Mini-course on K3 surfaces

Antonio Laface
Universidad de Concepción
antonio.laface@gmail.com

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Contents

CHAPTER 1

Projective properties

1. Preliminaries

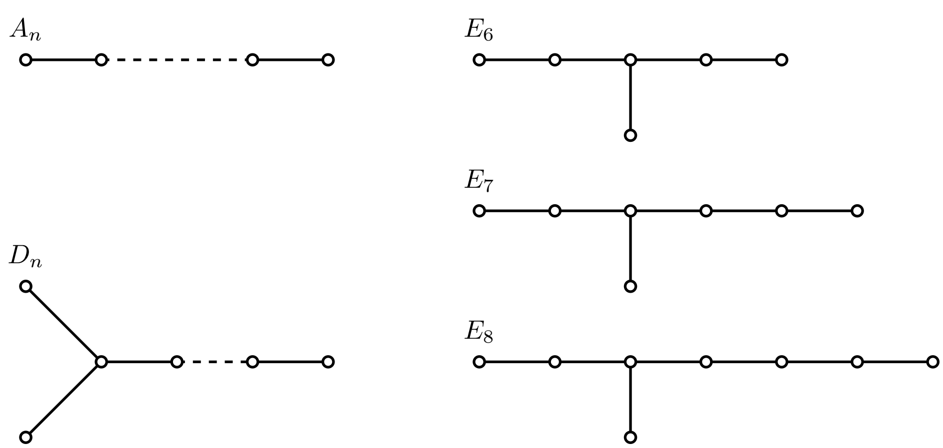
1.1. Zariski decomposition. Let X be a smooth projective surface and let D be an effective divisor of X. The *Zariski decomposition* of D is

$$D = P + N,$$

where both P and N are divisors with rational coefficients, P is nef, $P \cdot N = 0$ and N is a sum, with positive coefficients of prime divisors N_i, such that the intersection matrix $(N_i \cdot N_j)$ on the components of its support is negative-definite. As proved in [**Laz04**, Theorem 2.3.19] any effective divisor admits a unique Zariski decomposition.

1.2. Negative curves. Now, let us assume that D is an effective divisor on a K3 surface X and that $D = P + N$ is its Zariski decomposition. Since the components N_i of the support of N are prime divisors of negative self-intersection, by adjunction formula $2g(N_i) - 2 = N_i^2 < 0$, so that each N_i is a smooth rational curve with $N_i^2 = -2$. Such curves are called (-2)-*curves*. In particular each connected component Γ of the support of N is a union of (-2)-curves and the intersection matrix of such curves is negative definite. Due to this condition Γ must be a tree, since otherwise Γ contains a cycle and its components $N_1, \dots, N_r$ satisfy $(\sum_i N_i)^2 = 0$, a contradiction. In fact a lot more can be said about the structure of such a Γ.

Theorem 1.2.1. *Let Γ be a connected curve on a K3 surface X. Assume that the intersection form on the prime components of Γ is negative-definite. Then the lattice spanned by the classes of these components in* $\mathrm{Pic}(X)$ *is of type*



1.3. Morphisms. Recall that given a divisor D on a projective variety X, the *complete linear series* $|D|$ is the projective space whose points are the effective divisors D' of X linearly equivalent to D. By means of $|D|$ one can define a rational map, denoted by $\varphi_{|D|} : X \to \mathbb{P}^n = |D|^*$, defined by $p \mapsto |D - p|$, where the last symbol means the linear subspace of elements $D' \in |D|$ which contain p. Equivalently, given a basis $\{s_0, \dots, s_n\}$ of $H^0(X, \mathcal{O}_X(D))$, we have

$$\varphi_{|D|}(p) := (s_0(p) : \dots : s_n(p)).$$

Consider now a smooth irreducible curve C on a K3 surface X whose class $[C] \in \mathrm{Pic}(X)$ is ample. If $C^2 > 0$, then by the Kawamata-Viehweg vanishing theorem [**Laz04**, Theorem 4.3.1] the higher cohomology groups of $\mathcal{O}_X(C)$ vanish. Thus

$$h^0(\mathcal{O}_X(C)) = \frac{C^2}{2} + 2$$

by the Riemann-Roch theorem. It is not hard to prove that the same holds if $C^2 \leq 0$. Hence particular the complete linear series $|C|$ has dimension $C^2/2 + 1 = g(C)$, where $g(C)$ is the topological genus of C. Moreover, by adjunction formula and the triviality of K_X, the restriction of $\mathcal{O}_X(C)$ to C is the canonical divisor K_C of the curve. Hence there is an exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(C) \longrightarrow \mathcal{O}_C(K_C) \longrightarrow 0. \tag{1.3.1}$$

By passing to the long exact sequence in cohomology and recalling that $h^1(\mathcal{O}_X) = 0$, we observe that the restriction map $H^0(\mathcal{O}_X(C)) \to H^0(\mathcal{O}_C(K_C))$ is surjective. Equivalently this means that the rational map

$$\varphi_{|C|} : X \to \mathbb{P}^g,$$

where $g = g(C)$, defined by the complete linear series $|C|$, induces the canonical embedding on all the smooth members of $|C|$. In particular if C is non-hyperelliptic then φ is an embedding, so that C is a very ample divisor on X. More generally we have the following.

THEOREM 1.3.1. *Let C be a smooth curve on a K3 surface X with $C^2 > 0$. Then the complete linear series $|C|$ is base point free. The morphism $\varphi_{|C|} : X \to \mathbb{P}^g$ has degree 1 or 2. Moreover it has degree 2 if and only if any smooth member of $|C|$ is a hyperelliptic curve.*

EXAMPLE 1.3.2. Let X be a K3 surface which contains a smooth curve C with $C^2 = 2$, whose classe $[C] \in \mathrm{Pic}(X)$ is ample. By the Riemann-Roch theorem $h^0(\mathcal{O}_X(C)) = 3$, that is the dimension of the complete linear series $|C|$ is 2. By adjunction formula C has genus 2, so that it is hyperelliptic. Hence the morphism $\varphi_{|C|} : X \to \mathbb{P}^2$ is a double cover. If $B \in |C|$ is a general smooth member, the restriction of $\varphi_{|C|}$ to B is the canonical map of B, hence it is a double cover branched at six points. Since $\varphi_{|C|}(B)$ is a line this implies that the degree of the branch divisor of $\varphi_{|C|}$ is a plane curve of degree six.

1.4. Semiample divisors. We recall that a divisor D is *semiample* if the complete linear series $|nD|$ is base point free for some positive integer n, that is for any $p \in X$ there exists an element $D' \in |D|$ such that $x \notin D'$. The following theorem shows that any nef divisor on a K3 surface is semiample, the converse being obvious.

THEOREM 1.4.1. *Let D be a nef divisor on a smooth K3 surface, then D is semiample.*

PROOF. Since D is a nef divisor its class lies in the closure of the ample cone of X by Kleiman theorem [**Laz04**, Theorem 1.4.23]. Hence $D^2 \geq 0$ and $h^0(\mathcal{O}_X(D)) > 1$ by Riemann-Roch. Assume $D^2 > 0$. First of all we prove that the linear series $|D|$ does not contain fixed components. Indeed, if E is the fixed divisor of the linear series, then $h^2(\mathcal{O}_X(E)) = h^0(\mathcal{O}_X(-E)) = 0$, where the first equality is by Serre's duality and the second is because E is effective. Thus

$$1 = h^0(\mathcal{O}_X(E)) \geq \frac{E^2}{2} + 2,$$

where the inequality is by Riemann-Roch, implies $E^2 < 0$. Observe that for any nef and big divisor P, that is $P^2 > 0$, we have $h^0(\mathcal{O}_X(P)) = P^2/2 + 2$, by Riemann-Roch and the Kawamata-Viewheg vanishing theorem. Since both D and $D - E$ are nef and big divisors with linear series of the same dimension, then by the previous observation $D^2 = (D - E)^2$, so that $0 \leq 2D \cdot E = E^2 < 0$, a contradiction. Hence $|D|$ does not have fixed components and we conclude that D is semiample by Zariski's theorem [**Laz04**, Remark 2.1.32].

Assume now that $D^2 = 0$. It is enough to show, as above, that $|D|$ does not contain fixed components. If E is the fixed part of the linear series, then $E^2 < 0$ and $D - E$ is a nef divisor. Hence $0 \leq (D - E)^2 = -2D \cdot E + E^2 < 0$, a contradiction. □

It is possible to prove more in general that if D is a nef divisor on a K3 surface, then $|3D|$ is base point free [**SD74**].

2. The Mori cone

Let X be a K3 surface; recall that $\operatorname{Pic}(X)$ injects into $H^2(X, \mathbb{Z})$. Denote by $N_1(X)$ the image of $\operatorname{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ into $H^2(X, \mathbb{R})$, that is the real vector space of 1-cycles modulo homological equivalence. The *Mori cone* of X is the closure, in the Euclidean topology, of the cone of $N_1(X)$:

$$\mathrm{NE}(X) := \{\sum_i a_i [C_i] : C_i \text{ is a curve of } X \text{ and } a_i \geq 0\}.$$

We will denote the Mori cone by $\overline{\mathrm{NE}(X)}$. Since we are dealing with a surface, then curves are also divisors, so that the Mori cone of X coincides with the closure of the cone of effective divisors of X. A vector v of a cone V is said to span an *extremal ray* of V if v can not be written as a sum $v = v_1 + v_2$ of vectors $v_i \in V$ which are not multiples of v. Now observe that if E is an effective divisor such that, for any integer $n > 0$, the components of any reducible element of the linear series $|nE|$ are linearly equivalent to a multiple of E, then the class $[E]$ of E spans an extremal ray of $\mathrm{NE}(X)$. Indeed, if $[E] = x_1 + x_2$, with x_1 and x_2 in $\mathrm{NE}(X)$, then $nE \sim D_1 + D_2$, for some integer $n > 0$, where D_1 and D_2 are not linearly equivalent to a multiple of E. Hence $|nE|$ contains a reducible element whose prime components are not all linearly equivalent to a multiple of E, a contradiction. As a consequence, the class of a (-2)-curve spans an extremal ray of $\mathrm{NE}(X)$. If $\rho_X \geq 2$, denote by V the *light cone* of X, that is:

$$V := \{x \in \operatorname{Pic}(X)_{\mathbb{Z}} \otimes \mathbb{R} : x^2 \geq 0\}.$$

The reason for the name "light cone" is that the Picard lattice has a quadratic form of signature $(1, \rho_X - 1)$, like the Minkowski space-time. Let V_X^+ be the closure of the connected component of $V - \{0\}$ which contains the nef cone.

The following theorem has been proved in [**Kov94**]

THEOREM 2.0.2. *Let X be a K3 surface with $\rho_X \geq 2$. Then one of the following holds:*

(i) *$\rho_X = 1$ and the Mori cone is generated by an ample class;*
(ii) *$\rho_X = 2$ and the Mori cone is generated by the classes of a (-2)-curve and an elliptic curve;*
(iii) *$2 \leq \rho_X \leq 4$, the surface X does not contain elliptic curves and (-2)-curves, and the Mori cone is V_X^+;*
(iv) *$2 \leq \rho_X \leq 11$ and the Mori cone is V_X^+, which is also the closure of the cone spanned by classes of elliptic curves;*
(v) *$2 \leq \rho_X \leq 20$ and the Mori cone is the closure of the cone generated by classes of (-2)-curves.*

All the previous cases occur for any indicated value of ρ_X.

It is interesting to observe that, in case $\rho_X \geq 3$, if X contains a (-2)-curve then the Mori cone of X is the closure of the cone generated by the classes of (-2)-curves of X.

EXAMPLE 2.0.3. Let X be a K3 surface whose Picard lattice is isometric to the lattice $S = U \oplus (-4)$. Such a K3 surface exists since Nikulin [**Nik79**] proved that any even hyperbolic lattice of rank ≤ 10 can be embedded in the K3 lattice. Thus by the global Torelli theorem S is isometric to the Picard lattice of any K3 surface whose period $\omega \in S^\perp \otimes_{\mathbb{Z}} \mathbb{C}$ is very general. If $\{e_1, e_2, e_3\}$ is a basis of $\mathrm{Pic}(X)$ with the given intersection matrix, then the classes e_1, $e_2 - e_1$ and $e_1 + e_2 - e_3$ represents, respectively, an elliptic curve C and two (-2)-curves E_1 and E_2 of X. The elliptic fibration

$$\varphi_{|C|} : X \to \mathbb{P}^1$$

does not admits reducible fibers. To see this observe that a reducible fiber is a union of (-2)-curves whose classes are orthogonal to $e_1 = [C]$, but the square of any element of $e_1^\perp = \langle e_1, e_3 \rangle$ is divisible by 4. Since $\varphi_{|C}$ does not admit reducible fibers, then its *Mordell-Weil group* (this is $\mathrm{Pic}^0(X_\eta)$, where X_η is the generic fiber of $\varphi_{|C|}$) is infinite and isomorphic to $\mathbb{Z}$ by the Shioda-Tate formula. Any element of the Mordell-Weil group corresponds to a section of the elliptic fibration. Thus there are infinitely many sections. These are just smooth rational curves on X, so that $\mathrm{Nef}(X)$ is not polyhedral since X contains infinitely many (-2)-curves. Hence $\mathrm{Aut}(X)$ is not finite. See also [**Tod79**, Examples 5.1] for a description of the nef cone of X.

2.1. K3 Surfaces without (-2)-curves. If X does not contain (-2)-curves, then any effective class x has non-negative self intersection. The main point here is: does there exist a class $x \in \mathrm{Pic}(X)$ with $x^2 = 0$? The theorem gives an affirmative answer if $\rho_X \geq 5$. In this case by Riemann-Roch x or $-x$ is effective. Let us say x. Thus x must span an extremal ray of the effective cone of X, since otherwise $x = \sum_i \alpha_i x_i$, with x_i classes of effective integral curves of X and α_i positive rational coefficients. From

$$x \cdot (\sum_i \alpha_i x_i) = x^2 = 0$$

and the fact that X does not contain curves of negative self-intersection, we conclude $x = x_i$ by the Hodge index theorem. Let $[D]$ be a primitive generator of the ray spanned by x in $\mathrm{Pic}(X)_{\mathbb{R}}$, that is it has integer coefficients with greatest common divisor 1. Since x is an extremal ray, then D is an integral curve. By adjunction formula $2p_a(D)-2 = D^2 = 0$, so that either D is an elliptic curve or it is a singular rational curve. In both cases the morphism $\varphi_{|D|} : X \to \mathbb{P}^1$ defined by the complete linear series $|D|$ is an *elliptic fibration*, meaning with this that its general fiber is a smooth elliptic curve. The fact that x is an extremal ray of the Mori cone implies that the elliptic fibration $\varphi_{|D|}$ does not have reducible fibers.

2.2. The case $\rho_X \geq 12$. In this case it is possible to prove [**Kon86**, Lemma 4.1] that there exists a class $x \in \mathrm{Pic}(X)$ with $x^2 = -2$. By Riemann-Roch either x or $-x$ must be effective. Let us assume x to be effective. Then $x = \sum_i \alpha_i x_i$, with x_i classes of effective integral curves of X and α_i positive rational coefficients. From

$$x \cdot (\sum_i \alpha_i x_i) = x^2 = -2$$

we deduce that $x_i^2 < 0$ for some i, so that X contains the (-2)-curve whose class is x_i. According to Theorem 2.0.2 the Mori cone of X is generated by classes of (-2)-curves.

2.3. The case $\rho_X \leq 2$. If $\rho_X = 1$, the Mori cone is spanned by the primitive ample class of X and there is not much to say. If $\rho_X = 2$, by Theorem 2.0.2, the Mori cone has two extremal rays. which can be generated by the classes of two (-2)-curves, one (-2)-curve and an elliptic curve, two elliptic curves, two non-effective classes x_1, x_2 with $x_i^2 = 0$. The following are four examples of Gram matrices of Picard lattices for each of the four possibility.

$$\begin{bmatrix} -2 & 4 \\ 4 & -2 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \qquad \begin{bmatrix} 4 & 0 \\ 0 & -8 \end{bmatrix}$$

It is not difficult to show that each such lattice embeds into the K3 lattice Λ_{K3} so that by Theorem **??** there exists a K3 surface X in each case with that Picard lattice. Moreover it is possible to give a projective model in each case. In the first case X is a quartic surface of $\mathbb{P}^3$ which admit a hyperplane section which is the union of two conics C_1 and C_2. Since the C_i are smooth rational curves on X, then by adjunction formula they are (-2)-curves. Moreover two plane conics intersect at 4 points by Bezout's theorem.

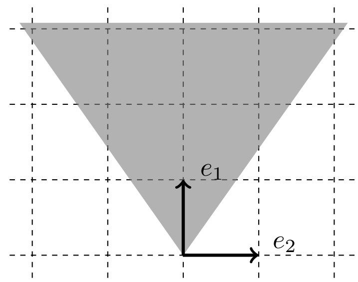
In the second case X contains two classes x_1 and x_2 which intersect at one point. One might be tempted to sat that both the x_i are classes of elliptic curves C_1 and C_2, but this can not be the case. Indeed if so, there would be two elliptic fibrations on X, given by $|C_1|$ and $|C_2|$. Since $C_1 \cdot C_2 = 1$, then $\varphi_{|C_1|}|_{C_2} : C_2 \to \mathbb{P}^1$ would be one to one, that is an isomorphism, a contradiction. Hence one of the two C_i must be reducible, for example $C_2 = C_1 + E$. Now E is a (-2)-curve of X and the Mori cone of X is spanned by the classes of E and C_1. A projective model is given by the double cover of a Hirzebruch surface $Y = \mathbb{F}_4$ branched along an element of $B + \Gamma \in |-2K_Y|$, where Γ is the (-4)-curve of Y. In this case the class of C_1 is the pul-back of the class of a element of the ruling of Y and the class of E is the pull-back of that of Γ.

The third case is the double cover of $Y = \mathbb{P}^1 \times \mathbb{P}^1$ branched along $|-2K_Y|$. The two elliptinc fibrations come from the two rulings of Y.

Finally in the fourth case $x^2 \in 4\mathbb{Z}$ for any $x \in \mathrm{Pic}(X)$ so that X does not contain (-2)-curves. Moreover, Moreover if e_1 and e_2 generate $\mathrm{Pic}(X)$ and $x = ae_1 + be_2$, with a, b integers, then

$$x^2 = 4a^2 - 8b^2$$

can not vanish, so that X does not contain elliptic curves. Hence we are in case (iv) of Theorem 2.0.2, so that the Mori cone of X is generated by the classes of non-effective curves.



2.4. Mori dream K3 surfaces. We now wish to deepen our knowledge of K3 surfaces which admit a finitely generated Mori cone. A smooth algebraic surface X is *Mori dream* if the following conditions hold:

(i) $h^1(\mathcal{O}_X) = 0$;
(ii) $\mathrm{Nef}(X)$ is generated by a finite number of semiample classes.

Observe that since the nef cone is dual to the Mori cone, then condition (ii) is equivalent to ask that the Mori cone is generated by finitely many classes of effective curves and that each nef divisor is semiample.

Theorem 2.4.1. *Let X be a K3 surface. The X is Mori dream if and only if its automorphism group is finite.*

Proof. We have already proved in Theorem 1.4.1 that on any K3 surface every nef divisor is semiample. Recall that in the previous chapter we showed that the homomorphism $\mathrm{Aut}(X) \to O(\mathrm{Nef}(X))$ has finite kernel and cokernel. As a consequence $\mathrm{Aut}(X)$ is finite if and only if $O(\mathrm{Nef}(X))$ is finite. This happens if and only if $\mathrm{Nef}(X)$ is polyhedral. □

Observe that if $\mathrm{Nef}(X)$ is polyhedral, to each maximal face F of this cone there corresponds an extremal ray of the effective cone. This ray has to be spanned by the class e of a curve E which is orthogonal to all the nef classes in F. If $\rho_X \geq 3$, then F contains at least two elements x_1 and x_2. Observe that $(x_1 + x_2)^2 > 0$, since Since the signature of $\mathrm{Pic}(X)$ is $(1, \rho_X - 1)$. By the same reason $e^2 < 0$, being orthogonal to a class of positive self intersection. hence E is a (-2)-curve. This shows that the Mori cone of X is spanned by a finite number of (-2)-curves.

3. Cox rings

In this last section we consider Cox rings of K3 surfaces. Briefly recall the definition of the *Cox ring* of X:

$$\mathcal{R}(X) := \bigoplus_{[D] \in \mathrm{Pic}(X)} H^0(X, \mathcal{O}_X(D)).$$

It is possible to show that $\mathcal{R}(X)$ is finitely generated if and only if X is Mori dream, hence if and only if $\mathrm{Aut}(X)$ is a finite group. The action of $\mathrm{Aut}(X)$ can be used in some cases to determine a presentation for $\mathcal{R}(X)$. In particular if $\varphi \in \mathrm{Aut}(X)$ is a non-symplectic involution one wish to relate the Cox rings of the two surfaces of the double covering:

$$\pi : X \to Y := X/\langle\varphi\rangle.$$

We know that Y is either a rational surface or an Enriques surface. If Y is an Enriques surface, then we have an injective homomorphism $\mathrm{Aut}(Y) \to \mathrm{Aut}(X)$ since X is the universal covering of Y. It is possible to show, by means of Theorem **??**, that if $\mathrm{Aut}(Y)$ is finite then $\mathrm{Aut}(X)$ is not finite. Hence there are no Mori dream K3 surfaces which double cover an Enriques surface. Observe that this does not mean that there are no Mori dream Enriques surfaces. Indeed it is possible to prove that Y is Mori dream if and only if $\mathrm{Aut}(Y)$ is finite [**AHL10**]. If Y is a rational surface it is possible to relate its Cox ring with that of X in the following case.

THEOREM 3.0.2. *Let X be a K3 surface which admits a double cover $\pi : X \to Y$ on a Mori dream rational surface Y. If $\pi^*(\mathrm{Pic}(Y))$ has finite index in $\mathrm{Pic}(X)$, then X is Mori dream. Moreover If $\pi^*(\mathrm{Pic}(Y)) = \mathrm{Pic}(X)$, then there is an isomorphism of $\mathrm{Pic}(X)$-graded rings:*

$$\mathcal{R}(X) \cong \mathcal{R}(Y)[t]/(t^2 - x_B),$$

where x_R is a defining section for the branch divisor B of π.

The proof makes use of the fact that there is an isomorphism of sheaves $\pi_*\mathcal{O}_X \cong \mathcal{O}_Y \oplus \mathcal{O}_Y(-1/2\,B)$, where B is the branch divisor of π. By the hypothesis, if D is a divisor of X, then $D = \pi^*L$, for some divisor L of Y. It s possible to show that there is an isomorphism of sheaves:

$$\mathcal{O}_X(D) \cong \pi^*\mathcal{O}_Y(L) \oplus \sqrt{x_B} \cdot \pi^*\mathcal{O}_Y(L - 1/2\,B).$$

By taking global sections and observing that $H^0(\pi^*\mathcal{O}_X(L)) \cong H^0(\mathcal{O}_X(L))$, one proves the statement.

3.1. Examples of Cox rings. Consider the K3 surface X whose Picard lattice has Gram matrix

$$\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}.$$

We have already seen that X is double cover of $Y := \mathbb{P}^1 \times \mathbb{P}^1$ branched along a smooth curve $B \in |-2K_Y|$. Since the Picard lattice of Y is generated by two classes f_1, f_2 of zero self intersection with $f_1 \cdot f_2$, if we set $e_i : 0\pi^*(f_i)$, then $\{e_1, e_2\}$ is a basis of $\mathrm{Pic}(X)$. Hence the condition $\mathrm{Pic}(X) = \pi^*(\mathrm{Pic}(Y))$ holds, so that

$$\mathcal{R}(X) \cong \mathbb{C}[x_1, \ldots, x_4, t]/(t^2 - x_B),$$

since the Cox ring of Y is a polynomial ring, being Y a toric variety (see [**ADHL**, Chapter II]). As a second example consider the K3 surface whose Picard lattice has Gram matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

As we have already explained X is double cover of a Hirzebruch surface $Y = \mathbb{F}_4$ branched along a smooth element $B \in |-2K_Y|$. Observe that B is a union of two disjoint curves $C \cup \Gamma$, where Γ is the unique rational curve of self intersection -4 of Y. Since Γ is in the branch locus of π, we have that $E := \pi^{-1}(\Gamma)$ is still a smooth rational curve, so that it is a (2)-curve. Observe that $\pi^*\Gamma = 2E$, so that the condition of Theorem 3.0.2 is not satisfied since the class of $[E]$ does not belong to $\pi^*(\mathrm{Pic}(Y))$. It is still possible [**AHL10**] to find a presentation for the Cox ring of X:

$$\mathcal{R}(X) \cong \mathbb{C}[x_1, \ldots, x_4, t]/(t^2 - x_C),$$

where one of the x_i, let us say the fourth is the square root of the generator of the Cox ring of Y which corresponds to a defining section of Γ.

Exercises

Exercise 3.1. Let D be a nef and big divisor on a K3 surface X. Show that a multiple of D defines a morphism $X \to X'$, where X' is a normal surface with *Du Val singularities*, that is singularities whose minimal resolution is a tree of rational curves of type A_n, D_n, E_6, E_7 or E_8.

Exercise 3.2. Let D be an elliptic curve on a K3 surface. Show that the complete linear series $|D|$ has dimension 1.

Exercise 3.3. Let D be a divisor on a K3 surface with $D^2 \geq -2$. Show that either $-D$ or D is linearly equivalent to an effective divisor.

Exercise 3.4. Show that if D is a nef divisor on a K3 surface, with $D^2 = 0$, then D is linearly equivalent to nE, where E is an elliptic curve and n is a positive integer.

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