad \subsetneq$$

$$\mathcal{K}_1 \qquad\qquad \mathcal{K}_2 = \mathcal{K}_q$$

$$\subsetneq \qquad \subset$$

$$\mathcal{K}_\infty$$

A problem by Hans Jarchow

Is it true that

$$\mathcal{K}_p \subseteq \mathcal{K}_\infty \subsetneq \mathcal{K}_2, \quad (1 < p < 2)?$$

That is, for $1 < p < 2$

$$\mathcal{L}(\ell^p, X) = \mathcal{K}(\ell^p, X) \Rightarrow \mathcal{L}(\ell^\infty, X) = \mathcal{K}(\ell^\infty, X)?$$

and

$$\mathcal{L}(\ell^2, X) = \mathcal{K}(\ell^2, X) \not\Rightarrow \mathcal{L}(\ell^\infty, X) = \mathcal{K}(\ell^\infty, X)?$$

The previous problem was

$$\mathcal{L}(\ell^2, X) = \mathcal{K}(\ell^2, X) \Rightarrow \mathcal{L}(\ell^\infty, X) = \mathcal{K}(\ell^\infty, X)?$$

which corresponds to $\mathcal{K}_2 \subseteq \mathcal{K}_\infty$.

Solution to the previous problems

We proof the following:

For $1 < p < \infty$

$$\mathcal{L}(\ell^p, X) = \mathcal{K}(\ell^p, X) \not\Rightarrow \mathcal{L}(\ell^\infty, X) = \mathcal{K}(\ell^\infty, X).$$

From where it follows that

$$\mathcal{K}_p \not\subset \mathcal{K}_\infty, \quad (1 < p \leq 2).$$

The result

Theorem (C. & Rodríguez-Piazza, 2014)

There exists a rearrangement invariant space X on $[0,1]$ such that, for every $p \in (1,\infty)$, if

$$T\colon \ell^p \to X$$

is a bounded linear operator then T is compact.

Consequences:

- $\mathcal{L}(\ell^p, X) = \mathcal{K}(\ell^p, X)$.
- Since X is rearrangement invariant

$$L^\infty([0,1]) \subseteq X \subseteq L^1([0,1]),$$

together with $\ell^\infty \approx L^\infty([0,1])$ implies there exists a non-compact operator $\ell^\infty \to X$.

The construction

- Given two sequences $a = (a_k)$ and $\varepsilon = (\varepsilon_k)$ satisfying
  - (ε_k) decreases to zero.
  - $a_k > 0$ for all $k \geq 1$.
  - $\sum_{k=1}^{\infty} \varepsilon_k a_k = 1$.
- We define

$$X(\varepsilon, a) := \left\{ f \in L^1([0,1]) : \|f\|_X := \sum_{k=1}^{\infty} a_k \int_0^{\varepsilon_k} f^*(t)\, dt < \infty \right\}.$$

General properties of X

The space X satisfies:

- X is a rearrangement invariant Banach function space,

$$\|f\|_1 \leq \|f\|_X \leq \|f\|_\infty.$$

- X is separable.
- The X-valued vector measure

$$A \in \mathcal{M}([0,1]) \longmapsto \chi_A \in X$$

has non relatively compact range in X.

Comments

- X is a Lorentz space $\Lambda(\varphi)$ space, for $\varphi' = \sum_{k=1}^{\infty} a_k \chi_{[0,\varepsilon_k]}$.
- Every Lorentz space is $\Lambda(\varphi) = X(\varepsilon, a)$ for adequate ε, and a:
  - $a_k = 2^k$.
  - ε_k such that $\{t \in [0,1] : \varphi'(t) > 2^k\} = [0, \varepsilon_k)]$.
- Lemma 1. A property of X:

If $f_n \to 0$ weakly in X and $\|f_n\|_X \geq C > 0$, then, for some $\delta > 0$,

$$\|f_n\|_{L^1} \geq \delta.$$

Proof: directly via sliding hump argument; or via the Subsequence Splitting Property of Kadec and Pełczyński.

A particular choice

- The space X which proves the theorem is given by the following choice:
  - $\varepsilon_k := \exp(-e^{2^{k+1}})$.
  - $a_k := 2^{-k}\exp(e^{2^{k+1}})$.
- Lemma 2: functions $f \in X$ satisfy:

$$m\Big(\big\{t \in [0,1] : |f(t)| > \lambda\|f\|_X\big\}\Big) \leq \exp(-e^{\lambda}).$$

The technical lemma

Lemma 3:

Let $T\colon \ell^p \to X$ *be bounded and linear operator. Suppose it satisfies*

$$\|T(e_n)\|_{L^q} \geq 1, \quad n \geq 1,$$

where $1/p + 1/q = 1$*.*
Then, there exists a constant $C_p > 0$*, a sequence* $(x_n) \subset \ell^p$ *with* $\|x_n\|_{\ell^p} \leq 1$*, and scalars* $\lambda_n \to \infty$ *such that:*

$$m\Big(\big\{t \in [0,1] : |T(x_n)(t)| > \lambda_n\big\}\Big) > \exp(-C_p \lambda_n^q).$$

The proof

Let $T\colon \ell^p \to X$ bounded linear. Suppose T is not compact.

1. There exists $S\colon \ell^p \to X$ with $\|S(e_n)\|_X \geq 1$.
2. From Lemma 1: $\|S(e_n)\|_{L^1} > \delta$. This implies $\|S(e_n)\|_{L^q} > \delta$.
3. From Lemma 3:

$$m\Big(\big\{t \in [0,1] : |S(x_n)(t)| > \lambda_n\big\}\Big) > \exp(-C_p\lambda_n^q),$$

   for some $C_p > 0$, $x_n \in \ell^p$, and $\lambda_n \to \infty$.

4. From Lemma 2:

$$m\Big(\big\{t \in [0,1] : |S(x_n)(t)| > \lambda_n\big\}\Big) \leq \exp(-e^{\lambda_n}).$$

Conclusion

Theorem

There exists a rearrangement invariant space X on $[0,1]$ such that, for every $p \in (1,\infty)$, if

$$T\colon \ell^p \to X$$

is a bounded linear operator then T is compact.

Theorem

$$\mathcal{L}(\ell^p, X) = \mathcal{K}(\ell^p, X) \not\Rightarrow \mathcal{L}(\ell^\infty, X) = \mathcal{K}(\ell^\infty, X).$$

Thank you for your attention.