Towards a bulk dual of the unitary Fermi gas

O(N)-like vs BCS-like models

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Outline

1. The unitary Fermi gas
   - What is it?
   - How to describe and how to prepare one?
   - Why is it of theoretical interest?
   - What might be an educated guess for its bulk dual?

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2. Relativistic vs non-relativistic higher-spin holography
   - O(N)-like vs BCS-like models
   - Higher-spin holography
   - Relativistic vs non-relativistic currents

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2. Relativistic vs non-relativistic higher-spin holography
    - O(N)-like vs BCS-like models
    - Higher-spin holography
    - Relativistic vs non-relativistic currents
3. Relativistic vs non-relativistic higher-spin symmetries
    - Conformal vs Schrödinger algebras
    - Relativistic vs non-relativistic singletons (massless vs massive fields)
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    - Light-like reduction and light-cone formalism

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What is the unitary Fermi gas?

The underlying motivation is the quest for a holographic dual description of the unitary Fermi gas (Son, Balasubramanian, McGreevy; 2008).

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- (cold): low-energy scattering

  Low-energy

  $\Leftrightarrow$ interaction range $\ll$ relative-momentum de Broglie wavelength

  $\Rightarrow$ (approximately) contact interaction

What is the unitary Fermi gas?

Definition (simplified)

- Fermi: fermionic particles

  Cold many-body system of fermions
  $\Rightarrow$ relative-momentum de Broglie wavelength $\approx$ Fermi wavelength $\approx$
  $\approx$ mean interparticle distance

  Two-body contact interactions
  $\Rightarrow$ (at least) two species of fermions in order to have interactions

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  Examples:

  - *BCS supraconductors*: electrons with spin "up" or "down"
  - *ultracold gases*: fermionic atoms (& electrically neutral $\Rightarrow$ odd number of neutrons) in two hyperfine states (e.g. alkali ^{6}Li or ^{40}K)

What is the unitary Fermi gas?

Definition (simplified)

- unitary: maximal (modulus of) scattering (amplitude) compatible with unitarity

  low-energy two-body scattering $\Rightarrow$ s-wave dominates and the unitarity bound on the scattering amplitude is saturated in the

  unitarity regime
  $\Leftrightarrow$ relative-momentum de Broglie wavelength $\ll$ $|$scattering length$|$

Reminder: The low-energy scattering matrix for the s-wave is

$$S = 1 + 2iT \approx \frac{\lambda - ia}{\lambda + ia}$$

where λ = relative-momentum de Broglie wavelength ($>$0) and a = scattering length.
Therefore, $0 \leqslant |T| \leqslant 1$ in general, while $|T| \overset{\lambda \gg |a|}{\approx} 0$ and $|T| \overset{\lambda \ll |a|}{\approx} 1$.

What is the unitary Fermi gas?

Definition (simplified)

$\Rightarrow$ unitary Fermi gas: cold many-body system of fermions with

$$\text{interaction range} \ll \text{interparticle distance} \ll |\text{scattering length}|$$

What is the unitary Fermi gas?

How to describe and how to prepare a unitary Fermi gas?

BCS model

Many-body system with two-body contact interactions between two-component fermions $\Longleftrightarrow$ **BCS model**

$$S[\psi\,;u_0]=\int dt\int d\mathbf{x}\left[\sum_{\alpha=\uparrow,\downarrow}\psi^*_\alpha\left(i\partial_t+\frac{\Delta}{2m}+\mu\right)\psi_\alpha\;-\;u_0\,\psi^*_\downarrow\psi^*_\uparrow\psi_\uparrow\psi_\downarrow\right]$$

Attractive (repulsive) interaction for negative (positive) coupling constant.

Terminology: We chose to refer to the above field theory as BCS *model* because the corresponding Hamiltonian is sometimes called "BCS Hamiltonian" in the condensed matter litterature. In order to avoid confusion let us stress that, strictly speaking, in the BCS *theory* of supraconductivity one must further minimal couple the fermions to an external $U(1)$ vector gauge field.

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Low-energy scattering theory for an attractive short-range potential states that the sign of the scattering length caracterises the existence of a (shallow, i.e. nearly zero-energy) two-body bound state in vacuum.

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Scatt length	Bound state	Interaction	Gas
> 0	$\exists$	Binding	Dimers
< 0	$\nexists$	Pairing	Cooper pairs
$= 0$	No	No	Ideal
$\pm\infty$	Zero energy	Feshbach resonance	Unitary

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Experimentally, the interaction strength (and thus the scattering length) can be controlled very accurately by tuning an external magnetic field, so a unitary gas can be prepared via fine tuning.

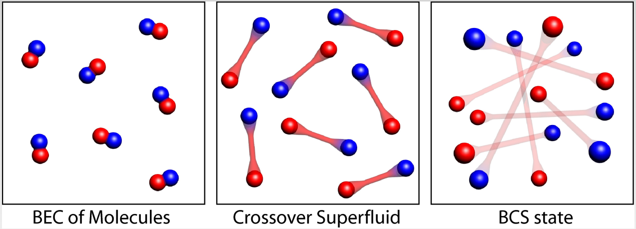
BEC-BCS crossover

Why is the unitary Fermi gas of theoretical interest?

BEC-BCS crossover

(Leggett, 1980) The ground state of the BCS model is always a superfluid (i.e. spontaneous breaking of rigid $U(1)$ symmetry).

Regime	interparticle/scattering length	Pair condensate
BEC	$\gg 1$	Tightly-bound pairs (dimers)
Unitarity	≈ 0	Crossover
BCS	$\ll -1$	Long-range (Cooper) pairs



BEC-BCS crossover

Hopefully, the unitarity regime might provide an analytically tractable model of

- BEC-BCS crossover (interparticle distance $\ll$ $|$scattering length$|$),
- High-temperature superfluidity (critical temperature close to Fermi temperature),
- Second-order quantum phase transition (at zero temperature),
- etc.

The unitary Fermi gas as a non-relativistic CFT

For simplicity, one assumes from now to be **at zero density and temperature**.

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$\Rightarrow$ Technical definition of the ideal/unitary Fermi gase used here.

The unitary Fermi gas as a non-relativistic CFT

Therefore, the unitary Fermi gas is a tantalising example of non-relativistic conformal field theory:

- "simple" interactions (two-body contact interactions of two-component fermions, i.e. BCS model),
- though challenging (strongly interacting, i.e. no small parameter).

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- though challenging (strongly interacting, i.e. no small parameter).

$\Rightarrow$ **What might be an educated guess for the bulk dual of the ideal/unitary Fermi gases?** (Son, 2008)

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A semi-classical description on the bulk side should correspond to a large-N limit on the boundary side.

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BCS-like model with N “flavors”

$$S[\psi\,;u_0,N]=\int dt\int d\mathbf{x}\left[\sum_{\alpha=\uparrow,\downarrow}\vec{\psi}^{\,*}_\alpha\cdot\left(i\partial_t+\frac{\Delta}{2m}+\mu\right)\vec{\psi}_\alpha-\frac{u_0}{N}\,|\vec{\psi}_\uparrow\cdot\vec{\psi}_\downarrow|^2\right]$$

where $\vec{\psi}_\alpha$ denotes an vector-like multiplet of 2N massive non-relativistic fermions. The large-N limit corresponds to the mean field approximation.

BCS-like model

Applying the general observation of (Gubser & Klebanov, 2003) on double-trace deformations of CFTs to the BCS-like model,

(BeMeMo, 2011) In the large-N limit, the free energies of the ideal and the unitary Fermi gases are related by a Legendre transformation (with respect to the source for the Cooper pair, and modulo proper rescalings).

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- Many properties of the unitary Fermi gas in the mean field approximation can be derived from the ideal Fermi gas.
- The two fixed points have the same infinite set of conserved currents and symmetries, most of which are broken by $1/N$ corrections in the interacting theory.
- Both fixed points should correspond to different choices of boundary conditions for the *same* bulk theory (holographic degeneracy).

O(N)-like vs BCS-like models

Many properties of the BCS-like model are reminiscent from the O(N) model, so let us compare these models in details.

O(N)-like vs BCS-like models

Models	O(N)-like	BCS-like
Space-time	Relativistic	Non-relativistic
Kinetic operator	$\Box - M^2$	$i\partial_t + \frac{\Delta}{2m} + \mu$
Scale-free	Massless $M = 0$	Zero chem pot $\mu = 0$
Fundamental fields	Bosons $\vec{\phi}$	Fermions $\vec{\psi}_\uparrow$, $\vec{\psi}_\downarrow$
N components	Real [or complex]	Complex (up/down)
Quartic interaction	$(\vec{\phi}^* \cdot \vec{\phi})^2$	$\lvert\vec{\psi}_\uparrow \cdot \vec{\psi}_\downarrow\rvert^2$
Internal symmetry	$O(N)$ $[\subset U(N)]$	$U(2)$ $[\subset U(1) \times Sp(2N)]$
Composite field	Particle density $\vec{\phi}^* \cdot \vec{\phi}$	Cooper pair $\vec{\psi}_\uparrow \cdot \vec{\psi}_\downarrow$
Dimension	$D = d + 2$ (spacetime)	$d = D - 2$ (space)
Non-triv fixed pt	Wilson-Fisher	Unitary Fermi gas
Kinematical sym	Conformal $\mathfrak{o}(d+2,2)$	Schrödinger $\mathfrak{sch}(d)$
Higher-spin sym	Vasiliev	Weyl

O(N)-like vs BCS-like models

Engineering scale dimensions of elementary fields ($\vec{\phi}$ vs $\vec{\psi}_\alpha$):

$$\Delta^{\text{elementary}} = (D-2)/2 = d/2$$

Bare and dressed (large-N approx vs vacuum exact) scale dimensions of composite two-body field ($\vec{\phi}^* \cdot \vec{\phi}$ vs $\vec{\psi}_\uparrow \cdot \vec{\psi}_\downarrow$) at the fixed points:

$$\Delta^{\text{free}} = 2\,\Delta^{\text{elementary}} = D-2 = d\,, \qquad \Delta^{\text{int}} = 2$$

(Non)relativistic scale dimensions $\Delta_+ \geqslant \Delta_-$:

$$\Delta_+ + \Delta_- = \Delta^{\text{free}} + \Delta^{\text{int}} = D = d+2$$

Δ composite	Fixed point
Δ^{free}	Gaussian
Δ^{int}	Non-trivial
Δ_+	IR
Δ_-	UV

O(N)-like vs BCS-like models

D	Δ_-	Δ_+	Property
$D=2$	Δ^{free}	Δ^{int}	saturation of unitarity bound (line of fixed pts)
$2<D<4$	Δ^{free}	Δ^{int}	pair of admissible fixed pts (asymptotic freedom)
$D=4$	$\Delta^{\text{free}}=\Delta^{\text{int}}$	$\Delta^{\text{int}}=\Delta^{\text{free}}$	fusion of fixed pts (triviality)
$4<D<6$	-	Δ^{free}	only single admissible fixed pt (unstable int)
$D=6$	-	Δ^{free}	saturation of unitarity bound
$D>6$	-	Δ^{free}	only single admissible fixed pt (non-unitary int)

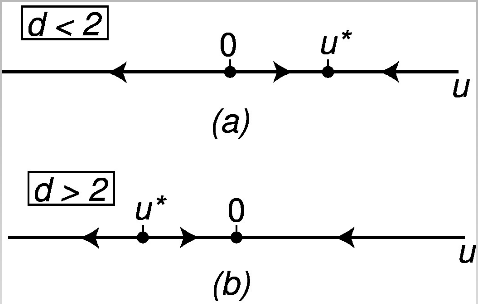
$D=3$: free/critical O(N) models

O(N)-like vs BCS-like models

d	Δ_-	Δ_+	Property
$d=0$	Δ^{free}	Δ^{int}	saturation of unitarity bound
$0<d<2$	Δ^{free}	Δ^{int}	pair of admissible fixed pts (asymptotic freedom)
$d=2$	$\Delta^{\text{free}}=\Delta^{\text{int}}$	$\Delta^{\text{int}}=\Delta^{\text{free}}$	fusion of fixed pts (triviality)
$2<d<4$	Δ^{int}	Δ^{free}	pair of admissible fixed pts (asymptotic safety)
$d=4$	Δ^{int}	Δ^{free}	saturation of unitarity bound
$d>4$	-	Δ^{free}	only single admissible fixed pt (non-unitary int)

$d=1$ or 3: ideal/unitary Fermi gases (repulsive or attractive interactions)

O(N)-like vs BCS-like models



Renormalisation group flow of the BCS model

Source of inspiration: higher-spin holography

Growing body of evidence (Petkou-Sezgin-Sundell, 2003; Giombi-Yin, 2010; Maldacena-Zhiboedov, 2011; ...) that
the bulk dual of the free/critical O(N) models in 3 dimensions should be Vasiliev's minimal higher-spin gravity on AdS_4
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In order to make this idea more precise, let us review the higher-spin holography.

Higher-spin holography

Δ composite	AdS bdy cond & symmetry	CFT fixed pt
Δ^{free}	Unbroken higher-spin	Gaussian
Δ^{int}	Broken higher-spin	Non-trivial
Δ_+	Standard ("Dirichlet")	IR
Δ_-	Exotic ("Neumann")	UV

Scaling dimension of the collective field

Higher-spin holography

Dictionary	AdS$_{D+1}$	CFT$_D$
Bulk/Boundary	Vasiliev theory	U(N) model
Symmetric phase	Unbroken	Free
Broken phase	Broken	Critical
Field/Operator	Symmetric tensor fundamental gauge field: adjoint-valued	Symmetric tensor composite conserved current: fund $\otimes$ antifund
Examples	Singlet scalar $U(1)$ vector Metric tensor Higher-spin fields	Particle density Charge current Energy-momentum-stress Higher-spin currents

Holographic dictionary

Higher-spin conserved currents

Set of symmetric currents of all ranks (Berends, Burgers, van Dam; 1986)

$$J^{AB}_{\mu_1\dots\mu_s}(x) \;=\; i^s\,\phi^{A*}(x)\;\overset{\leftrightarrow}{\partial}_{\mu_1}\cdots\overset{\leftrightarrow}{\partial}_{\mu_s}\;\phi^B(x)$$

$$(A,B=1,\dots,N\,;\,\mu=0,\dots,D-1)$$

Features:

- $\mathfrak{u}(N)$-valued, $J^*_{AB}=J_{BA}$
- Bilinear in the scalar fields ϕ and its conjugate
- Number of derivatives = Rank
- Conserved (on-shell) for $s\geqslant 1$

$$\partial^\mu J^{AB}_{\mu_1\dots\mu_s}(x)\;\approx 0$$

where the weak equality $\approx$ stands for "on the free mass shell", i.e. modulo $\Box\phi(x)\approx 0$.

Higher-spin conserved currents

The Berends-Burgers-vanDam currents can be packed in the generating function

$$\begin{aligned} J^{AB}(x,q) &:= \sum_{s\geqslant 0} \frac{1}{s!}\, q^{\mu_1}\dots q^{\mu_s}\, J^{AB}_{\mu_1\dots\mu_s}(x) \\ &= \phi^*(x-iq)\cdot\phi(x+iq) \;=\; |\phi(x+iq)|^2 \end{aligned}$$

which is a bi-local function of the scalar field, c.f. the collective field of (Das, Jevicki; 2003) and (de Mello Koch, Jevicki, Jin, Rodrigues; 2010).

Higher-spin conserved currents

In terms of the generating function

$$J^{AB}(x,q) \quad := \quad \sum_{s\geqslant 0} \frac{1}{s!}\, q^{\mu_1}\dots q^{\mu_s}\, J^{AB}_{\mu_1\dots\mu_s}(x)\,, \tag{1}$$

the conservation and tracefulness conditions for the complete set of symmetric currents are equivalent to the two compact equations

$$\Big(\eta^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial q^\nu}\Big) J^{AB}(x,q) \approx 0 \quad \Longleftrightarrow \quad \partial^\mu J^{AB}_{\mu_1\dots\mu_s}(x) \approx 0\,,\ \forall s$$

$$\Big(\eta^{\mu\nu}\frac{\partial}{\partial q^\mu}\frac{\partial}{\partial q^\nu}\Big) J^{AB}(x,q) \not\approx 0\,. \quad \Longleftrightarrow \quad \eta^{\mu_1\mu_2} J^{AB}_{\mu_1\mu_2\dots\mu_s}(x) \not\approx 0\,,\ \forall s\,.$$

Higher-spin conserved currents

When the scalar field is massless, a set of symmetric currents which are conserved & traceless exists in any dimension (D. Anselmi; 1998).

Remarks:
Such bilinear currents are the conformal primaries appearing in the higher-spin holographic conjecture.

Traceless conserved currents

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Remarks:
An equivalent set can be obtained from the BBvD currents (X.B., E. Joung, E. Mourad; 2010).

$$\bar{J}^{AB}(x,q) = \sum_{n=0}^{\infty} \frac{1}{4^n\, n!\,(-q\cdot\partial_q - \frac{d-5}{2})_n} \left(q^2\, \partial_x^2 - (q\cdot\partial_x)^2 \right)^n\; J^{AB}(x,q)$$

where $(a)_n = \Gamma(a+n)/\Gamma(a)$ is the Pochhammer symbol.

The on-shell conservation & tracelessness conditions are equivalent to

$$\Big(\eta^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial q^\nu}\Big)\bar{J}^{AB}(x,q)\,\approx 0\,,\quad \Big(\eta^{\mu\nu}\frac{\partial}{\partial q^\mu}\frac{\partial}{\partial q^\nu}\Big)\bar{J}^{AB}(x,q)\,\approx 0\,.$$

Traceless conserved currents

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Remarks:
Other equivalent generating functions are known in specific dimensions, e.g. in $d = 4$ (Gelfond, Skvortsov, Vasiliev; 2006) and in $d = 3$ (Giombi, Yin; 2009).

Non-relativistic higher-spin currents

For U(N) model, one usually focuses on composite fields which are singlets of the internal symmetry group, as the particle density

$$J = \vec{\phi}^* \cdot \vec{\phi} = J^* \,.$$

The generic U(N)-singlet bilocal (ϕ^* and ϕ at distinct points) composite field generates all the traceless conserved symmetric currents, which are the Noether currents for the higher-spin symmetries of the free massless scalar field ("singleton").

Non-relativistic higher-spin currents

For BCS-like models, one would instead focus on composite fields which are flavor-singlets but form the adjoint multiplet of the internal symmetry subgroup $U(2)$:

$$J = \begin{pmatrix} -\vec{\psi}^*_\uparrow \cdot \vec{\psi}_\uparrow & \vec{\psi}_\uparrow \cdot \vec{\psi}_\downarrow \\ \vec{\psi}^*_\downarrow \cdot \vec{\psi}^*_\uparrow & \vec{\psi}^*_\downarrow \cdot \vec{\psi}_\downarrow \end{pmatrix} = J^\dagger$$

because it includes the Cooper pair field $\vec{\psi}_\uparrow \cdot \vec{\psi}_\downarrow$.

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(BeMeMo, 2011) The above O(N)-singlet bilocal (ψ's at distinct points) composite field generates all U(1)-neutral non-relativistic currents of all integer spins generalising the up/down particle numbers $\vec{\psi}^*_\alpha \cdot \vec{\psi}_\alpha$ together with U(1)-charged tensors generalising the Cooper pair $\vec{\psi}_\uparrow \cdot \vec{\psi}_\downarrow$.

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because it includes the Cooper pair field $\vec{\psi}_\uparrow \cdot \vec{\psi}_\downarrow$.

$\Rightarrow$ A bulk/boundary dictionary would identify these $\mathfrak{u}(2)$-valued non-relativistic symmetric "currents" of all integer spins as the boundary data of a tower of $\mathfrak{u}(2)$-valued higher-spin bulk gauge fields.

Non-relativistic higher-spin currents

For BCS-like models, one would instead focus on composite fields which are flavor-singlets but form the adjoint multiplet of the internal symmetry subgroup $U(2)$:

$$J = \begin{pmatrix} -\vec{\psi}^*_\uparrow \cdot \vec{\psi}_\uparrow & \vec{\psi}_\uparrow \cdot \vec{\psi}_\downarrow \\ \vec{\psi}^*_\downarrow \cdot \vec{\psi}^*_\uparrow & \vec{\psi}^*_\downarrow \cdot \vec{\psi}_\downarrow \end{pmatrix} = J^\dagger$$

because it includes the Cooper pair field $\vec{\psi}_\uparrow \cdot \vec{\psi}_\downarrow$.

$\Rightarrow$ A bulk/boundary dictionary would identify these $\mathfrak{u}(2)$-valued non-relativistic symmetric "currents" of all integer spins as the boundary data of a tower of $\mathfrak{u}(2)$-valued higher-spin bulk gauge fields.

Of which higher-spin symmetries are they Noether currents?

What are non-relativistic singletons?

Vasiliev bosonic higher-spin algebras are known to be maximal symmetry algebras of free relativistic singletons.

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$\Rightarrow$ **What are free non-relativistic singletons? What are their maximal symmetry algebras?**

What are non-relativistic singletons?

Group-theoretical definitions

Free relativistic singleton
UIR of the Poincaré algebra that can be lifted to a UIR of the conformal algebra.

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In other words, the free non-relativistic singletons can be identified with the solutions of free Schrödinger equation with zero chemical potential

$$\left(i\partial_t + \frac{\Delta}{2m} \right) \psi(t, \mathbf{x}) = 0$$

Schrödinger algebra

Schrödinger algebra: $\mathfrak{sch}(d) = \mathfrak{h}_d \ni \big(\mathfrak{o}(d) \oplus \mathfrak{sl}(2,\mathbb{R})\big)$

Standard representation as order-one differential operators acting on wave functions $\psi(t, \mathbf{x})$

Rigid symmetries: relativistic vs non-relativistic

Schrödinger algebra: $\mathfrak{sch}(d) = \mathfrak{h}_d \ni \big(\mathfrak{o}(d) \oplus \mathfrak{sl}(2,\mathbb{R})\big)$

$\mathfrak{h}_d$:

$$\hat{P}_i = -i\partial_i, \qquad \hat{K}_i = m x_i + i t \partial_i, \qquad \hat{m} = m,$$

$\mathfrak{o}(d)$:

$$\hat{M}_{ij} = -i(x_i\partial_j - x_j\partial_i),$$

$\mathfrak{sl}(2,\mathbb{R})$:

$$\hat{P}_t = i\partial_t,$$

$$\hat{D} = i\left(2\,t\,\partial_t + x^i\partial_i + \frac{d}{2}\right),$$

$$\hat{C} = i\left(t^2\partial_t + t\Big(x^i\partial_i + \frac{d}{2}\Big)\right) + \frac{m}{2}\,x^2.$$

Rigid symmetries: maximal algebra

Theorem: (Eastwood, 2002) The maximal algebra of infinitesimal symmetry generators for a free massless scalar field, i.e. differential operators $\hat{A}$ such that $\Box\hat{A} = \hat{B}\Box$ and modulo trivial generators $\hat{A} = \hat{C}\Box$, is generated algebraically by the conformal Killing vectors.

The maximal Lie algebra of symmetries for conformal scalar field in a flat D-dimensional spacetime is isomorphic to the gauge algebra of Vasiliev higher-spin gravity around AdS_{D+1} (Vasiliev, 2003).

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$\Rightarrow$ **What is its non-relativistic analogue?**

Rigid symmetries: maximal algebra

Theorem: The maximal algebra of infinitesimal symmetry generators for a free non-relativistic massive scalar field, i.e. differential operators $\hat{A}$ such that

$$\left(i\partial_t + \frac{\Delta}{2m}\right)\hat{A} = \hat{B}\left(i\partial_t + \frac{\Delta}{2m}\right)$$

and modulo trivial generators $\hat{A} = \hat{C}(i\partial_t + \frac{\Delta}{2m})$, is generated algebraically by the space translations and by the Galilean boosts.

$\Rightarrow$ This maximal Lie algebra of symmetries for non-relativistic particle on a flat d-dimensional space is isomorphic to the Weyl algebra of quantum observables.

Remark: This isomorphism also follows as a corollary from the general results on global symmetries of $Sp\,(2d,\mathbb{R})$-covariant unfolded equations (Vasiliev, 2001) upon the identification of the space coordinates with the twistor variables and of the time coordinate with the trace of the $\mathfrak{sp}(2d,\mathbb{R})$-matrix coordinates.

Rigid symmetries: relativistic vs non-relativistic

Schrödinger algebra: $\mathfrak{sch}(d) = \mathfrak{h}_d \ni \big(\mathfrak{o}(d) \oplus \mathfrak{sl}(2,\mathbb{R})\big)$

Observation: (M. Valenzuela, 2009) Alternative representation as degree-two polynomials in the momenta and Galilean boost generators acting on wave functions solutions of free Schrödinger equation $(i\partial_t + \frac{\Delta}{2m})\psi(t,\mathbf{x}) = 0$.

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Schrödinger algebra: $\mathfrak{sch}(d)=\mathfrak{h}_d \ni \big(\mathfrak{o}(d)\oplus\mathfrak{sl}(2,\mathbb{R})\big)$

$\mathfrak{h}_d$:

$$\hat{P}_i=\hat{P}_i(-t),\qquad \hat{K}_i=m\hat{X}_i(-t),\qquad \hat{m}=m,$$

$\mathfrak{o}(d)$:

$$\hat{M}_{ij}=\hat{X}^i(-t)\hat{P}^j(-t)-\hat{X}^j(-t)\hat{P}^i(-t),$$

$\mathfrak{sl}(2,\mathbb{R})$:

$$\hat{P}_t=\frac{\hat{P}^2(-t)}{2m},$$

$$\hat{D}=-\hat{X}^i(-t)\hat{P}_i(-t)+\frac{d}{2},$$

$$\hat{C}=\frac{m}{2}\hat{X}^2(-t).$$

Rigid symmetries: relativistic vs non-relativistic

Weyl algebra: $\mathfrak{A}(d) = \mathcal{U}(\mathfrak{h}_d)$

Higher-derivative generators:

The Weyl algebra of infinitesimal symmetry generators for the free non-relativistic particle is generated algebraically by the space translation and the Galilean boost generators.

Rigid symmetries: relativistic vs non-relativistic

Weyl algebra: $\mathfrak{A}(d) = \mathcal{U}(\mathfrak{h}_d)$

Maximal symmetry algebra: (idea of the proof)

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Maximal symmetry algebra: (idea of the proof)

Acting with any time-reversed (Heisenberg picture) observable $\hat{A}\big(\hat{\mathbf{X}}(-t), \hat{\mathbf{P}}(-t)\big)$ on a time-evolved (Schrödinger picture) state $\psi(t, \mathbf{x})$ is equivalent to acting with any initial observable $\hat{A}\big(\hat{\mathbf{X}}(0), \hat{\mathbf{P}}(0)\big)$ on the initial state $\psi(0, \mathbf{x})$.

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Therefore any quantum observable of $\mathfrak{A}(d)$ maps solutions on solutions of the Schrödinger equation.

Light-like dimensional reduction

Main idea behind the light-like dimensional reduction:

The kinetic operator of a relativistic theory

$$\Box - M^2 = -2\partial_+\partial_- + \Delta - M^2$$

when acting on eigenmodes of a light-like component of the momentum,

$$\Psi(x) = e^{-imx^-}\psi(x^+, x^i),$$

is proportional to the kinetic Schrödinger operator of a non-relativistic theory

$$i\partial_t + \Delta/2m + \mu$$

via the identification $x^+ = t$ and $M^2 = -\mu/2m$.

Light-like dimensional reduction

Main idea behind the light-like dimensional reduction:

(Group theory) The quadratic Casimir operators of the Poincaré and the Bargmann algebras are related

$$\hat{P}^\mu \hat{P}_\mu/2 = -\hat{P}_+\hat{P}_- + \hat{P}^i\hat{P}_i/2 = -\hat{m}\hat{P}_t + \hat{P}^i\hat{P}_i/2$$

upon the standard light-cone identification of the non-relativistic mass and Hamiltonian operators

$$\hat{P}_+ = \hat{m}\,, \qquad \hat{P}_- = \hat{P}_t\,.$$

The Bargmann (Schrödinger) algebra is isomorphic to the subalgebra of the Poincaré (conformal) algebra that commutes with $\hat{P}_+ = \hat{m}$.

Rigid symmetries: relativistic vs non-relativistic

Embedding diagram

$$
\begin{array}{ccc}
\mathfrak{o}(d+2,2) & \subset & \text{Vasiliev algebra (d+2,2)} \\
\cup & & \cup \\
\mathfrak{sch}(d) & \subset & \text{Weyl algebra (d)}
\end{array}
.
$$

where the embeddings stand for:

- $\subset$: first-order generator subalgebra
- $\cup$: centraliser subalgebra of $\hat{P}_+ = \hat{m}$.

Conclusion

Summary and outlook

Summary

Some hints toward a bulk dual of ideal/unitary Fermi gases

Boundary side:

1. Similarities between the free/critical $O(N)$ models and the ideal/unitary Fermi gases.
2. Non-relativistic symmetries (Schrödinger and Weyl alg) embedded in relativistic symmetries (respectively, conformal and Vasiliev alg).
3. Uniform treatment of generating functionals for relativistic (or not) scalar theories with quartic (two-body) contact interaction, e.g. O(N)-like (or BCS-like) models.
4. Non-relativistic theories as light-like dimensional reduction of relativistic theories, in the semi-classical (mean-field) regime.

Summary

Some hints toward a bulk dual of ideal/unitary Fermi gases

Bulk side:

1. Sezgin-Sundell/Klebanov-Polyakov conjecture (2002/2003) & its various tests (Petkou, 2003; Sezgin and Sundell, 2003; Giombi and Yin, 2010)
2. Newton-Cartan gravity as light-like dimensional reduction of Einstein-Cartan gravity (Duval et al, 1985; Julia and Nicolai, 1995)
3. AdS/CFT dictionary in the light-cone formalism (Metsaev, 1999)

Proposal

Towards an educated guess for the bulk dual of ideal/unitary Fermi gases

At least in the large-N (mean field) approximation, the following diagram may commute (c.f. Goldberger, Barbon-Fuertes, Lin-Wu, 2008):

$$\begin{array}{ccc} \text{HS on AdS spacetime (d+3)} & \leftrightarrow & \text{CFT on flat spacetime (d+2)} \\ \downarrow\uparrow & & \downarrow\uparrow \\ \text{NRHS on space-time (d+2)} & \leftrightarrow & \text{NRCFT on flat space-time (d+1)} \end{array}.$$

where the arrows stand for:

- $\leftrightarrow$ holographic duality (Sezgin-Sundell-Klebanov-Polyakov like)
- $\downarrow$ light-like reduction (Bargmann framework)
- $\uparrow$ light-like oxydation (Eisenhart lift)

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with

- $d = 1$: free/critical O(N) models
- $d = 1, 3$: ideal/unitary Fermi gases

Proposal

A candidate for the holographic description of fermions at unitarity is the light-like reduction of a Vasiliev higher-spin gravity. More precisely,

The $O(N)$-invariant sector of the large-N ideal/unitary Fermi gas in d spatial dimensions might be dual to the light-like dimensional reduction of the Vasiliev bosonic theory on AdS_{d+3} with $U(2)$ internal symmetry.

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In particular, the gravity dual of the "physical" three-dimensional ($d = 3$) two-component ($N = 1$) UV-stable ($\Delta_- = 2$) unitary Fermi gas would be the *light-like reduction of Vasiliev theory describing interacting $\mathfrak{u}(2)$-valued higher-spin gauge fields on AdS_6 with exotic boundary condition for the bulk scalar field dual to the Cooper pair.*

Proposal

By construction, kinematical properties such as

- spectrum of fields/operators
- (un)broken symmetries
- two-point functions

are matched (at tree level, i.e. large-N).

Open issues

Many issues remain open:

- Clarify the representation theory of Schrödinger algebra (Perroud, 1977) e.g.
    - holographic dictionary from AdS/CFT in the light-cone formalism (Metsaev, 1999),
    - status of non-relativistic massless representations,
    - non-relativistic analogue of Flato-Frønsdal theorem.
- Perform the light-like reduction of Vasiliev equations, e.g.
    - explicit them in light-cone gauge, and/or
    - generalise the works (Duval et al, 1985; Julia and Nicolai, 1995).
- Check the proposal beyond two-point functions.
- Etc ...

Thank you for your attention