Mini-course on K3 surfaces

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Contents

Preliminaries on algebraic surfaces

With the word *surface* we will always mean a smooth compact complex connected surface. Most of the time we will be interested in projective surfaces, even if sometimes we will deal with non-projective K3 surfaces, when studying the period domain of such surfaces.

1. Geometry

1.1. Divisors. By a *divisor* on a smooth compact complex variety X we mean a formal finite sum $D := \sum_i a_i C_i$, where the a_i are integers and the C_i are irreducible hypersurfaces of X. The *support* of D is the union $\cup_i C_i$. The divisor D is *effective* if all the $a_i \geq 0$. We say that the divisor is *prime* if it contains just one summand with coefficient 1. In this case we will often identify the divisor with the hypersurface itself. We recall that given a rational function f on X its *principal divisor* is

$$\operatorname{div}(f) := \sum_{C \subset X} \nu_C(f) C,$$

where the sum runs over all the hypersurfaces in X and $\nu_C(f) \in \mathbb{Z}$ is the order of zero/pole of f at C. Two divisors D and D' of X are *linearly equivalent* if $D - D' = \operatorname{div}(f)$ for some rational function f on X. In this case we write $D \sim D'$ to denote that D is linearly equivalent to D'. The set of divisors of X form a free abelian group denoted by $\operatorname{Div}(X)$. It contains the subgroup $\operatorname{PDiv}(X)$ of principal divisors. The quotient

$$\operatorname{Pic}(X) := \operatorname{Div}(X) / \operatorname{PDiv}(X)$$

is the *Picard group* of X. Elements of the Picard group will be called *classes*.

EXAMPLE 1.1.1. As easy examples one can keep in mind $\operatorname{Pic}(\mathbb{P}^n) = \mathbb{Z}[H]$, where H is a hyperplane, and $\operatorname{Pic}(\mathbb{P}^1 \times \mathbb{P}^1) = \mathbb{Z}[F_1] \oplus \mathbb{Z}[F_2]$, where each F_i is a fiber of the i-th projection $\pi_i : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$.

1.2. Intersection of divisors. Given a divisor D on a surface X there exists an open covering $\{U_i\}$ of X such that the restriction of D to each U_i is a principal divisor $\operatorname{div}(f_i)$. Given an irreducible curve C of X, not contained in the support of D, we define the *restriction* $D|_C$ to be the divisor of C locally defined by $\operatorname{div}(f_i|C)$ on the open subset $U_i \cap C$ of C. Given a curve C and a divisor D on a surface X their *intersection* is the number:

$$D \cdot C := \deg(D|_C),$$

where the right hand side is the degree of a divisor on a curve. Observe that from the previous definition we immediately have $D \cdot C = D' \cdot C$ if D is linearly equivalent to D' and the support of D' does not contain C. We use this property to define the intersection $D \cdot C$ without restrictions on D: if the support of D contains C, then we choose a $D' = D + \mathrm{div}(f)$, where f is a rational function such that $-\nu_C(f)$ is equal to the multiplicity of D at C, and define $D \cdot C := D' \cdot C$. The intersection number of two divisors is defined as $D \cdot \sum_i a_i C_i := \sum_i a_i D \cdot C_i$. It is possible to prove that $A \cdot B = B \cdot A$ for any pair of divisors A and B of X. Since $A' \cdot B' = A \cdot B$ if $A' \sim A$ and $B' \sim B$, then the intersection is well defined on the Picard group of X, that is it induces a bilinear map

$$\mathrm{Pic}(X) \times \mathrm{Pic}(X) \to \mathbb{Z}.$$

The Picard group, modulo torsion, equipped with the quadratic form defined by the intersection pairing is called the *Picard lattice* of X.

EXAMPLE 1.2.1. The surface $X = \mathbb{P}^1 \times \mathbb{P}^1$ has a Picard group of rank 2. The two generators $[F_1]$ and $[F_2]$ have intersections $F_i \cdot F_j = \delta_{ij}$. Hence the Picard lattice of X is represented by the Gram matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

The quadratic form on $\mathbb{Z}^2$ represented by the above matrix is denoted by U.

1.3. The canonical class. Let X be a surface and let ω be a meromorphic 2-form on X. If U is an open affine subset of X with coordinates z_1, z_2, then

$$\omega|_U = f_U\, dz_1 \wedge dz_2$$

were f_U is a meromorphic function on U. If we consider an open affine covering $\{U_i\}$ of X and let $\omega|_{U_i} = f_{U_i} dz_1^i \wedge dz_2^i$, then the collection of principal divisors $\mathrm{div}(f_{U_i})$ defines a divisor of X called a *canonical divisor* of X and denoted by K_X. Now consider two meromorphic forms ω and ω' of X. If we write these forms on two affine open subsets U and V of X, we get $\omega|_U = \alpha_U\, dz_1 \wedge dz_2$ and $\omega'|_U = \alpha'_U\, dz_1 \wedge dz_2$ and similarly on V. Observe that $\alpha_V = J\,\alpha_U$ and $\alpha'_V = J\,\alpha_U$, where J is the Jacobian of the coordinate change from U to V. In particular the quotient α'/α does not change, so that $\mathrm{div}(\alpha'/\alpha)$ is a principal divisor. Hence the class of K_X in $\mathrm{Pic}(X)$ is independent on the choice of the meromorphic form.

EXAMPLE 1.3.1. We can cover $\mathbb{P}^1$ with two affine coordinate charts U_0 and U_1. The meromorphic form dz_0 on U_0 glues on $U_0 \cap U_1$ with the form $-1/z_1^2\, dz_1$ of U_1 since $z_1 = 1/z_0$ on $U_0 \cap U_1$. Hence a canonical divisor of $\mathbb{P}^1$ is $K_{\mathbb{P}^1} = -2p$, where p is the zero locus of z_1 in U_1. Similarly one can prove that $K_{\mathbb{P}^n} = -(n+1)H$, where H is a hyperplane of $\mathbb{P}^n$.

1.4. Adjunction formula. If C is a smooth curve on a surface X a canonical divisor for C can be obtained by means of the *adjunction formula*:

$$K_C = (K_X + C)|_C.$$

Since C is a curve, the degree of a canonical divisor is $2g(C) - 2$, where $g(C)$ is the topological genus of C. Thus we have

$$2g(C) - 2 = \deg(K_C) = (K_X + C) \cdot C.$$

It follows that the right hand side intersection number is always even.

EXAMPLE 1.4.1. If F_i is the fiber of the i-th projection $\pi_i : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$, then F_i is a smooth rational curve with $F_i^2 = 0$. Hence $K_X \cdot F_i = -2$. Let $X = \mathbb{P}^1 \times \mathbb{P}^1$. If $[K_X] = a[F_1] + b[F_2]$, then by using the previous observation and our knowledge of the intersections between the F_i's, we get $[K_X] = -2[F_1] - 2[F_2]$.

EXAMPLE 1.4.2. If $K_X \sim 0$ and C is a smooth curve, then $C^2 = 2g(C) - 2$. In particular $C^2 = -2$ if C is rational and $C^2 = 0$ if C is elliptic.

1.5. Riemann-Roch formula. Given a divisor D on a surface X we can form the sheaf $\mathcal{O}_X(D)$, locally defined, on an open subset U of X, as the complex vector space of rational functions f of U such that $\operatorname{div}(f) + D$ is an effective divisor of U. The dimension of the cohomology groups $H^i(X, \mathcal{O}_X(D))$ are denoted by $h^i(\mathcal{O}_X(D))$ and the Euler characteristic by $\chi(\mathcal{O}_X(D)) := h^0(\mathcal{O}_X(D)) - h^1(\mathcal{O}_X(D)) + h^2(\mathcal{O}_X(D))$. The *Riemann-Roch formula* is

$$\chi(\mathcal{O}_X(D)) = \frac{1}{2}(D - K_X) \cdot D + \frac{1}{12}(K_X^2 + e(X)),$$

where $e(X)$ denotes the topological Euler characteristic of X, that is the alternating sum of the ranks of the singular homology groups of X. Observe that $h^0(\mathcal{O}_X) = 1$ since X is a complete variety. Moreover $h^2(\mathcal{O}_X) = h^0(\mathcal{O}_X(K_X))$ due to an important theorem of Serre, called *Serre's duality theorem*. The Euler characteristic of the sheaf of regular function $\mathcal{O}_X$ is related to the Euler characteristic of the surface X by the following *Noether formula*:

$$\chi(\mathcal{O}_X) = \frac{1}{12}(K_X^2 + e(X)),$$

which is easily obtained by putting $D = 0$ in the Riemann-Roch formula.

EXAMPLE 1.5.1. The Euler characteristic of the projective plane is $e(\mathbb{P}^2) = 3$, since it has cohomology only in even dimension and all these groups are isomorphic to $\mathbb{Z}$. Since $K_{\mathbb{P}^2} = -3H$, then $K_{\mathbb{P}^2}^2 = 9$, so that we have $\chi(\mathcal{O}_{\mathbb{P}^2}) = 1$. Hence the Riemann-Roch formula for the projective plane gives:

$$\chi(\mathcal{O}_{\mathbb{P}^2}(dH)) = \frac{1}{2}(dH + 3H) \cdot dH + 1 = \frac{1}{2}(d+3)d + 1$$

since $H^2 = 1$. Observe that the last formula gives exactly the dimension of the degree d part of the graded polynomial ring $\mathbb{C}[x_0, x_1, x_2]$, when $d \geq 0$. Hence $\chi(\mathcal{O}_{\mathbb{P}^2}(dH)) = h^0(\mathcal{O}_{\mathbb{P}^2}(dH))$. Indeed it is possible to prove that both the higher cohomology groups of the sheaf $\mathcal{O}_{\mathbb{P}^2}(dH)$ vanish for $d > 0$.

2. Topology

2.1. Poincaré duality. Given a compact complex connected surface X we will denote by $H^i(X, \mathbb{Z})$ its i-th singular cohomology group and by $H_i(X, \mathbb{Z})$ the i-th singular homology group . All of them are finitely generated abelian groups. Recall the content of *Poincaré duality* for surfaces [**GH94**, Pag. 53]: for each $i \geq 0$ there is a natural isomorphism

$$H_i(X, \mathbb{Z}) \to H^{4-i}(X, \mathbb{Z}).$$

By the definition of singular homology and cohomology there is a natural map $H^i(X, \mathbb{Z}) \to H_i(X, \mathbb{Z})^*$ coming from the corresponding map at the level of cochain. The *universal coefficient theorem* asserts that the following sequence of abelian group is exact:

$$0 \longrightarrow \operatorname{Ext}^1(H_{i-1}(X, \mathbb{Z}), \mathbb{Z}) \longrightarrow H^i(X, \mathbb{Z}) \longrightarrow H_i(X, \mathbb{Z}) \longrightarrow 0.$$

In particular $H^i(X,\mathbb{Z})/\operatorname{Tors} \cong H_i(X,\mathbb{Z})^*$ and $\operatorname{Tors} H^i(X,\mathbb{Z}) \cong \operatorname{Tors} H_{i-1}(X,\mathbb{Z})$, where Tors denotes the torsion part of an abelian group. As a consequence there is a perfect bilinear symmetric pairing, called the *intersection pairing*:

$$H^2(X,\mathbb{Z})/\operatorname{Tors} \times H^2(X,\mathbb{Z})/\operatorname{Tors} \to \mathbb{Z}.$$

Here the word perfect means that the matrix defining the pairing has determinant ± 1. Other two easy consequences of Poincaré duality and the universal coefficient theorem are

$$\operatorname{Tors} H_2(X,\mathbb{Z}) \cong \operatorname{Tors} H_1(X,\mathbb{Z}) \qquad \operatorname{Tors} H_3(X,\mathbb{Z}) = (0).$$

It is worth noticing that $H^i(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R} \cong H^i(X,\mathbb{Z})_{\mathrm{DR}}$, where the right hand side is the De Rham cohomology of X, that is the real vector space of closed forms modulo exact forms. The intersection pairing on $H^i(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$ can be thus expressed at the level of forms as:

$$\omega_1 \cdot \omega_2 := \int_X \omega_1 \wedge \omega_2.$$

Example 2.1.1. Let $X := \mathbb{C}^2/\Gamma$ be a complex torus obtained by taking the quotient of $\mathbb{C}^2$ with a maximal subgroup $\Gamma \cong \mathbb{Z}^4$. The group Γ acts by translation, so if dx_i is a 1-form on $\mathbb{C}^2$, then it descends to a 1-form on X since it is invariant with respect to the action of Γ, that is $d(x_i + a) = dx_i$ for any $a \in \Gamma$. Moreover one can prove that it is a closed form and that $\{dx_i \wedge dx_j : 1 \le i < j \le 4\}$ gives a basis of $H^2(X,\mathbb{Z})$. Observe that for example $dx_1 \wedge dx_2 \wedge dx_i \wedge dx_j = 0$ whenever dx_i or dx_j is linearly dependent with dx_1, dx_2. Thus the Gram matrix with respect to the given basis is the block matrix:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

The standard notation for this type of lattice is $U \oplus U \oplus U$, where U is the rank two lattice whose Gram matrix is the up-left two by two submatrix of the previous matrix.

2.2. The topological index theorem. The intersection form on $H^2(X,\mathbb{Z})/\operatorname{Tors}$ defines a quadratic form $q : H^2(X,\mathbb{Z})/\operatorname{Tors} \to \mathbb{Z}$ by $q(x) := x \cdot x$. Taking tensor product with the real numbers we obtain a real vector space $H^2(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$ equipped with a non-degenerate quadratic form. Its signature is a topological invariant of X. If we denote by K_X the canonical divisor of X, with K_X^2 its self-intersection and by $e(X) := \sum_{i=0}^4 (-1)^i \operatorname{rk} H_i(X,\mathbb{Z})$ the Euler characteristic of X, then we have the following.

Theorem 2.2.1. *Let b^+ and b^- be respectively the number of positive and negative eigenvalues of the quadratic form q on the real vector space $H^2(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$. Then*

$$b^+ - b^- = \frac{1}{3}(K_X^2 - 2e(X)).$$

Example 2.2.2. Let X be a smooth cubic surface of $\mathbb{P}^3$. Since X is birational to the projective plane, then it is possible to show that $h^1(\mathcal{O}_X) = h^2(\mathcal{O}_X) = 0$, so that $\chi(\mathcal{O}_X) = 1$. By adjunction formula we have $K_X = -H|_X$, where H is a plane. Hence $K_X^2 = 3$. By using the previous facts and Noether's formula we get $e(X) = 9$, which gives $b^+ - b^- = -5$. Still by the Euler characteristic of X we deduce that $H^2(X, \mathbb{Z})$ has rank 7, so that $b^+ = 1$ and $b^- = 6$.

2.3. The exponential sequence. Let X be a smooth compact complex variety and denote by $\mathbb{Z}_X$ the sheaf of locally constant integer functions on X. The exponential sequence of X is

$$0 \longrightarrow \mathbb{Z}_X \longrightarrow \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \longrightarrow 0,$$

where $\exp(f) := e^{2\pi i f}$. We briefly recall that the singular cohomology groups $H^i(X, \mathbb{Z})$ and the sheaf cohomology groups $H^i(X, \mathbb{Z}_X)$ are isomorphic and we will often identify them in the future. Moreover the Picard group of X is isomorphic to the cohomology group $H^1(X, \mathcal{O}_X^*)$. Hence taking cohomology we obain the long exact sequence

$$\begin{array}{ccccccccc} H^1(X,\mathbb{Z}_X) & \longrightarrow & H^1(X,\mathcal{O}_X) & \xrightarrow{\exp} & H^1(X,\mathcal{O}_X^*) & \xrightarrow{\tau} & H^2(X,\mathbb{Z}_X) & \longrightarrow & H^2(X,\mathcal{O}_X) \\ \| & & & & \| & & \| & & \\ H^1(X,\mathbb{Z}) & & & & \mathrm{Pic}(X) & & H^2(X,\mathbb{Z}). & & \end{array}$$

The image of the above exponential map is isomorphic to the quotient of $H^1(X, \mathcal{O}_X)$ by the image of the subgroup $H^1(X, \mathbb{Z})$. It is possible to see that this quotient is an abelian variety, that is a projective complex torus, also denoted by $\mathrm{Pic}(X)^0$, which when X is a curve is exactly the group of degree zero divisors modulo linear equivalence. The image of τ is the *Nerón-Severi group* of X, denoted $\mathrm{NS}(X)$. Hence we can summarize the previous observations in the following exact sequence:

$$0 \longrightarrow \mathrm{Pic}(X)^0 \longrightarrow \mathrm{Pic}(X) \xrightarrow{\tau} \mathrm{NS}(X) \longrightarrow 0.$$

It is important to remark that the homomorphism τ is an isometry with respect to the two quadratic forms defined in $\mathrm{Pic}(X)$ and $H^2(X, \mathbb{Z}) \cong H^2(X, \mathbb{Z})$, that is

$$\tau([D_1]) \cdot \tau([D_2]) = [D_1] \cdot [D_2].$$

2.4. The Lefschetz theorem on hyperplane sections. Let X be a smooth algebraic complex subvariety of dimension n of $\mathbb{P}^N$. Let H be a hyperplane and let $Y = X \cap H$. Then the inclusion map $Y \to X$ induces isomorphisms

$$H_i(Y, \mathbb{Z}) \to H_i(X, \mathbb{Z})$$

for any $i < n - 1$ and is surjective for $i = n - 1$. A similar statement holds for the induced homomorphism

$$\pi_1(Y) \to \pi_1(X).$$

It is an isomorphism when $n \geq 3$ and is surjective when $n = 2$ (see [**EoMa**]).

Exercises

Exercise 2.1. Calculate a basis of the Picard group of $\mathbb{P}^a \times \mathbb{P}^b$.

Exercise 2.2. Let X be a smooth cubic surface of $\mathbb{P}^3$ which contains a line L. Calculate L^2.

Exercise 2.3. Let $X = \mathbb{C}^2/\Gamma$ be a complex torus. Prove that $K_X \sim 0$.

Exercise 2.4. Let $X = \mathbb{P}^1 \times \mathbb{P}^1$ and let C be a smooth curve of X whose class in $\mathrm{Pic}(X)$ is $a[F_1] + b[F_2]$. Find the genus of C and $\chi(\mathcal{O}_X(C))$.

Exercise 2.5. Let X be a smooth projective surface with $h^1(\mathcal{O}_X) = h^2(\mathcal{O}_X) = 0$. Prove that $\mathrm{Pic}(X)/\,\mathrm{Tors}$ is unimodular. Moreover, if $n\,K_X \sim 0$ for some positive integer n, show that the previous lattice has signature $(1, 9)$.

The period domain

3. Topological properties

3.1. K3 surfaces. A *K3 surface* is a smooth complex compact surface X which satisfies the following:

$$H^0(X, \mathcal{O}_X(K_X)) = \mathbb{C}\,\omega_X \qquad H^1(X, \mathcal{O}_X) = (0). \tag{3.1.1}$$

The first condition tells us that, modulo scalar multiplication, X admits a unique holomorphic 2-form ω_X, while the second condition is equivalent to ask for the vanishing of the first Betti number of X. As an example of K3 surface consider a smooth quartic surface $X \subset \mathbb{P}^3$, like the Fermat surface defined by the equation

$$x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0.$$

By adjunction formula the fact that $X \sim 4H$ and the fact that $-K_{\mathbb{P}^3} = -4H$, where H is a plane, we deduce that the canonical class of X is trivial. Hence the first condition in (3.1.1) is satisfied. It is possible to prove that also the second condition is satisfied, by using the Lefschetz theorem on hyperplane sections, which gives the vanishing of $H^1(X, \mathbb{C})$, and the Hodge decomposition that we will introduce later in this chapter. By an iterated application of the previous argument, one can prove that a smooth complete intersection of a quadric and a cubic in $\mathbb{P}^4$ is again a K3 surface. The same holds for the complete intersection of three quadrics in $\mathbb{P}^5$.

Aim of this mini-course is to introduce the basic theory of K3 surfaces which from many perspectives represent a 2-dimensional generalization of elliptic curves.

3.2. Singular cohomology I. Let X be a K3 surface. Then $h^1(\mathcal{O}_X) = 0$ by definition and $h^2(\mathcal{O}_X) = h^0(\mathcal{O}_X(K_X)) = h^0(\mathcal{O}_X) = 1$ by Serre's duality. So the Euler characteristic of the structure sheaf $\mathcal{O}_X$ is 2. Hence by Noether's formula and the fact that K_X is trivial we get

$$e(X) = 12\,(\chi(\mathcal{O}_X) + K_X^2) = 24.$$

Since $h^1(\mathcal{O}_X) = 0$, then by the exponential sequence the rank of $H^1(X, \mathbb{Z})$ is zero. Hence the same is true for $H_1(X, \mathbb{Z})$, so that by Poincaré duality also $H^3(X)$ has zero rank. Since X is connected $H^0(X) \cong \mathbb{Z}$ and $H^4(X) \cong \mathbb{Z}$ being X orientable. Thus by our previous calculation of the Euler characteristic of X we deduce that $H^2(X, \mathbb{Z})$, or equivalently $H^2(X, \mathbb{Z})$, has rank 22.

If C is a smooth curve on X, and K_C is the canonical divisor of C, by adjunction formula

$$2g(C) - 2 = \deg(K_C) = (K_X + C) \cdot C = C^2,$$

where $g(C)$ is the topological genus of C. In particular C^2 is an even number. Recall that the curve C has a representative class $[C]$ in $\mathrm{Pic}(X)$ and a class $\tau([C])$ in $H^2(X, \mathbb{Z})$, defined by means of the exponential sequence. Thus we have just shown that all the elements of the Nerón-Severi group of X have even square. It is possible to extend this observation to the whole cohomology group, that is x^2 is an even number for any $x \in H^2(X, \mathbb{Z})$ (see [**BHPVdV04**]).

3.3. The fundamental group. The proof that any K3 surface is simply connected is not easy since it makes use of the full knowledge of the period domain. To sketch the idea, the proof is in two steps. First of all one proves that all K3 surfaces are diffeomorphic. This result depends on the fact that the period domain of K3 surfaces is connected (see Theorems 5.4.1 and 5.5.1) and the fact that a holomorphic family of complex manifolds is a trivial family from the differential point of view [**BHPVdV04**]. In particular it is enough to show that a smooth quartic surface X of $\mathbb{P}^3$ is simply connected.

PROPOSITION 3.3.1. *Any smooth quartic surface of $\mathbb{P}^3$ is simply connected.*

PROOF. Consider the degree four Veronese embedding $\nu : \mathbb{P}^3 \to \mathbb{P}^{34}$ and observe that it maps quartic surfaces of $\mathbb{P}^3$ to hyperplane sections of $\nu(\mathbb{P}^3)$ so that $X \cong \nu(\mathbb{P}^3) \cap H$ for some hyperplane H of $\mathbb{P}^{34}$. Then one applies the Lefschetz theorem on hyperplane sections to get

$$\pi_1(X) \cong \pi_1(\nu(\mathbb{P}^3) \cap H) \cong \pi_1(\nu(\mathbb{P}^3)) \cong \pi_1(\mathbb{P}^3),$$

showing that X is simply connected. □

REMARK 3.3.2. If X is a K3 surface then, by the exponential sequence and $h^1(\mathcal{O}_X) = 0$, we know that $H^1(X, \mathbb{Z})$ has rank zero as already observed before. By the universal coefficient theorem also $H_1(X, \mathbb{Z})$ has rank zero. Observe that this argument is not enough to conclude that X is simply connected. Indeed consider the Godeaux surface Y, defined as the quotient of the Fermat quintic S:

$$x_0^5 + x_1^5 + x_2^5 + x_3^5 = 0$$

with respect to the action $x_i \mapsto \varepsilon^i x_i$, where ε is a 5-th root of the unity. Since the action has no fixed points, then Y is a smooth surface. Moreover S is simply connected by Lefschetz theorem on hyperplane sections. Hence it is the universal covering space of Y so that $\pi_1(Y) \cong \mathbb{Z}/5\mathbb{Z}$ and $h^1(\mathcal{O}_Y) = 0$ due to the Hodge decomposition (4.3.1) of $H^1(X, \mathbb{C})$.

4. Hodge theory

4.1. Exterior forms. Let V be a complex vector space with basis $\{v_1, \dots, v_n\}$. The space of (p, q)-forms on V is the complex vector space $V^{p,q}$ generated by the symbols

$$v_{i_1} \wedge \cdots \wedge v_{i_p} \wedge \bar{v}_{j_1} \wedge \cdots \wedge \bar{v}_{j_q},$$

where $v \wedge w = -w \wedge v$ and the indices i_k and j_s vary over all the possible subsets of $\{1, \dots, n\}$ of cardinalities p and q respectively. The symbol $\wedge V$ denotes the *exterior algebra* of V, meaning with this the vector space

$$\wedge V := \bigoplus_{p+q=0}^{2n} V^{p,q}$$

together with the antisymmetric product $(w, w') \mapsto w \wedge w'$.

EXAMPLE 4.1.1. The exterior algebra of $\mathbb{C}$ is $\wedge\mathbb{C} = \mathbb{C}^{0,0} \oplus \mathbb{C}^{1,0} \oplus \mathbb{C}^{0,1} \oplus \mathbb{C}^{1,1}$ where for example $\mathbb{C}^{1,0} = \langle v \rangle$ and $\mathbb{C}^{1,1} = \langle v \wedge \bar{v} \rangle$. Observe that $\mathbb{C}^{2,0} = \langle 0 \rangle$, since $v \wedge v = 0$ by antisymmetry.

4.2. Dolbeault cohomology. Given a smooth compact complex surface X with cotangent bundle $\mathcal{E}_X$, we define its *exterior bundle* $\wedge\mathcal{E}_X$ to be the vector bundle whose fibers are the exterior algebras $\wedge{\mathcal{E}_X}_p$, for $p \in X$. Its transition functions on the intersection $U_i \cap U_j$ of two open subsets of a trivializing covering of X, are the matrices $\wedge g_{ij}$, where g_{ij} are the transition matrices of the bundle Ω_X. Now, since X has dimension 2, then

$$\wedge\mathcal{E}_X = \bigoplus_{p+q=0}^{4} \mathcal{E}_X^{p,q},$$

moreover the right hand side summands vanish whenever $p > 2$ or $q > 2$. An interesting property of the exterior bundle is that if we have a holomorphic map $f : X \to Y$ of compact complex varieties, then the pull-back $f^* : \wedge\mathcal{E}_Y \to \wedge\mathcal{E}_X$ maps each $\mathcal{E}_Y^{p,q}$ into the corresponding $\mathcal{E}_X^{p,q}$. This property, together with the existence of a linear differential operator

$$\bar{\partial} : \Gamma(X, \mathcal{E}_X^{p,q}) \to \Gamma(X, \mathcal{E}_X^{p,q+1})$$

such that $\bar{\partial} \circ \bar{\partial} = 0$, gives a cohomology theory for compact complex varieties. This is the *Dolbeault cohomology* whose groups are denoted by $H^{p,q}(X)$ and their dimensions by $h^{p,q}(X)$.

Remark 4.2.1. Denote by Ω_X^p the sheaf of *holomorphic p-forms*, that is the sheaf of forms which locally can be written as $\alpha\, dz_{i_1} \wedge \cdots \wedge dz_{i_k}$, with α holomorphic. The sheaf admits the following *acyclic resolution*:

$$\Omega_X^p \longrightarrow \mathcal{E}_X^{p,0} \xrightarrow{\bar{\partial}} \mathcal{E}_X^{p,1} \xrightarrow{\bar{\partial}} \cdots$$

meaning with this that the sequence is exact and the higher cohomology ($i > 0$) of all the $\mathcal{E}_X^{p,k}$ vanishes. The exactness of the sequence is due to Poincaré Lemma for the operator $\bar{\partial}$. Hence by considering the spectral sequence of the double complex $\check{\mathcal{C}}^i(\mathcal{E}_X^{p,j})$, given by Čech cocycles of the sheaf $\Omega_X^{p,j}$, one proves that

$$H^q(X, \Omega_X^p) \cong H^{p,q}(X).$$

4.3. Hodge decomposition. Let V be a finitely generated free abelian group. a *Hodge structure* of level n, with $n \in \mathbb{Z}$, on $V \otimes_{\mathbb{Z}} \mathbb{C}$ is a direct sum decomposition

$$V \otimes_{\mathbb{Z}} \mathbb{C} = \bigoplus_{p+q=n} V^{p,q}$$

such that $\overline{V^{p,q}} = V^{q,p}$. Here the overline means the complex conjugation. Denote by $b_i(X)$ the i-th Betti number of X, that is the rank of the singular homology group $H_i(X, \mathbb{Z})$. In case X is a smooth projective variety, or just smooth Kähler variety [**EoMb**], the n-th singular cohomology group of X admits the following Hodge structure of level n:

$$H^n(X, \mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}(X). \tag{4.3.1}$$

In particular each odd Betti number $b_{2k+1}(X)$ is an even number. Observe that by definition a K3 surface is not necessarily projective, but it is always Kähler, as shown in [**Siu83**]. Hence the cohomology of any K3 surface admits a Hodge structure.

We conclude the section by applying the result of the previous proposition and Serre's duality, to the description of the *Hodge diamond* of a K3 surface X. This is a picture containing all the dimensions of the spaces $h^{p,q}(X)$.

$$\begin{array}{ccccc} & & 1 & & \\ & h^{1,0}(X) & & h^{0,1}(X) & \\ h^{2,0}(X) & & h^{1,1}(X) & & h^{0,2}(X) \\ & h^{2,1}(X) & & h^{1,2}(X) & \\ & & 1 & & \end{array} \qquad \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

FIGURE 1. The Hodge diamond of a K3 surface

4.4. Singular cohomology II. As a consequence of the previous proposition we have the following [**BHPVdV04**].

PROPOSITION 4.4.1. *Let X be a K3 surface. Then the groups $H^1(X,\mathbb{Z})$ and $H^3(X,\mathbb{Z})$ are trivial. Moreover $H^2(X,\mathbb{Z})$ is a free $\mathbb{Z}$-module of rank 22 which, endowed with the quadratic form given by the cup product, is an even lattice of signature* $(3,19)$.

PROOF. By the exponential sequence we already know that the first and third Betti numbers of X are zero. Hence $H^1(X,\mathbb{Z}) = (0)$ and

$$\operatorname{Tors} H^2(X,\mathbb{Z}) \cong \operatorname{Tors} H_2(X,\mathbb{Z}) \cong \operatorname{Tors} H_1(X,\mathbb{Z}) \cong \operatorname{Tors} H^3(X,\mathbb{Z})$$

by Poincaré duality and the universal coefficient theorem. Hence it is enough to show that $H_1(X,\mathbb{Z})$ has no torsion (and thus it is trivial). Assume the countrary, then $\pi_1(X)$ would contain a torsion element. This is equivalent to say that X admits a degree $n > 1$ finite unbranched cover $\pi : Y \to X$, where Y is a compact complex surface. Now $e(Y) = n \cdot e(X) = 24\,n$ and $K_Y = \pi^* K_X \sim 0$ so that $h^2(\mathcal{O}_Y) = 1$. Hence by Noether formula we get

$$2 - h^1(\mathcal{O}_Y) = \chi(\mathcal{O}_Y) = \frac{1}{12}(K_Y^2 + e(Y)) = 2\,n,$$

which gives $n = 1$, a contradiction. To conclude the proof, let b^+ and b^- be, respectively, the number of positive and negative eigenvalues of the quadratic form defined by the intersection form on $H^2(X,\mathbb{Z})$. By the topological index theorem and our calculation of the Euler characteristic of X we get

$$b^+ - b^- = \frac{1}{3}(K_X^2 - 2e(X)) = -16.$$

Since $H^2(X,\mathbb{Z})$ has rank 22, then the signature of its quadratic form is $(3,19)$. We have already seen that it is an even lattice. □

REMARK 4.4.2. Observe that even if the argument adopted in the previous proposition shows that $\pi_1(X)_{\mathrm{ab}} \cong H_1(X,\mathbb{Z}) = (0)$, this is not enough to conclude that X is simply connected. There are examples of topological spaces with trivial homology and non-trivial fundamental group, like the Poincaré Homology 3-sphere (`http://goo.gl/sV1Ds`).

4.5. Lattice structure in cohomology. Given a K3 surface X we can consider the Hodge decomposition of its second cohomology group:

$$\begin{array}{ccccccc} H^2(X,\mathbb{C}) & = & H^{2,0}(X) & \oplus & H^{1,1}(X) & \oplus & H^{0,2}(X) \\ & & \parallel & & & & \parallel \\ & & \mathbb{C}\,\omega_X & & & & \mathbb{C}\bar{\omega}_X, \end{array}$$

where the two vertical isomorphisms are given at the end of the subsection 4.2 of Chaper 1. Observe that $H^2(X,\mathbb{C})$ is equipped with a quadratic form coming from the cup product defined on singular cohomology of X. This product can be written in terms of differential forms as (see Subsection 2.1 of the Preliminaries):

$$(\omega_1,\omega_2) \mapsto \omega_1 \cdot \omega_2 := \int_X \omega_1 \wedge \omega_2, \tag{4.5.1}$$

where ω_1 and ω_2 are closed 2-forms on X. In this way, if z_1 and z_2 are local coordinates on X, then a local expression of the holomorphic 2-form ω_X is $\alpha\, dz_1 \wedge dz_2$, with α holomorphic. Thus we immediately deduce the *Riemann relations*:

$$\omega_X \cdot \omega_X = 0 \qquad \omega_X \cdot \bar{\omega}_X > 0.$$

Moreover both ω_X and $\bar{\omega}_X$ are orthogonal to any element of $H^{1,1}(X)$, since such an element is locally written as $\beta\, dz_1 \wedge d\bar{z}_2$ or as $\gamma\, d\bar{z}_1 \wedge dz_2$. Observe that if V is the complex vector space spanned by ω_X and $\bar{\omega}_X$, then the two-dimensional real vector space $V_{\mathbb{R}} := \{x \in V : x = \bar{x}\}$ has a basis made by $\omega_X + \bar{\omega}_X$ and $i(\omega_X - \bar{\omega}_X)$. With respect to this basis the intersection form is diagonal and positive-definite. Also $V = V_{\mathbb{R}} \otimes \mathbb{C}$ and the quadratic form on V is that induced by the complexification of $V_{\mathbb{R}}$. We have already seen that the intersection form on $H^2(X,\mathbb{Z})$ is *even*, meaning with this that x^2 is even for any $x \in H^2(X,\mathbb{Z})$, and *unimodular*, which means that the induced map $H^2(X,\mathbb{Z}) \to H_2(X,\mathbb{Z})^*$ is an isomorphism, with signature $(3,19)$. By Milnor Theorem **??** there is a unique such lattice, modulo isomorphism. We will denote it by Λ_{K3}.

4.6. The Picard lattice. If X is a K3 surface, the long exact cohomology sequence of the exponential sequence of X gives

$$\begin{array}{ccccccc} H^1(X,\mathcal{O}_X) & \longrightarrow & H^1(X,\mathcal{O}_X^*) & \xrightarrow{\tau} & H^2(X,\mathbb{Z}) & \xrightarrow{\pi} & H^2(X,\mathcal{O}_X) \\ \parallel & & \parallel & & & & \parallel \\ (0) & & \mathrm{Pic}(X) & & & & \mathbb{C}\,\omega_X. \end{array}$$

This description fits well with the fact that $\tau(\mathrm{Pic}(X))$ is orthogonal to $\mathbb{C}\,\omega_X$ in $H^2(X,\mathbb{C}) \cong H^2(X,\mathbb{Z}) \otimes \mathbb{C}$ once we interpret the map π of the previous exact sequence as the projection over the first factor in the Hodge decomposition of the cohomology of X.

Since bot ω_X and $\bar{\omega}_X$ are orthogonal to the elements of $H^{1,1}(X)$ with respect to the intersection product (4.5.1) we have that $\psi(\mathrm{Pic}(X))$ is contained in the intersection of $H^{1,1}(X)$ with $H^2(X,\mathbb{Z})$. In fact by the Lefschetz theorem on cohomology [**EoMa**], after identifying $\psi(\mathrm{Pic}(X))$ with $\mathrm{Pic}(X)$, we have:

$$\mathrm{Pic}(X) = H^{1,1}(X) \cap H^2(X,\mathbb{Z}),$$

where we are considering $H^2(X,\mathbb{Z})$ embedded into $H^2(X,\mathbb{C})$. Thus the Picard lattice of a K3 surface can be thought as the sublattice of $H^2(X,\mathbb{Z})$ which is orthogonal to $\omega_X \in H^2(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} = H^2(X,\mathbb{C})$. In particular $\mathrm{Pic}(X)$ is an even lattice of rank

$$0 \leq \rho_X \leq 20$$

and signature $(1, \rho_X - 1)$ if X is projective. The number ρ_X is called the *Picard rank* of X. We conclude by recalling that a class $[D] \in \mathrm{Pic}(X)$ is *nef* if $D \cdot C \geq 0$ for any integral curve C of X. The set of nef classes forms the *nef cone*

$$\mathrm{Nef}(X) \subset \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

5. Torelli theorem

In this section we briefly describe the period domain of marked K3 surfaces.

5.1. Deformation theory. A *deformation* of a complex manifold X is a smooth proper flat morphism $\pi : \mathcal{X} \to S$, where both $\mathcal{X}$ and S are connected complex varieties and moreover X is isomorphic to $\mathcal{X}_0 := \pi^{-1}(0)$, where $0 \in S$ is a distinguished point. An *infinitesimal deformation* is defined in a similar way, but this time $S = \mathrm{Spec}(\mathbb{C}[\varepsilon])$, where $\varepsilon^2 = 0$.

Given a morphism $S' \to S$ which maps a distinguished point $0' \in S'$ to $0 \in S$ one can construct the *pull-back* of the deformation as the fibre product

$$\begin{array}{ccc} \mathcal{X}' := \mathcal{X} \times_S S' & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S. \end{array}$$

The deformation $\mathcal{X} \to S$ of X is *complete* if any other deformation of X is isomorphic to a pull-back by a morphism $S' \to S$. If moreover the morphism is unique then $\mathcal{X} \to S$ is the *universal deformation* of X. If a deformation is complete and just the tangent of the map $S' \to S$ is unique, then the deformation is called *versal*. Observe that a universal deformation, if it exists, is a versal one. The versal deformation of X is denoted by $\mathcal{X} \to \mathrm{Def}(X)$. Hence with $\mathrm{Def}(X)$ we will denote the complex manifold whose points "represent" the deformations of X. We refer to [**Huy12**, Theorem 2.5, pag. 76] for more details about the following.

THEOREM 5.1.1. *Every compact complex manifold X has a versal deformation. Moreover $T_0 \mathrm{Def}(X) \cong H^1(X, T_X)$.*

i) If $H^2(X, T_X) = (0)$, then a smooth versal deformation exists.
ii) If $H^0(X, T_X) = (0)$, then a universal deformation exists.
iii) The versal deformation of X is versal and complete for any of its fibers $\mathcal{X}_t$ if $h^1(\mathcal{X}_t, T_{\mathcal{X}_t})$ is constant.

It is possible to prove that the infinitesimal deformations of X are in bijection with the elements of $H^1(X, T_X)$. Hence they represent the tangent vectors to $\mathrm{Def}(X)$ at the point $0 \in \mathrm{Def}(X)$.

Now if X is a K3 surface the existence of a holomorphic 2-form ω_X, which vanishes nowhere, gives an isomorphism between the tangent and the cotangent sheaf:

$$T_X \to \Omega_X \qquad \tau \mapsto \omega_X(\tau, -).$$

Thus $H^0(X, T_X)$ vanishes being isomorphic to $H^0(X, \Omega_X)$, whose dimension is $h^{0,1}(X) = h^{1,0}(X) = h^1(\mathcal{O}_X) = 0$. Hence X has a universal deformation. By a similar argument one proves that $H^2(X, T_X)$ vanishes, so that the universal deformation of X is smooth. Moreover

$$h^1(X, T_X) = -\chi(T_X) = 10\,\chi(\mathcal{O}_X) = 20,$$

where the middle equality is by the Riemann-Roch theorem for vector bundles on an algebraic surface [**Fri98**, Theorem 2(ii), pag. 31] (see below for another calculation when X is a quartic surface). Observe that since $\mathrm{Def}(X)$ is smooth, then its dimension is the dimension of its tangent space at $0 \in \mathrm{Def}(X)$, so that $\dim \mathrm{Def}(X) = 20$, by our previous calculation and Theorem 5.1.1. It is possible to show that fibers $\mathcal{X}_t$ in a sufficiently small neighborhood of $0 \in \mathrm{Def}(X)$ are still K3 surfaces. Hence by Theorem 5.1.1 the universal deformation of X is also a universal deformation of any such fiber $\mathcal{X}_t$.

Example 5.1.2. If we consider smooth quartic surfaces of $\mathbb{P}^3$, they form a $\mathbb{P}^{34}$, the dimension being obtained just by counting the elements of a monomial basis of quartics minus one. Two such quartics X and Y are isomorphic if there exists an element f of the projective linear group $G := \mathrm{PGL}(3, \mathbb{C})$ such that $f(X) = Y$. Since G has dimension 15, then the GIT quotient $\mathbb{P}^{34}//G$ is 19-dimensional. This in particular implies that not all K3 surfaces are quartic surfaces. A similar conclusion can be obtained by considering the map γ coming from the exact sequence of the normal sheaf of X:

$$\begin{array}{ccccccccccc} H^0(T_X) & \longrightarrow & H^0(T_{\mathbb{P}^3}|_X) & \longrightarrow & H^0(\mathcal{N}_X) & \xrightarrow{\gamma} & H^1(T_X) & \longrightarrow & H^1(T_{\mathbb{P}^3}|_X) & \longrightarrow & H^1(\mathcal{N}_X) \\ \| & & \| & & \| & & \| & & \| & & \| \\ (0) & & \mathbb{C}^{15} & & \mathbb{C}^{34} & & \mathbb{C}^{20} & & \mathbb{C} & & (0) \end{array}$$

We have already seen that the first space vanishes. The third and sixth equalities are due to $\mathcal{N}_X \cong \mathcal{O}_X(4)$ and Riemann-Roch. The second and fifth equalities are due to the Euler sequence of $\mathbb{P}^3$ for the tangent sheaf $T_{\mathbb{P}^3}$ tensored with $\mathcal{O}_X$. The space $\gamma(H^0(\mathcal{N}_X))$ represents the infinitesimal deformations of X inside $\mathbb{P}^3$, that is it can be regarded as the space of embedded infinitesimal deformations of X. Hence X has a 19-dimensional family of such deformations, which corresponds to the tangent space at the point $[X]$ of the GIT quotient $\mathbb{P}^{34}//G$.

5.2. The period domain. Recall that the second cohomology of any K3 surface X is isometric to the K3 lattice Λ_{K3}. A *marking* is an isometry:

$$\Phi : H^2(X, \mathbb{Z}) \to \Lambda_{\mathrm{K3}}.$$

Taking the complexification of Φ we obtain a $\mathbb{C}$-linear map which we will denote by the same symbol. Thus we can consider the image of the period line $\Phi(\mathbb{C}\,\omega_X)$ in $\mathbb{P}(\Lambda_{\mathrm{K3}} \otimes_{\mathbb{Z}} \mathbb{C})$. The *period domain* is the open subset of the 20-dimensional projective quadric hypersurface:

$$\mathcal{Q} := \{\mathbb{C}\,\omega \in \mathbb{P}(\Lambda_{\mathrm{K3}} \otimes \mathbb{C}) : \omega \cdot \omega = 0 \text{ and } \omega \cdot \bar{\omega} > 0\}.$$

Observe that due to the Riemann conditions $\Phi(\mathbb{C}\,\omega_X) \in \mathcal{Q}$ for any K3 surface X and any marking Φ. Consider now the universal deformation $\mathcal{X} \to \mathrm{Def}(X)$ of X. A marking Φ for X induces a marking for all the fibers of the deformation. This allows us to define the *period map* to be the holomorphic map:

$$\mathcal{P}_X : \mathrm{Def}(X) \to \mathcal{Q} \qquad t \mapsto \mathbb{C}\,\omega_t,$$

where ω_t is the image, via the marking induced by Φ, of a holomorphic 2-form of $\mathcal{X}_t = \pi^{-1}(t)$.

5.3. The local Torelli theorem.

PROPOSITION 5.3.1 (local Torelli theorem). *Let X be a K3 surface and let $\mathcal{X} \to \mathrm{Def}(X)$ be the universal deformation of X. Then the period map $\mathcal{P}_X : \mathrm{Def}(X) \to \mathcal{Q}$ is a local isomorphism.*

For a complete proof of this proposition see [**Huy12**, Proposition 2.9, pag. 77]. Here we just observe that it is possible to show that the differential $d\mathcal{P}_X$ at the point $0 \in \mathrm{Def}(X)$ is given by the $\mathbb{C}$-linear map induced by the contraction homomorphism (contraction by means of the holomorphic symplectic form ω):

$$\begin{array}{ccc} H^1(X, T_X) & \xrightarrow{\text{contr.}} & H^1(X, \Omega_X) \\ \cong \uparrow & & \downarrow \cong \\ T_0 \, \mathrm{Def}(X) & & \mathrm{Hom}(H^{2,0}(X), H^{2,0}(X)^{\perp}/H^{2,0}(X)), \end{array}$$

by showing that the right bottom expression is the tangent space of $\mathcal{Q}$ at $\mathcal{P}(0)$. Hence $d\mathcal{P}_X$ is an isomorphism at $0 \in \mathrm{Def}(X)$ and the local period map is a local isomorphism since in a small analytic neighborhood of $0 \in \mathrm{Def}(X)$ all the fibers are K3 surfaces with the same universal deformation space. Observe that we already knew that both $\mathrm{Def}(X)$ and $\mathcal{Q}$ have dimension 20, which is the dimension of $H^{1,1}(X)$.

5.4. The global Torelli theorem. The following theorem has been proved by Pjateckiĭ-Šapiro, Šhafarevič [**PŠŠ71**].

THEOREM 5.4.1 (global Torelli theorem). *Let X and X' be two K3 surfaces and let $\sigma : H^2(X, \mathbb{Z}) \to H^2(X', \mathbb{Z})$ be an isometry such that*

(i) $\sigma(\mathbb{C}\,\omega_X) = \mathbb{C}\,\omega_{X'}$*;*
(ii) $\sigma(\mathrm{Nef}(X)) = \mathrm{Nef}(X')$.

Then there exists a unique isomorphism $\varphi : X' \to X$ such that $\varphi^* = \sigma$.

Condition (ii) of the theorem is usually formulated in terms of the Kähler cone of X. Instead of introducing this cone here, we prefer to use the nef cone of X which is its closure. When X is projective, the nef cone is defined as done at the end of the previous section.

Let us denote now with $\mathrm{Def}(X)'$ the moduli space of pairs (X, Φ), where X is a K3 surface and Φ is a marking for X modulo the natural notion of isomorphism between pairs. This can be formally obtained as $\mathrm{Isom}(R^2 f_* \mathbb{Z}_{\mathcal{X}}, \Lambda_{\mathrm{K3}})$, where $f : \mathcal{X} \to \mathrm{Def}(X)$ is the universal deformation of X. It is possible to show (see [**Huy12**]) that the forgetful map $\mathrm{Def}(X)' \to \mathrm{Def}(X)$ is an infinite étale covering and that the period map lifts to the *global period map* $\mathcal{P} : \mathrm{Def}(X)' \to \mathcal{Q}$.

5.5. Surjectivity of the global period map. The following theorem is due to Todorov [**Tod79**].

THEOREM 5.5.1 (Surjectivity of the global period map). *Let $\mathbb{C}\,\omega \in \mathcal{Q}$. Then there exists a K3 surface X and a marking $\Phi : H^2(X, \mathbb{Z}) \to \Lambda_{\mathrm{K3}}$ such that $\mathbb{C}\Phi(\omega_X) = \mathbb{C}\,\omega$.*

Observe that the global period map $\mathcal{P}$ is locally injective due to the local injectivity of the period map $\mathcal{P}_X$ but it is not necessarily injective. Indeed if $\sigma : H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{Z})$ is an isometry which satisfies condition (i) but not condition

(ii) of Theorem 5.4.1, then the pairs (X, Φ) and $(X, \Phi \circ \sigma)$ are not isomorphic but $\mathcal{P}((X, \Phi)) = \mathcal{P}((X, \Phi \circ \sigma))$. Thus two such pairs are in the same fiber of the global period map. It is possible to show that if the K3 surface is very general and σ is any isometry of its second cohomology group which satisfies condition (i) of Theorem 5.4.1, then either σ or $-\sigma$ satisfies condition (ii) of the Theorem.

PROPOSITION 5.5.2. *For any positive integer* $0 \leq n \leq 20$ *there exists a K3 surface* X *with* $\rho_X = n$.

PROOF. Let $\mathbb{C}\,\omega \in \mathcal{Q}$ be a period such that $S := \omega^\perp \cap \Lambda_{\mathrm{K3}}$ is a lattice of rank n. Observe that this depends just on the coefficients of ω with respect to a basis of Λ_{K3}. Now by Theorem 5.5.1 there exists a K3 surfaces X and a marking $\Phi : H^2(X, \mathbb{Z}) \to \Lambda_{\mathrm{K3}}$ such that $\Phi(\mathbb{C}\,\omega_X) = \mathbb{C}\,\omega$. The Picard lattice of X is $\omega_X^\perp \cap H^{1,1}(X)$, so that its image in Λ_{K3} is S. Thus X has Picard number n. □

Exercises

EXERCISE 5.1. Shows that any non-isotrivial family of K3 surfaces, that is a family $\pi : \chi \to S$ whose period map is non-constant, admits a dense subset of K3 surfaces of rank 20.

EXERCISE 5.2. Show that for any even, positive integer number n there exists a K3 surface whose Picard lattice is generated by a class x with $x^2 = n$.

EXERCISE 5.3. Let X be the *Fermat quartic* surface of $\mathbb{P}^3$ defined by:

$$x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0.$$

(i) Show that X contains 48 lines contained in the 12 planes of equations:

$$x_3 = \pm\zeta x_i \qquad x_3 = \pm\zeta^3 x_i,$$

where $i \in \{0, 1, 2\}$ and ζ is a 8-th primitive root of unity.

(ii) Show that the intersection matrix of the classes of the lines in $\mathrm{Pic}(X)$ has rank 20 and signature $(1, 19)$. In particular $\mathrm{Pic}(X)$ is a maximal sub lattice of the 20-dimensional vector space $H^{1,1}(X)$.

(iii) Deduce that the intersection form on $H^2(X, \mathbb{Z})$ has signature $(3, 19)$.

(iv) Reproduce the calculation of the intersection matrix of the lines of S by means of the following Magma code [**BCP97**].

```
K<a>:=CyclotomicField(8);
P<x,y,z,w>:=ProjectiveSpace(K,3);
X:=Scheme(P,x^4+y^4+z^4+w^4);
lines:=&cat[PrimeComponents(Scheme(X,x+p*q)): p in [a,-a,a
    ^3,-a^3], q in [y,z,w]];
M:=Matrix(#lines ,[Degree(p meet q): p,q in lines]);
for i in [1..#lines] do M[i,i]:=-2; end for;
Rank(M);
```

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