Mini-course on K3 surfaces

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Lattices

1. Even lattices

A lattice is a finitely generated free abelian group Λ together with a quadratic form $q : \Lambda \times \Lambda \to \mathbb{Z}$. Basic invariants of a lattice Λ are its *rank*, defined as the dimension of the real vector space $\Lambda \otimes_{\mathbb{Z}} \mathbb{R}$, and its *signature*, defined to be the pair of numbers of positive and negative eigenvalues of the extension of the quadratic form q to $\Lambda \otimes_{\mathbb{Z}} \mathbb{R}$. A lattice is *even* if $q(x) \in 2\mathbb{Z}$ for any $x \in \Lambda$. Recall that a lattice is *unimodular* if the determinant of a Gram matrix of q with respect to a basis is ± 1. An *isometry* of lattices is an homomorphism of abelian groups $\sigma : \Lambda_1 \to \Lambda_2$ such that $q_2(\sigma(x)) = q_1(x)$, for any $x \in \Lambda_1$, where q_i is the quadratic form of Λ_i. A good reference for the whole section is [**Dol83**].

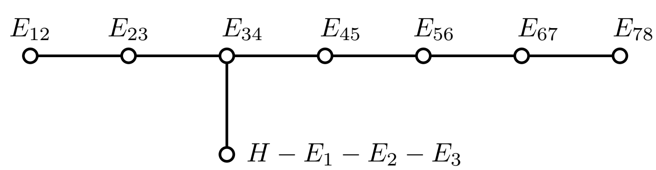
1.1. The U and E_8 lattices. The *U lattice* is the rank two unimodular lattice of signature $(1, 1)$, whose Gram matrix is

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

We define the E_8 lattice by means of the following geometric construction. Let $\pi : Y \to \mathbb{P}^2$ be the blow-up of the projective plane at $r \leq 8$ distinct general points. The surface Y is a *del Pezzo* surface, that is its anticanonical class is ample, and its Picard group is a lattice of signature $(1, r)$. It is not difficult to see that $\mathrm{Pic}(Y)$ is unimodular, as it admits a basis done by the classes of the pull-back of a line plus the exceptional divisors, whose Gram matrix is diagonal with determinant ± 1. Inside $\mathrm{Pic}(Y)$ consider the sublattice

$$K_Y^{\perp} := \{x \in \mathrm{Pic}(Y) : x \cdot K_Y = 0\}.$$

Since $K_Y^2 > 0$, then $K_Y^{\perp}$ is a negative definite lattice. If we concentrate on the case $r = 8$, we see that $K_Y^{\perp}$ is the lattice spanned by the classes of the vertices of the following diagram:



Each vertex E_{ij} is the class of the difference $E_i - E_j$ of the i-th and j-th exceptional divisor of the blow-up. The vertex $H - E_1 - E_2 - E_3$ is the class of the pull-back of a line minus the first three exceptional divisors. Finally each edge represents an intersection between the classes of the corresponding vertices. For example we have an edge from E_{12} to E_{23} since $(E_1 - E_2) \cdot (E_2 - E_3) = -E_2^2 = 1$. The vertices of the above picture form a basis of the lattice. Its Gram matrix with respect to the given basis is

$$\begin{bmatrix} -2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{bmatrix}.$$

It is called the E_8-*lattice*. Since the previous matrix has determinant 1, the E_8-lattice is unimodular.

1.2. The K3 lattice. Whenever we have two lattices Λ_1 and Λ_2, we can form their direct sum $\Lambda_1 \oplus \Lambda_2$. This is a lattice with respect to the product $(x_1, x_2) \cdot (y_1, y_2) := x_1 \cdot y_1 + x_2 \cdot y_2$. With Λ^n we wil mean the direct sum of n copies of Λ. Recall the following theorem of J. Milnor.

THEOREM 1.2.1 ([**Mil58**]). *Let Λ be an indefinite unimodular lattice. If Λ is even, then $\Lambda \cong E_8(\pm 1)^m \oplus U^n$ for some m and n integers. If Λ is odd, then $\Lambda \cong (1)^m \oplus (-1)^n$ for some m and n integers.*

A remark about notation is due here. Our notation for the lattice E_8 is not the standard one adopted in the theory of Lie Groups. To relate with this notation we should write $E_8(-1)$ instead, meaning with this the lattice whose entries of the Gram matrix are the opposite of those that we have given for our E_8. As a consequence of the previous theorem we have the following.

PROPOSITION 1.2.2. *The K3 lattice Λ_{K3} is isometric to $E_8^2 \oplus U^3$.*

PROOF. Since both U and E_8 are unimodular and even, then also their sum is. Moreover the lattice $E_8^2 \oplus U^3$ has signature $(3, 19)$, so that it is not definite. Hence we conclude by Theorem 1.2.1, recalling that the K3 lattice Λ_{K3}, which is isomorphic to $H^2(X, \mathbb{Z})$ for any K3 surface X, is even unimodular with signature $(3, 19)$. □

1.3. The discriminant group. Given a lattice Λ we define its *dual lattice* to be the subset of elements of $\Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$ which have integer intersection with any element of Λ. In symbols it is:

$$\Lambda^* := \{x \in \Lambda \otimes_{\mathbb{Z}} \mathbb{Q} : x \cdot z \in \mathbb{Z} \text{ for any } z \text{ in } \Lambda\}.$$

Observe that the dual lattice may be not a lattice with respect to our original definition, since it can contain elements whose intersection is not integer. By abuse of language we will keep calling it lattice. For example consider the rank one lattice whose Gram matrix is (2). Then its dual lattice has Gram matrix $(1/2)$. With abuse of language we will still call it lattice. Given a non-degenerate even lattice Λ, its *discriminant group* is the quotient

$$\mathrm{d}(\Lambda) := \Lambda^*/\Lambda$$

equipped with the quadratic form $q_\Lambda : \mathrm{d}(\Lambda) \to \mathbb{Q}/2\mathbb{Z}$, induced by the quadratic form q on Λ. Observe that if M is a Gram matrix for Λ, then the order of the discriminant group $d(\Lambda)$ is the absolute value of the determinant of M. In particular $\Lambda^* = \Lambda$ if and only if Λ is unimodular.

EXAMPLE 1.3.1. Consider the rank 2 lattice Λ whose Gram matrix with respect to a basis $\{e_1, e_2\}$ is

$$M := \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}.$$

The vector $v := (e_1 - e_2)/3 \in \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$ has integer intersection with e_1 and e_2, hence with all the elements of Λ, so that $v \in \Lambda^*$. Since M has determinant 3, then the discriminant group $d(\Lambda)$ has order three by the previous observation, so that it is generated by v. Since $q_d(v) = -2/3$, a Gram matrix for the discriminant is $(-2/3)$.

1.4. Primitive embeddings. An inclusion of lattices $\Lambda_1 \subset \Lambda$ is a *primitive embedding* if the quotient Λ/Λ_1 is a torsion-free abelian group. For example if Λ is a lattice with basis $\{e_1, e_2\}$, then the sublattice Λ_1 spanned by $\{e_1 + e_2, e_1 - e_2\}$ is not primitive in Λ, as the quotient $\Lambda/\Lambda_1 \cong \mathbb{Z}/2\mathbb{Z}$. Given a sublattice $\Lambda_1 \subset \Lambda$ we define its *orthogonal lattice* to be $\Lambda_1^\perp := \{x \in \Lambda : x \cdot y = 0 \text{ for any } y \in \Lambda_1\}$. Observe that $\Lambda_1^\perp$ is always primitive in Λ.

PROPOSITION 1.4.1. *Let Λ be a unimodular lattice, $\Lambda_1 \subset \Lambda$ be a primitive embedding and $\Lambda_2 := \Lambda_1^\perp$ be its orthogonal complement. Then, for $i = 1, 2$, there are natural isomorphisms of abelian groups*

$$\gamma_i : \Lambda/(\Lambda_1 \oplus \Lambda_2) \to \mathrm{d}(\Lambda_i).$$

In particular $d(\Lambda_1) \cong d(\Lambda_2)$.

PROOF. First of all observe that an element $x \in \Lambda$ can be written in a unique way as $x = x_1 + x_2$, with $x_i \in \Lambda_i^*$, since $\Lambda_1 \oplus \Lambda_2$ has finite index in Λ. Consider now the homomorphism $\varphi_i : \Lambda \to \Lambda_i^*$ defined by $\varphi_i(x) = x_i$. We want to prove that it is surjective. Since Λ_i is primitive in Λ, then the inclusion map $i : \Lambda_i \to \Lambda$ admits a projection $\pi : \Lambda \to \Lambda_i$, that is $\pi \circ i = \mathrm{id}$. This implies that the map $i^* : \Lambda^* \to \Lambda_i^*$, which coincides with φ_i since Λ is unimodular, is surjective. Moreover if for example $\varphi_2(z) = 0$, then $z \in \Lambda_2^\perp = \Lambda_1$, since Λ_1 is primitive in Λ. Hence we get an isomorphism of abelian groups $\Lambda/\Lambda_1 \to \Lambda_2^*$ and similarly exchanging 1 with 2. Hence the induced maps $\Lambda/(\Lambda_1 \oplus \Lambda_2) \to \Lambda_i^*/\Lambda_i = d(\Lambda_i)$ are isomorphisms. □

1.5. Lifting isometries. Consider now a primitive embedding $L \subset \Lambda$ of non-degenerate even lattices of the same rank. This gives inclusions $L \subset \Lambda \subset \Lambda^* \subset L^*$. The quadratic form q_L on $\mathrm{d}(L)$ restricts to the null form on Λ/L since $q(x)$ is an even integer for any $x \in \Lambda$. On the other hand, if we have an *isotropic subgroup* H of $\mathrm{d}(L)$, that is a subgroup such that $q_L|_H \equiv 0$, then there exists a non degenerate lattice $\Lambda \supset L$ such that $\Lambda/L \cong H$. Hence there is a bijection

$$\{\Lambda : \Lambda \supset L \text{ with } \mathrm{rk}(\Lambda) = \mathrm{rk}(L)\} \leftrightarrow \{\text{Subgroups } H \subset \mathrm{d}(L) : q_L|_H \equiv 0\}.$$

We are interested in understanding when an isometry σ of such an L extends to an isometry η of a lattice $\Lambda \supset L$ of the same rank. Observe that an isometry σ of L induces an isometry of its dual lattice L^*, which in turns gives an isometry σ^* of the discriminant lattice $\mathrm{d}(L)$. Consider the inclusions

$$L \subset \Lambda \subset \Lambda^* \subset L^*.$$

It is not difficult to show that σ admits an extension η if and only if $\sigma^*(\Lambda/L) = \Lambda/L$.

1.6. Gluing isometries. We have already seen by Proposition 1.4.1 that given a primitive sublattice Λ_1 of a unimodular lattice Λ and its orthogonal Λ_2, there is a natural isomorphism $\gamma : \mathrm{d}(\Lambda_1) \to \mathrm{d}(\Lambda_2)$ which allows us to identify the two discriminant groups. Now we consider when a pair of isometries of Λ_1 and Λ_2 give an isometry of Λ. More precisely we have the following.

PROPOSITION 1.6.1. *Let $\Lambda_1 \subset \Lambda$ be a primitive sublattice of a unimodular lattice and let $\Lambda_2 := \Lambda_1^{\perp}$ be its orthogonal sublattice. Let σ_1 and σ_2 be two isometries of Λ_1 and Λ_2 respectively. Then the following are equivalent.*

(i) *There exists a unique isometry σ of Λ such that $\sigma|_{\Lambda_1} = \sigma_1$ and $\sigma|_{\Lambda_2} = \sigma_2$.*
(ii) *If σ_i^* is the isometry of the discriminant lattice induced by σ_i, then the following diagram is commutative*

$$\begin{CD} \mathrm{d}(\Lambda_1) @>{\gamma}>> \mathrm{d}(\Lambda_2) \\ @V{\sigma_1^*}VV @VV{\sigma_2^*}V \\ \mathrm{d}(\Lambda_1) @>>{\gamma}> \mathrm{d}(\Lambda_2). \end{CD}$$

PROOF. Recall that the elements of the quotient group $H := \Lambda/(\Lambda_1 \oplus \Lambda_2)$ are of the form $x + \gamma(x)$, with $x \in \mathrm{d}(\Lambda_1)$. Assume that (i) holds. Then $\sigma^*(x + \gamma(x)) = \sigma_1^*(x) + \sigma_2^*(\gamma(x))$ is an element of H, so that $\sigma_2^*(\gamma(x)) = \gamma(\sigma_1^*(x))$, which proves (ii).

Assume now that (ii) holds. Then given an element $x + \gamma(x)$ of H we have that $(\sigma_1^* \oplus \sigma_2^*)(x + \gamma(x)) = \sigma_1^*(x) + \sigma_2^*(\gamma(x)) = \sigma_1^*(x) + \gamma(\sigma_1^*(x))$ is again in H. Hence we conclude by our previous discussion. □

As a last remark, observe that the two quadratic forms q_{Λ_1} and q_{Λ_2} on the two discriminant groups are related. If $x + \gamma(x)$ is an element of $\Lambda/(\Lambda_1 \oplus \Lambda_2)$, then $0 = q(x + \gamma(x)) = q_{\Lambda_1}(x) + q_{\Lambda_2}(\gamma(x))$. Hence we have

$$q_{\Lambda_1} = -q_{\Lambda_2} \circ \gamma.$$

2. Automorphisms

Now we want to apply our knowledges of even lattices and Torelli theorem to the study of automorphisms of K3 surfaces. To this aim we will denote by $\mathrm{Aut}(X)$ the group of automorphisms of X and by $\mathrm{Aut}(X)_0$ the subgroup of $\mathrm{Aut}(X)$ which induces the identity on the Picard group. Given an automorphism φ of X, denote by φ^* its action on $H^2(X, \mathbb{Z})$. If Φ is a marking for X, we get a commutative diagram:

$$\begin{CD} \Lambda_{\mathrm{K3}} @>{\sigma}>> \Lambda_{\mathrm{K3}} \\ @A{\Phi}AA @AA{\Phi}A \\ H^2(X,\mathbb{Z}) @>>{\varphi^*}> H^2(X,\mathbb{Z}) \end{CD}$$

where σ is an isometry of the K3 lattice Λ_{K3} which maps the period line $\mathbb{C}\,\omega = \Phi(\mathbb{C}\,\omega_X)$ into itself and preserves the image of the nef cone. Conversely, given such a σ, by the global Torelli theorem, there exists a unique automorphism φ of X such that $\sigma = \varphi^*$. Hence, after identifying $H^2(X, \mathbb{Z})$ with the K3 lattice Λ_{K3}, we have

$$\mathrm{Aut}(X) = \{\sigma \in O(\Lambda_{\mathrm{K3}}) : \sigma(\mathbb{C}\,\omega_X) = \mathbb{C}\,\omega_X, \sigma \text{ preserves the ample cone of } X\}.$$

2.1. The transcendental lattice. If we denote by $S \subset \Lambda_{\mathrm{K3}}$ the Picard lattice of X and by $T := S^{\perp}$ its *transcendental lattice*, then we can apply the results of the previous section to construct automorphisms of a given X. Observe that $\omega_X \in T \otimes_{\mathbb{Z}} \mathbb{C}$.

EXAMPLE 2.1.1. Assume that S is isomorphic to U. Since S is unimodular, then also T is. Thus $T \cong U^2 \oplus E_8^2$, by Theorem 1.2.1. Let $\sigma_T = -\operatorname{id}$ and $\sigma_S = \operatorname{id}$. Since the discriminant groups of S and T are trivial, then the hypothesis of Proposition 1.6.1 is automatically satisfied, so that there exists an isometry σ of Λ_{K3} inducing both σ_S and σ_T. Moreover $\sigma(\omega_X) = -\omega_X$ and σ is the identity on the whole Picard lattice, so that in particular it preserves the nef cone. Whence there exists an isomorphism φ of X which induces σ in cohomology. It is possible to prove that the quotient surface $Y := X/\langle\varphi\rangle$ is smooth projective. In particular, since ω_X is not preserved by φ^*, then $H^{2,0}(Y) = (0)$. Moreover the Picard lattice of Y has rank 2. Hence by the classification of smooth algebraic surfaces Y is a rational surface. In the next chapter we will see that Y is a Hirzebruch surface $\mathbb{F}_4$.

2.2. Symplectic automorphisms. Given an automorphism φ of a K3 surface X it must preserve the period line. Hence we have

$$\varphi^*(\omega_X) = \zeta\,\omega_X,$$

for some complex number ζ. if $\zeta = 1$, the automorphism φ is *symplectic* and *non-symplectic* otherwise. Assume that φ is symplectic. Given an element $z \in T$ in the transcendental lattice we have $\varphi^*(z)\cdot\omega_X = z\cdot\varphi^*(\omega_X) = z\cdot\omega_X$, so that $\varphi^*(z) - z$ is orthogonal to the period ω_X. hence $\varphi^*(z) - z$ belongs to both the Picard and the transcendental lattices of X so that $\varphi^*(z) - z = 0$. Thus

$$\varphi^*|_T = \operatorname{id}.$$

On the other hand if φ is an automorphism which induces the identity on the transcendental lattice, then it is obviously symplectic as $\omega_X \in T \otimes \mathbb{C}$.

If we denote by $\mathrm{G}(X)$ the subgroup of $\operatorname{Aut}(X)$ whose elements are symplectic automorphisms, then we get an exact sequence

$$0 \longrightarrow \mathrm{G}(X) \longrightarrow \operatorname{Aut}(X) \longrightarrow \operatorname{Aut}(X)|_{\mathbb{C}\,\omega_X} \longrightarrow 0. \tag{2.2.1}$$

Mukai proved in [**Muk88**] that if $G(X)$ is finite then it is isomorphic to a subgroup of the Mathieu group M_{23}.

2.3. Nikulin involutions. An important example of symplectic automorphism is the case of involutions, that is $\varphi^2 = \operatorname{id}$. These are also called *Nikulin involutions*, after the work of Nikulin [**Nik79**]. In this case the only possible eigenvalues of φ^* are ± 1. We have already seen that φ^* restricts to the identity on the transcendental lattice T. This implies that the induced action of the discriminant lattice $\mathrm{d}(T)$ is the identity. Hence $\varphi^*|_S$ must induce the identity on $\mathrm{d}(S)$, where S is the Picard lattice. Observe that if (z_1, z_2) is a fixed point, in local coordinates, of a Nikulin involution φ, then $\varphi(z_1, z_2) = (-z_1, -z_2)$ since $\varphi^*(\omega_X) = \omega_X$, where $\omega_X = \alpha\, dz_1 \wedge dz_2$ in local coordinates. Thus any such fixed point is isolate. By applying the *holomorphic Lefschetz fixed point formula* [**EoMa**]:

$$\sum_{p\in \operatorname{Fix}(\varphi)} \frac{1}{\det(I - d\varphi_p)} = \sum_{q=0}^{2} (-1)^q \operatorname{Tr}(\varphi^*|_{H^{0,q}(X)})$$

we conclude that φ has exactly 8 fixed points, since the right hand side has just two summands equal to 1, while the left hand side has n summands equal to $1/4$, where n is the number of fixed points of φ. In particular the quotient surface $Y = X/\langle\varphi\rangle$ is singular exactly at the images of these points, where it has ordinary double points. A minimal resolution of singularities $Y' \to Y$ gives another K3 surface.

2.4. 2-elementary lattices. A lattice is 2-*elementary* if its discriminant group is isomorphic to a direct sum of copies of $\mathbb{Z}/2\mathbb{Z}$. Let X be a K3 surface with 2-*elementary* Picard lattice, and let $S \subset \Lambda_{\mathrm{K3}}$ be the image of the Picard lattice via a marking so that

$$\mathrm{d}(S) \cong (\mathbb{Z}/2\mathbb{Z})^r.$$

If σ_S is an involution of S then the corresponding σ_S^* acts as the identity on the discriminant group $d(S)$, since $\sigma_S^*(x) = \pm x = x$. Hence by Proposition 1.6.1 there is an isometry σ of the K3 lattice Λ_{K3} which induces σ_S on S and $\sigma_T = \mathrm{id}$ on T. Thus as soon as σ_S^* preserves the nef cone, it induces an automorphism of X. Since the eigenvalues of σ_S are ± 1, then we are just asking for the ample cone of X to have non-empty intersection with the eigenspace of σ_S corresponding to eigenvalue 1. In particular we have

$$\{\text{Nikulin involutions of } X\} = \{\sigma_S \in O(S) : \sigma_S(\mathrm{Nef}(X)) = \mathrm{Nef}(X)\}.$$

Example 2.4.1. As an explicit example one can consider the involution of the Fermat quartic surface $V(x_0^4 + x_1^4 + x_2^4 + x_3^4)$ given by $(x_0 : x_1 : x_2 : x_3) \mapsto (-x_0 : -x_1 : x_2 : x_3)$, a direct calculation shows that it is symplectic and its fixed points are $(0 : 0 : -\zeta : 1)$, $(0 : 0 : \zeta : 1)$, $(0 : 0 : -\zeta^3 : 1)$, $(0 : 0 : \zeta^3 : 1)$, $(-\zeta : 1 : 0 : 0)$, $(\zeta : 1 : 0 : 0)$, $(-\zeta^3 : 1 : 0 : 0)$, $(\zeta^3 : 1 : 0 : 0)$, where ζ is a primitive 8-th root of unity.

2.5. Non-symplectic automorphisms. Given an automorphism φ of a K3 surface X, it induces an isometry φ^* of the transcendental lattice T. This gives a homomorphism $\gamma : \mathrm{Aut}(X) \to O(T)$ whose kernel is the subgroup of symplectic automorphisms $\mathrm{G}(X)$, as we have already observed. Now we are interested in the image of the previous homomorphism.

Theorem 2.5.1. *Let X be a K3 surface with transcendental lattice T. The image of the homomorphism $\mathrm{Aut}(X) \to O(T)$ is a finite group. In particular $\mathrm{Aut}(X)$ is finite if and only if $G(X)$ is finite.*

Proof. Let σ be the image of an automorphism φ of X. Then σ preserves the two-dimensional complex vector space $\langle \omega_X, \bar{\omega}_X \rangle$ and its orthogonal. The restriction of the quadratic form to both spaces is definite (positive on the first and negative on the second). Hence the eigenvalues of σ have module 1. On the other hand σ is an isometry of an integer lattice T, so its eigenvalues are algebraic integers. Thus they are roots of unity. Since the degree of the characteristic polynomial of σ equals the rank of T, in particular it is bounded, then only a finite number of roots of unity can appear as eigenvalues of σ. Thus the representation of $\mathrm{Aut}(X)$ on $\mathbb{C}\,\omega_X$, given by (2.2.1), assumes only a finite number of roots of unity. Hence we get the statement. □

Given a non-symplectic automorphism φ of a K3 surface X we know that $\varphi^*(\omega_X) = \zeta\,\omega_X$ for some $\zeta \neq 1$. If φ has finite order p, then ζ must be a p-th root of unity, non necessarily primitive since some proper power of φ can be symplectic.

It is possible to prove [**MO98**] that the transcendental lattice T has the structure of free $\mathbb{Z}[\zeta]$-module induced by the multiplication $\zeta \cdot x := \varphi^*(x)$.

Example 2.5.2. Let X be a K3 surface whose Picard lattice S has rank 20, so that the transcendental lattice T has rank 2. If φ is a non-symplectic automorphism of X of finite order, then its action on the transcendental lattice T is represented by a 2×2 matrix with integer entries. Thus the eigenvalue ζ of $\sigma := \varphi^*$ relative to ω_X is a root of unity which lives in a degree 2 extension of $\mathbb{Q}$. Thus $\zeta \in \{-1, \varepsilon, \varepsilon^2, \pm i\}$, where ε is a primitive third root of unity. Assume $\zeta = \varepsilon$, and let $\{e, \sigma(e)\}$ be a basis of the transcendental lattice T. If $e^2 = 2n$, then, by using the fact that $\sigma^2 + \sigma + \mathrm{id} = 0$, we get $e \cdot \sigma(e) = \sigma(e) \cdot \sigma^2(e) = \sigma(e) \cdot (-e - \sigma(e)) = -e \cdot \sigma(e) - 2n$, so that $e \cdot \sigma(e) = -n$. One can reason in a similar way when $\zeta = i$ obtaining the Gram matrices

$$\begin{bmatrix} 2n & -n \\ -n & 2n \end{bmatrix} \qquad \begin{bmatrix} 2n & 0 \\ 0 & 2n \end{bmatrix}$$

of transcendental lattices which admit respectively a non-symplectic involution of order three and a non-symplectic involution of order four (acting on the period as the multiplication by i, that is φ^2 is still non-symplectic). As an example, consider the Fermat surface X. In Exercise **??** you showed that the lines of X span a lattice of rank 20. It is not hard to show that the discriminant group is $(\mathbb{Z}/8\mathbb{Z})^2$. Observe that X admits non-symplectic automorphisms of order four, like for example $(x_0 : x_1 : x_2 : x_3) \mapsto (i\, x_0 : x_1 : x_2 : x_3)$. Hence by our previous argument the transcendental lattice of X is diagonal with eigenvalues $2n$. In particular its determinant $4n^2 \mid 64$, so that $n \in \{1, 2, 4\}$. It is possible to show that $n = 4$, so that the transcendental lattice of X has Gram matrix:

$$\begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}.$$

We conclude by observing that since T has determinant 64, which is not divisible by 3, then by our previous observations X does not admit a non-symplectic automorphism of order three.

Non-symplectic automorphisms of order two have been extensively studied. In particular if $\varphi \in \mathrm{Aut}(X)$ is such an automorphism, then the quotient surface $Y := X/\langle\varphi\rangle$ is either an Enriques surface (if φ does not have fixed points) or a rational surface. In the next section we will analyze the case of Enriques surfaces in more detail (see also Example 2.1.1).

2.6. The Weyl group. An element e of a lattice S is a *root* if $e^2 = -2$. Given a root $e \in S$ define the *Picard-Lefschetz reflection* associated to e as the isometry $s_e : S \to S$ given by $x \mapsto x + (x \cdot e)\, e$. The *Weyl group* of the lattice S is:

$$\mathrm{W}(S) := \langle s_e : e \text{ is a root of } \Lambda_{\mathrm{K3}} \rangle.$$

If X is a K3 surface, then no element of the Weyl group of the Picard lattice S can be induced by an automorphism. Indeed $s_e(e) = -e$ and it is not difficult to show, as a consequence of the Riemann-Roch theorem, that either e or $-e$ has to be an effective class, but an automorphism can not map an effective class into its (non-effective) opposite. Also, if h is an ample class of X and e is effective, then $h \cdot e > 0$, so that $s_e(h) \cdot e < 0$, which implies $s_e(h)$ non-ample. The effect of applying a Picard-Lefschetz reflection with respect to a root e is to make a reflection with respect to the hyperplane $e^\perp$. This reflection moves the whole ample cone, by our previous observation. It can be proved that the closure of the ample cone, that is the nef cone $\mathrm{Nef}(X)$, is a *fundamental chamber* for the action of $\mathrm{W} := \mathrm{W}(\mathrm{Pic}(X))$ on the Picard lattice, meaning with this that $\mathrm{W} \cdot \mathrm{Nef}(X)$ defines a decomposition of the positive light cone $\{x \in \mathrm{Pic}(X) \otimes \mathbb{C} : x^2 > 0 \text{ and } x \cdot h > 0 \text{ with } h \text{ ample}\}$ into chambers which are congruent to $\mathrm{Nef}(X)$ and W acts freely and transitively on this set of chambers. On the other hand an isometry of the Picard lattice coming from an automorphism clearly preserves the nef cone. Hence if we denote by $O(\mathrm{Nef}(X))$ the isometries of $\mathrm{Pic}(X)$ which preserve the nef cone, we have a map

$$\mathrm{Aut}(X) \to O(\mathrm{Nef}(X)).$$

We want to show that this map has finite kernel and cokernel. The first is an immediate consequence of Theorem 2.5.1. To prove the second, observe that the set of $\sigma \in O(\mathrm{Nef}(X))$ which induce the identity on the discriminant lattice $\mathrm{d}(\mathrm{Pic}(X))$ has finite index in $O(\mathrm{Nef}(X))$. Each such σ admits a lifting to an isometry σ' of $H^2(X, \mathbb{Z})$, just choosing the identity on the transcendental lattice. By the Global Torelli Theorem, σ' is induced by an automorphism of X, since σ' preserves both the period and the nef cone of X. This proves what claimed.

2.7. Finite automorphisms groups. Observe that W is a normal subgroup of $O(\mathrm{Pic}(X))$. Moreover an element $\sigma \in O(\mathrm{Pic}(X))$ preserves the set of roots of $\mathrm{Pic}(X)$, so that it moves the nef cone into one of the chambers in the orbit of the action of the Weyl group on the nef cone. Thus there exists an element $s \in W$ such that $\sigma(\mathrm{Nef}(X)) = s(\mathrm{Nef}(X))$. In other words $O(\mathrm{Pic}(X))$ is a semidirect product of W with $O(\mathrm{Nef}(X))$, which gives

$$O(\mathrm{Pic}(X)) \doteq W \rtimes \mathrm{Aut}(X),$$

where the symbol $\doteq$ means isomorphism "up to a finite group". In particular we have the equivalence between the first two conditions of the following [**Dol08**, Corollary 5.1]

THEOREM 2.7.1. *Let X be a K3 surface. Then the following are equivalent:*
(i) $\mathrm{Aut}(X)$ *is finite;*
(ii) *the Weyl group W of* $\mathrm{Pic}(X)$ *has finite index in* $O(\mathrm{Pic}(X))$*;*
(iii) $\mathrm{Nef}(X)$ *is polyhedral with maximal faces orthogonal to smooth rational curves of X.*

A main ingredient for proving the equivalence $(i) \iff (iii)$ is to show that $\mathrm{Aut}(X)$ acts on the nef cone with a polyhedral fundamental domain. By using the previous theorem it is possible to classify all the Picard lattices of $K3$ surfaces which admit a finite automorphism group (see [**Nik79**, **Nik75**, **Nik79**]). The number n of these lattices for any Picard rank ρ_X is given in the following table (see [**Dol83**, Theorem 2.2.2]).

ρ_X	3	4	5 − 6	7	8	9	10	11 − 12	13 − 14	15 − 19	20
n	27	17	10	9	12	10	9	4	3	1	0

The great part of these lattices are 2-elementary, that is the discriminant group is a sum of copies of $\mathbb{Z}/2\mathbb{Z}$. This is related with the fact that a K3 surface X with that Picard lattice admits a non-symplectic involution.

EXAMPLE 2.7.2. Let X be a K3 surface whose Picard lattice is isometric to the lattice $S = U \oplus (-2)$. Such a K3 surface exists since Nikulin [**Nik79**] proved that any even hyperbolic lattice of rank ≤ 10 can be embedded in the K3 lattice. Thus by the global Torelli theorem S is isometric to the Picard lattice of any K3 surface whose period $\omega \in S^\perp \otimes_{\mathbb{Z}} \mathbb{C}$ is very general. Since $d(S^\perp) \cong d(S) \cong \mathbb{Z}/2\mathbb{Z}$, then the pair $(\mathrm{id}_S, -\mathrm{id}_{S^\perp})$ induces an isometry σ of the K3 lattice. By the global Torelli theorem $\sigma = \varphi^*$, where φ is a non-symplectic involution of X. This gives a double cover

$$\pi : X \to Y = X/\langle\varphi\rangle,$$

with Y smooth toric surface. Since $\pi^*(\mathrm{Pic}(Y))$ has finite index in $\mathrm{Pic}(X)$, the nef cone of X is pull-back of the nef cone of Y. The last cone is polyhedral with maximal faces orthogonal to classes of rational curves. Hence the same holds for $\mathrm{Nef}(X)$ so that $\mathrm{Aut}(X)$ is finite by Theorem 2.7.1.

3. Enriques surfaces

An *Enriques surface* is a smooth projective surface Y with $H^1(Y, \mathcal{O}_Y)$ trivial, $2K_Y \sim \mathcal{O}_Y$ and K_Y not linearly equivalent to zero.

3.1. Topological invariants. If Y is an Enriques surface, then from the exponential sequence and $h^1(\mathcal{O}_Y) = 0$ we deduce that both $H^1(X, \mathbb{Z})$ and $H^3(X, \mathbb{Z})$ have zero rank. Moreover the first group is trivial since it is always torsion-free. Now recall that $\mathrm{Tors}\, H_1(X, \mathbb{Z}) \cong \mathrm{Tors}\, H_2(X, \mathbb{Z}) \cong \mathrm{Tors}\, H^2(X, \mathbb{Z})$, so that it is enough to determine the first group. Since the class of K_X is non trivial but $2K_X \sim 0$, then $[K_X]$ is a 2-torsion element of $\mathrm{Pic}(X)$ which gives 2-torsion element of $H^2(X, \mathbb{Z})$ by the injectivity of the map τ in the exponential sequence of Y. By the previous isomorphisms between torsion groups we deduce that $H_1(X, \mathbb{Z})$ contains a 2-torsion element which in turn implies that $\pi_1(Y)$ contains such an element. Hence Y admits an unbranched double covering

$$\pi : X \to Y,$$

where X is a compact complex surface with $K_X \cong \pi^* K_Y \sim 0$ and $h^1(\mathcal{O}_X) = 0$. Thus X is a K3 surface. In particular $\pi_1(Y) \cong \mathbb{Z}/2\mathbb{Z}$ since X is simply connected. Hence

$$\mathrm{Tors}\, H^2(Y, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$$

is generated by the class of K_Y. By Noether formula $e(Y) = 12(\chi(\mathcal{O}_Y) + K_Y^2) = 12$, since K_Y is numerically trivial, $h^1(\mathcal{O}_Y) = 0$ and $h^2(\mathcal{O}_Y) = h^0(K_Y) = 0$, where the last equality is due to the fact that K_Y is not linearly equivalent to zero. Hence $H^2(Y, \mathbb{Z})/\mathrm{Tors}$ is a unimodular lattice of rank 10 which, by Poincaré duality and the universal coefficient theorem. Moreover $H^2(Y, \mathbb{Z}) \cong \mathrm{Pic}(Y)$, by the exponential sequence. Hence it is an even lattice by the adjunction formula and the fact that K_X is numerically trivial. Hence the signature of $H^2(Y, \mathbb{Z})$ is $(1, 9)$ by the Hodge index theorem. Thus by Milnor theorem 1.2.1 this lattice has to be $U \oplus E_8$. We summarize the previous observations in the following proposition.

Proposition 3.1.1. *Let Y be an Enriques surface. Then* $\mathrm{Pic}(Y) \cong \Lambda \oplus \mathbb{Z}/2\mathbb{Z}$, *where Λ is isomorphic to the rank 10 even unimodular lattice $U \oplus E_8$.*

3.2. The Enriques lattice. Let Y be an Enriques surface. The Picard lattice of the K3 surface X, in the double cover $\pi : X \to Y$, contains the pull-back $\pi^* \mathrm{Pic}(Y)$. This is the following 2-elementary lattice called the *Enriques lattice*:

$$\Lambda_{\mathrm{En}} := U(2) \oplus E_8(2).$$

By the global Torelli theorem the moduli space of Enriques surfaces is birational to the moduli space of pairs (X, σ), where X is a K3 surface such that $\mathrm{Pic}(X)$ contains a lattice isomorphic to Λ_{En} and σ is a non-symplectic involution whose induced homomorphism σ^* on $\mathrm{Pic}(X)$ is the identity on Λ_{En}.

3.3. Projective constructions. Let $Q = \mathbb{P}^1 \times \mathbb{P}^1$ be a smooth quadric surface of $\mathbb{P}^3$. Consider the involution τ of Q given by

$$((x_0 : x_1), (y_0 : y_1)) \mapsto ((x_0 : -x_1), (y_0 : -y_1)).$$

It has 4 fixed points p_1, p_2, p_3, p_4. Choose now an irreducible curve B of Q cut out by a quartic surface, that is B has class $(4, 4)$ in $\mathrm{Pic}(Q)$, which passes through the p_i's and is invariant with respect to τ. It is possible to show, by using the Riemann-Hurwitz formula, that the double cover of Y branched along B is a K3 surface X. Due to our choice of B the involution τ lifts to an involution τ' of X. If ν is the double cover automorphism of $X \to Q$, then $g = \nu \circ \tau'$ is an involution of X without fixed points. The quotient surface $Y = X/\langle g\rangle$ is an Enriques surface. We summarize the construction in the following diagram

$$\begin{array}{ccc} X & \xrightarrow{\tau'} & X \\ {\scriptstyle /g}\downarrow & & \downarrow{\scriptstyle /\nu} \\ Y & & \mathbb{P}^1 \times \mathbb{P}^1 \end{array}$$

Remark 3.3.1. The very general Enriques surface Y is double covered by a K3 surface X whose Picard lattice is isomorphic to $U(2) \oplus E_8(2)$. Since an element x of this lattice has square $x^2 \in 4\mathbb{Z}$, then X does not contain (-2)-curves, so that the same is true for Y.

Now, if Y is not very general then the Picard lattice of the K3 surface X which double covers Y can have rank > 10 so that X may contain a (-2)-curve C. The image Γ of C in Y is a (-2)-curve of Y. Observe that even if the Picard rank of X is bigger than 10, that of Y remains constant, since any Enriques surface has Picard lattice of rank 10. What happened is that the class of Γ, which in the very general case was not effective, now becomes effective. Hence deforming an Enriques surface one expects to change the shape of the cone of effective divisors without changing the lattice structure on the Picard lattice.

3.4. Automorphisms. We conclude the section by discussing finite automorphism groups of Enriques surfaces. If ψ is an automorphism of an Enriques surface Y and $\pi : X \to Y$ is the K3 double covering, then $\psi \circ \pi$ lifts to a covering automorphism $\varphi \in \mathrm{Aut}(X)$, since X is simply connected. This means that if σ is the involution of X which exchanges the two sheets of the covering π, then $\sigma \circ \varphi = \varphi \circ \sigma$. On the other hand, any automorphism φ of X which commutes with σ induces an automorphism of Y. Hence we have an isomorphism

$$\mathrm{Aut}(Y) \to \{\varphi \in \mathrm{Aut}(X) : \varphi \circ \sigma = \sigma \circ \varphi\}.$$

This representation of $\mathrm{Aut}(Y)$ into a subgroup of automorphisms of a K3 surface, allows one to use the global Torelli theorem to classify which Y admit a finite automorphism group. The complete result, found by Kondo, is contained in the following theorem.

THEOREM 3.4.1 ([**Kon86**]). *Let Y be an Enriques surface whose automorphism group is finite. Then the transcendental lattice T_X of the general K3 surface X which double covers Y belongs to the following list.*

type	T_X	$\mathrm{Aut}(Y)$
I	$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix}$	D_4
II	$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 8 \end{bmatrix}$	S_4
III	$\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$	$D_4 \ltimes (\mathbb{Z}/2\mathbb{Z})^4$
IV	$\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$	$N \ltimes (\mathbb{Z}/2\mathbb{Z})^4$
V	$\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$	$S_4 \ltimes \mathbb{Z}/2\mathbb{Z}$
VI	$\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$	S_5
VII	$\begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix}$	S_5

Exercises

EXERCISE 3.1. Let $L \subset \Lambda$ be an inclusion of non-degenerate even lattices. Show that $L^{\perp}$ is primitive in Λ and that $(L^{\perp})^{\perp} = L$ if and only if L is primitive in Λ.

EXERCISE 3.2. Let Λ_1 and Λ_2 be non-degenerate even lattices. Prove that $\mathrm{d}(\Lambda_1 \oplus \Lambda_2) = \mathrm{d}(\Lambda_1) \oplus \mathrm{d}(\Lambda_2)$.

EXERCISE 3.3. Let L be a non-degenerate even lattice and let H be a subgroup of its discriminant $\mathrm{d}(L)$ such that $q_L|_H \cong 0$. Show that $\Lambda := \{x \in L \otimes_{\mathbb{Z}} \mathbb{Q} : x \bmod L \in H\}$ is a lattice which contains L and such that $\Lambda/L \cong H$.

EXERCISE 3.4. Let L be the lattice $(-2)^{16}$ with basis $\{e_1, \dots, e_{16}\}$. Consider the set $\mathcal{K}$ of affine functions $(\mathbb{Z}/2\mathbb{Z})^{16} \to \mathbb{Z}/2\mathbb{Z}$. Find the discriminant group of the *Kummer lattice*:

$$\Lambda_{\mathrm{Km}} := \left\{ \frac{1}{2} \sum_i a(i) e_i : a \in \mathcal{K} \right\}.$$

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