A feature of averaging

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Authorship

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based on the work "Factorizing the Classical Inequalities"

by G. Bennett (1996)

Outline

1. Some classical inequalities: Hardy and Copson

2. A third inequality: Hölder

3. A striking property of $ces(p)$

Two inequalities

- Hardy (1920), for $p > 1$:

$$\sum_{n=1}^{\infty} |a_n|^p < \infty \Longrightarrow \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=0}^{n} |a_k| \right)^p < \infty.$$

- Copson (1928), for $0 < p < 1$:

$$\sum_{n=1}^{\infty} \left(\sum_{k=n}^{\infty} \frac{|a_k|}{k} \right)^p < \infty \Longrightarrow \sum_{n=1}^{\infty} |a_n|^p < \infty.$$

Two sequence spaces

- Hardy, for $1 < p < \infty$:

$$ces(p) := \left\{ (a_n) : \|a\|_{ces(p)} := \left(\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |a_k| \right)^p \right)^{1/p} < \infty \right\}$$

- Copson, for $0 < p < \infty$:

$$cop(p) := \left\{ (a_n) : \|a\|_{cop(p)} := \left(\sum_{n=1}^{\infty} \left(\sum_{k=n}^{\infty} \frac{|a_k|}{k} \right)^p \right)^{1/p} < \infty \right\}$$

Inequalities via inclusion of spaces

- Hardy, for $p > 1$:

$$\ell^p \subseteq ces(p)$$

- Copson, for $0 < p \leq 1$:

$$cop(p) \subseteq \ell^p$$

- Copson, for $p \geq 1$:

$$\ell^p \subseteq cop(p)$$

Multipliers and factorization of inequalities

- Multipliers from ℓ^p into $ces(p)$:

$$a \in \ell^p \longmapsto a \cdot b \in ces(p),$$

via ***coordinate-wise*** multiplication

$$a \cdot b := (a_n \cdot b_n).$$

- We look for the sequence space Z such that

$$b \in Z : \quad a \cdot b = (a_n \cdot b_n) \in ces(p), \quad \forall y \in \ell^p$$

and we write this as

$$\ell^p \cdot Z \subset ces(p)$$

- Can this inclusion can be an equality?

Factorization of inequalities I

Theorem (Bennett, 1996)

Let $p > 1$ and $1/p + 1/p' = 1$. A sequence $x \in ces(p)$ if and only if

$$x = a \cdot b,$$

with

$$a \in \ell^p, \quad \sum_{i=1}^{n} |b_i|^{p'} = O(n)$$

- It contains the optimal Hardy inequality.
- We write this as

$$ces(p) = \ell^p \cdot g(p')$$

for

$$g(q) := \left\{ (b_n) : \sum_{i=1}^{n} |b_i|^q = O(n) \right\}$$

Factorization of inequalities I

- We define, for $x \in ces(p)$, another norm:

$$|||x|||_{ces(p)} := \inf \left\{ \|a\|_{\ell^p} \cdot \|b\|_{g(p')} \right\},$$

where

$$\|b\|_{g(q)} := \sup_n \left(\frac{1}{n} \sum_{i=1}^{n} |b_i|^q \right)^{1/q}.$$

Theorem (B., 1996)

$$\frac{1}{(p-1)^{1/p}} |||x|||_{ces(p)} \leq \|x\|_{ces(p)} \leq p' |||x|||.$$

- The constants are best possible.

Factorization of inequalities II

Theorem (B., 1996)

Let $p > 1$ *and* $1/p + 1/p' = 1$.

$$cop(p) = \ell^p \cdot g(p'),$$

and, moreover,

$$|||x|||_{cop(p)} \leq \|x\|_{ces(p)} \leq p|||x|||.$$

where the constants are best possible.

- A *somewhat surprising* consequence is that

$$ces(p) = cop(p).$$

Factorization of inequalities II

The case $0 < p < 1$ for Copson inequality is more involved.

Theorem (B., 1996)

Let $0 < p < 1$. A sequence $x \in cop(p)$ if and only if

$$x = a \cdot b,$$

with $a \in \ell^1$ and b satisfying

$$b_n = \frac{n}{\frac{1}{z_1} + \frac{1}{z_2} \cdots + \frac{1}{z_n}}, \quad \textit{for some } z = (z_n) \in \ell^{p/(1-p)}.$$

Hölder's classical inequality

- A classical fact: for $0 < p, 1 \leq \infty$ and s given by

$$\frac{1}{s} = \frac{1}{p} + \frac{1}{q},$$

  we have (in terms of factorization)

$$\ell^s = \ell^p \cdot \ell^q.$$

- Next, we consider a similar situation but where the partial sums $\sum_1^n |b_i|^q$ are growing at a previously prescribe rate.

Some sequence spaces

- For $p > 0$, and sequence $a = (a_1, a_2 \dots)$ with non-negative terms, and

$$A_n = a_1 + a_2 + \cdots + a_n,$$

consider the sequence spaces:

$$g(a,p) := \left\{ b = (b_n) : \|b\|_{g(a,p)} := \sup_n \left(\frac{1}{A_n} \sum_{i=1}^{n} |b_i|^p \right)^{1/p} \right\},$$

$$d(a,p) := \left\{ b = (b_n) : \|b\|_{d(a,p)} := \left(\sum_{n=1}^{\infty} a_n \sup_{k \geq n} |b_k|^p \right)^{1/p} \right\}.$$

Factorization of inequalities I

Theorem (B., 1996)

A sequence x admits a factorization $\mathbf{x} = \mathbf{a} \cdot \mathbf{b}$ with $a \in \ell^p$ and $b \in g(a,q)$ if and only if

$$\sum_{n=1}^{\infty} a_n \left(\sum_{k=n}^{\infty} \frac{|x_k|^s}{A_k} \right)^{p/s} < \infty.$$

Moreover, for

$$|||x||| := \inf \left\{ \|a\|_{\ell^p} \cdot \|b\|_{g(a,q)} \right\}$$

we have, with best possible constants,

$$|||x||| \leq \left(\sum_{n=1}^{\infty} a_n \left(\sum_{k=n}^{\infty} \frac{|x_k|^s}{A_k} \right)^{p/s} \right)^{1/p} \leq \left(\frac{p}{s} \right)^{1/s} |||x|||.$$

A striking property of *ces*(p)

"A sequence of non-negative terms belongs to the space ces(p) precisely when its sequence of averages does.
This property is shared (except for the set of all sequences) by ***none*** *of the classical sequence spaces. "*

More precisely,

Theorem (Bennett (Theorem 20.31), 1996)

Fix $1 < p < \infty$, and let x be an arbitrary sequence. The $x \in ces(p)$ if and only of $y \in ces(p)$, where

$$y_n = \frac{|x_1| + |x_2| + \cdots + |x_n|}{n} \quad (n = 1, 2, \ldots).$$

A striking property of *ces*(*p*)

- The proof is quite involved.
- The proof is based ultimately on an important result on summability theory: the *Knopp-Schnee-Hausdorff* Theorem on the equivalence (as summability methods) of the Hölder and Cesàro matrices of the same order.
- This last result is Theorem 211 in the book “Divergent Series” of Hardy.

A short proof

- The following observation provides a short and direct proof of the necessity part of Bennett's result (sufficiency is Hardy's inequality).
- Recall that the Cesàro operator $\mathcal{C}$ on $\mathbb{C}^{\mathbb{N}}$, which assigns to any sequence its sequence of averages, is given by

$$x = (x_n)_1^\infty \longmapsto \mathcal{C}(x) := \Big(\frac{1}{n}\sum_{k=1}^{n} x_k\Big)_{n=1}^{\infty}.$$

Lemma (C., Ricker, 2013)

Let x be any sequence and set $|x| := (|x_n|)_1^\infty$. Then

$$\big(\mathcal{C}\,\mathcal{C}|x|\big)_n \geq \frac{1}{6}(\mathcal{C}|x|)_{[n/2]}, \quad n = 2, 3, \dots,$$

where $[\cdot]$ denotes the integer part.

A short proof

The ‘proof’ is elementary. Let $n \geq 2$, then,

$$
\begin{aligned}
\frac{1}{n}\sum_{k=1}^{n}\frac{1}{k}\sum_{j=1}^{k}|x_j| &= \frac{1}{n}\sum_{j=1}^{n}|x_j|\sum_{k=j}^{n}\frac{1}{k} \\
&\geq \frac{1}{n}\sum_{j=1}^{[n/2]}|x_j|\sum_{k=j}^{n}\frac{1}{k} \\
&\geq \frac{1}{n}\sum_{j=1}^{[n/2]}|x_j|\sum_{k=[n/2]}^{n}\frac{1}{k} \\
&\geq \frac{1}{2n}\sum_{j=1}^{[n/2]}|x_j| \geq \frac{1}{6}\frac{1}{[n/2]}\sum_{j=1}^{[n/2]}|x_j|.
\end{aligned}
$$

A short proof

- Note that
$$x \in ces(p) \iff \mathcal{C}|x| \in \ell^p,$$
and that
$$y = \mathcal{C}|x| \in ces(p) \iff \mathcal{C}\,\mathcal{C}|x| \in \ell^p$$
- Then, from the Lemma,
$$\sum_{n=1}^{\infty} (\mathcal{C}|x|)_n^p \leq 6^p \sum_{n=1}^{\infty} \left(\mathcal{C}\,\mathcal{C}|x|\right)_{2n}^p.$$
So,
$$\mathcal{C}\,\mathcal{C}|x| \in \ell^p \Longrightarrow \mathcal{C}|x| \in \ell^p.$$

Consequences

Extension of the necessity part of Bennett's Theorem beyond the class of ℓ^p-spaces.

- Let $\mathbb{X}$ be a vector subspace of $\mathbb{C}^{\mathbb{N}}$ satisfying:
  - Is solid: $|y_n| \leq |x_n|$ and $x \in \mathbb{X}$ implies $y \in \mathbb{X}$,
  - There exists $\varrho\colon \mathbb{C}^{\mathbb{N}} \to [0, \infty]$, monotone: $\varrho(y) \leq \varrho(x)$ if $|y_n| \leq |x_n|$, such that
  $$\mathbb{X} = \{x \in \mathbb{C}^{\mathbb{N}} : \varrho(x) < \infty\}$$
  - $(a_1, a_1, a_2, a_2, \dots, a_n, a_n, \dots) \in \mathbb{X}$ implies $(a_1, a_2, \dots, a_n, \dots) \in \mathbb{X}$.
- Define the sequence space
$$ces(\mathbb{X}) := \left\{x \in \mathbb{C}^{\mathbb{N}} : \mathcal{C}|x| \in \mathbb{X}\right\}.$$

Theorem (C., Ricker, 2013)

$$\mathcal{C}|x| \in ces(\mathbb{X}) \Rightarrow x \in ces(\mathbb{X})$$

Examples

- Examples of $\mathbb{X}$ include the following classical spaces:
  - The Orlicz spaces ℓ_M associated to an Orlicz function M
  - The Lorentz spaces $d(w, p)$ for $p \geq 1$ and a non-increasing, positive sequence $w = (w_n)_1^\infty$
  - The space s of rapidly decreasing sequences
  - Certain power series spaces, as $\Lambda_0(\alpha), \Lambda_\infty(\alpha)$
- In the event that $\mathcal{C}$ maps $\mathbb{X}$ into itself, we recover the full Bennett result:

$$x \in ces(\mathbb{X}) \iff \mathcal{C}|x| \in ces(\mathbb{X}).$$

- In the general case, a consequence can be drawn:

when applying the process of averaging to non-negative terms, the result obtained after one iteration is not improved by further iterations of averaging. This is due to the fact, via the Lemma, that $\varrho(\mathcal{C}\,\mathcal{C}|x|)$ is not essentially smaller than $\varrho(\mathcal{C}|x|)$.

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Thank you