Introduction to nonrelativistic conformal symmetries

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J. Negro, M. A. del Olmo and A. Rogriguez-Marco
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Outline

Kinematical symmetries

The homogeneity of space-time implies that the generators P_i of *spatial displacements* $(i = 1, 2, \ldots, d)$ and the generator $\mathsf{P}_t = \mathsf{H}$ of *time translations* correspond to infinitesimal symmetries of any free particle.

Moreover, the isotropy of space implies that the generators J_{ij} of *rotations* are also symmetry generators of free particles.

Furthermore, a modern statement of the relativity principle is that the generators K_i of *inertial transformations* (or *boosts*) also generate symmetries for free particles.

Space and time reversals

The *space-reversal*

$$\Pi \; : \qquad \mathsf{H} \mapsto +\mathsf{H}\,, \quad \mathsf{P}_i \mapsto -\mathsf{P}_i\,, \quad \mathsf{J}_{ij} \mapsto \mathsf{J}_{ij}\,, \quad \mathsf{K}_i \mapsto -\mathsf{K}_i\,,$$

and *time-reversal*

$$\mathsf{T} \; : \qquad \mathsf{H} \mapsto -\mathsf{H}\,, \quad \mathsf{P}_i \mapsto +\mathsf{P}_i\,, \quad \mathsf{J}_{ij} \mapsto \mathsf{J}_{ij}\,, \quad \mathsf{K}_i \mapsto -\mathsf{K}_i\,,$$

induce involutive transformations of the kinematical generators.

Definition (Bacry & Lévy-Leblond, 1968)

A kinematical algebra is a Lie algebra spanned by the generators P_i of spatial displacements, $\mathsf{P}_t = \mathsf{H}$ of time translations, J_{ij} of rotations and K_i of inertial transformations such that

(i) the adjoint action of the subalgebra $\mathfrak{o}(d) = \text{span}_{\mathbb{R}}\{\mathsf{J}_{ij}\}$ decomposes into the sum of the irreducible

- trivial (= scalar) representation on the module spanned by H:

$$[\mathsf{J}, \mathsf{H}] = 0\,,$$

- fundamental (= vector) representation on the module spanned by the P_i or the K_j:

$$[\mathsf{J}, \mathsf{P}] \sim \mathsf{P}\,, \qquad [\mathsf{J}, \mathsf{K}] \sim \mathsf{K}\,,$$

- adjoint (= antisymmetric) representation on the module spanned by the J_{ij}:

$$[\mathsf{J}, \mathsf{J}] \sim \mathsf{J}\,,$$

(ii) space and time reversal Π and T are automorphisms.

Classification (Bacry & Lévy-Leblond, 1968)

Theorem

For space of dimension $d \geqslant 3$, there are only 4 types of noncompact kinematical algebras such that time translations and inertial transformations do not commute:

- ***Relativistic kinematical algebras***
  - *(c,Λ) The (anti) de Sitter isometry algebras $\mathfrak{o}(d+1,1)$ and $\mathfrak{o}(d,2)$,*
  - *(c,0) The Poincaré algebra $\mathfrak{io}(d,1) = \mathbb{R}^{d+1} \ni \mathfrak{o}(d,1)$,*
- ***Nonrelativistic kinematical algebras***
  - *(∞,Λ) The Newton-Hooke algebras $\mathfrak{nh}_\pm(d)$,*
  - *(∞,0) The Galilei algebra $\mathfrak{gal}(d)$.*

All algebras can be obtained from the (anti) de Sitter isometry algebras (via Inönu-Wigner contractions).
Any (non)relativistic algebra admits only (one non)trivial central extension.

Poincaré vs (anti) de Sitter algebras

Inserting the (rescaled) cosmological constant $\Lambda(=\pm\frac{1}{R^2})$, the nontrivial commutators (besides the one involving the rotation generators) of the relativistic kinematical algebras read

$$[\mathsf{P}_0,\mathsf{K}_i] = i\,\mathsf{P}_i\,, \qquad [\mathsf{K}_i,\mathsf{K}_j] = -\,i\,\mathsf{J}_{ij}\,, \qquad [\mathsf{P}_i,\mathsf{K}_j] = i\,\mathsf{P}_0\,\delta_{ij}$$

$$[\mathsf{P}_0,\mathsf{P}_i] = i\,\Lambda\,\mathsf{K}_i\,, \qquad [\mathsf{P}_i,\mathsf{P}_j] = i\,\Lambda\,\mathsf{J}_{ij}\,,$$

where the sign of the cosmological constant is

- $\Lambda > 0$ for de Sitter isometry algebra $\mathfrak{o}(d+1,1)$,
- $\Lambda = 0$ for Poincaré algebra $\mathfrak{io}(d,1)$,
- $\Lambda < 0$ for anti de Sitter isometry algebra $\mathfrak{o}(d,2)$.

Poincaré vs (anti) de Sitter algebras

Introducing the cosmological time τ, the speed of light c and the rest mass M, the nontrivial commutators (besides the one involving the rotation generators) of the trivial central extension by the generator M via

$$\mathsf{P}_0 = \mathsf{M}c^2 + \mathsf{H}$$

are

$$[\mathsf{H}, \mathsf{K}_i] = i\,\mathsf{P}_i\,, \qquad [\mathsf{K}_i, \mathsf{K}_j] = -\,i\,\frac{k}{c^2}\,\mathsf{J}_{ij}\,, \qquad [\mathsf{P}_i, \mathsf{K}_j] = i\,(\mathsf{M} + \frac{1}{c^2}\,\mathsf{H})\,\delta_{ij}$$

$$[\mathsf{H}, \mathsf{P}_i] = i\,\frac{k}{\tau^2}\,\mathsf{K}_i\,, \qquad [\mathsf{P}_i, \mathsf{P}_j] = i\,\frac{k}{c^2\tau^2}\,\mathsf{J}_{ij}\,,$$

where the sign k of the cosmological constant k is

- $+1$ for de Sitter isometry algebra $\mathfrak{o}(d+1,1)$,
- 0 for Poincaré algebra $\mathfrak{io}(d,1)$,
- -1 for anti de Sitter isometry algebra $\mathfrak{o}(d,2)$.

Galilei vs Newton-Hooke algebras

Fixing the cosmological time τ and sending the speed of light c to infinity, the nontrivial commutators (besides the one involving the rotation generators) are

$$[\mathsf{H},\mathsf{K}_i] = i\,\mathsf{P}_i\,, \qquad [\mathsf{H},\mathsf{P}_i] = i\,\frac{k}{\tau^2}\,\mathsf{K}_i\,, \qquad [\mathsf{P}_i,\mathsf{K}_j] = i\,\mathsf{M}\,\delta_{ij}$$

where the sign k of the cosmological constant is

- $+1$ for the central extension of the expanding Newton-Hooke algebra $\mathfrak{nh}_+(d)$,
- 0 for the central extension of the Galilei algebra $\mathfrak{gal}(d) = \mathfrak{nh}_0(d)$, called the Bargmann algebra $\mathfrak{bar}(d)$,
- -1 for the central extension of the oscillating Newton-Hooke algebra $\mathfrak{nh}_-(d)$.

Commuting diagram

Contraction diagram

	Cosmological		**Flat**
Relativistic	$\mathfrak{o}(d,2)$	$\xrightarrow{\tau\to\infty}$	$\mathfrak{io}(d,1)$
	$c\to\infty\downarrow$		$\downarrow c\to\infty$
Nonrelativistic	$\mathfrak{nh}_-(d)$	$\xrightarrow{\tau\to\infty}$	$\mathfrak{gal}(d)$

Fundamental representation of the Galilei group

The Galilei group $Gal(d)$ acts on the spatial coordinates $\mathbf{x}$ and time t as

$$(t, \mathbf{x}) \rightarrow g(t, \mathbf{x}) = (t + \beta, \mathscr{R}\mathbf{x} + \mathbf{v}t + \mathbf{a}),$$

where $\beta \in \mathbb{R}$; $\mathbf{v}, \mathbf{a} \in \mathbb{R}^d$ and $\mathscr{R} \in O(d)$.

Standard realization of Galilei algebra

Galilei algebra: $\mathfrak{gal}(d) = \mathbb{R}^{2d} \ni \left(\mathfrak{o}(d) \oplus \mathbb{R}\right)$

$\mathbb{R}^{2d}$: Space-translation and Galilean boost

$$\hat{P}_i = -i\partial_i, \qquad \hat{K}_i = it\partial_i,$$

$\mathfrak{o}(d)$: Rotation

$$\hat{J}_{ij} = -i(x_i\partial_j - x_j\partial_i),$$

$\mathbb{R}$: Time-translation

$$\hat{P}_t = i\partial_t.$$

Schrödinger equation for a free particle

In quantum mechanics, the Galilei group $Gal(d)$ acts by projective representations on the Hilbert space of solutions to the Schrödinger equation when the potential is space and time translation invariant.
For a single particle such a potential must be constant and is sometimes called the internal energy U:

$$i\,\partial_t\psi(t,\mathbf{x}) = (-\frac{\Delta}{2m} + U)\,\psi(t,\mathbf{x})\,,$$

so the particle is free.

Schrödinger equation for a free particle

The *projective* representation is

$$\psi(t,\mathbf{x}) \to \gamma\big(g(t,\mathbf{x})\big)\,\psi\big(g^{-1}(t,\mathbf{x})\big)\,,$$

where $\gamma \in U(1)$, e.g. under a pure Galilei boost $g_{\mathbf{v}}$

$$\psi(t,\mathbf{x}) \to \exp\left[-\frac{im}{2}(\mathbf{v}^2 t - 2\,\mathbf{v}\cdot\mathbf{x})\right]\psi\big(g_{\mathbf{v}}^{-1}(t,\mathbf{x})\big).$$

The presence of the mass-dependent phase factor in the transformation law implies a superselection rule forbidding the superposition of states of different masses, known as the Bargmann superselection rule.

Schrödinger equation for a free particle

The *projective* representation is

$$\psi(t,\mathbf{x}) \to \gamma\big(g(t,\mathbf{x})\big)\,\psi\big(g^{-1}(t,\mathbf{x})\big)\,,$$

where $\gamma \in U(1)$, e.g. under a pure Galilei boost $g_{\mathbf{v}}$ as

$$\psi(t,\mathbf{x}) \to \exp\left[-\frac{im}{2}(\mathbf{v}^2 t - 2\,\mathbf{v}\cdot\mathbf{x})\right]\psi\big(g_{\mathbf{v}}^{-1}(t,\mathbf{x})\big).$$

By enlarging the Galilei group $Gal(d)$ to the Bargmann group $Bar(d)$, the representation becomes *unitary*.

Standard realization of Bargmann algebra

Bargmann algebra: $\mathfrak{bar}(d) = \mathfrak{h}_d \ni (\mathfrak{o}(d) \oplus \mathbb{R})$

$\mathfrak{h}_d$: Space-translation, Galilean boost and mass

$$\hat{P}_i = -i\partial_i, \qquad \hat{K}_i = mx_i + it\partial_i, \qquad \hat{M} = m,$$

$\mathfrak{o}(d)$: Rotation

$$\hat{J}_{ij} = -i(x_i\partial_j - x_j\partial_i),$$

$\mathbb{R}$: Time-translation

$$\hat{P}_t = i\partial_t.$$

Standard realization of Newton-Hooke algebras

Newton-Hooke algebras: $\mathfrak{nh}(d) = \mathbb{R}^{2d} \ni \left(\mathfrak{o}(d) \oplus \mathbb{R}\right)$

$\mathbb{R}^{2d}$: Space-translation and inertial transformation

$$\hat{P}_i = -i\cosh\left(\sqrt{k}\,\frac{t}{\tau}\right)\partial_i, \qquad \hat{K}_i = i\,\frac{\tau}{\sqrt{k}}\sinh\left(\sqrt{k}\,\frac{t}{\tau}\right)\partial_i,$$

$\mathfrak{o}(d)$: Rotation

$$\hat{J}_{ij} = -i(x_i\partial_j - x_j\partial_i),$$

$\mathbb{R}$: Time-translation

$$\hat{P}_t = i\partial_t.$$

Schrödinger equation for a harmonic oscillator

In quantum mechanics, the Newton-Hooke group $NH(d)$ acts by projective representations on the Hilbert space of solutions to the Schrödinger equation for a harmonic oscillator:

$$i\,\partial_t\psi(t,\mathbf{x}) = \left(\frac{1}{2m}(-\Delta - \frac{k}{\tau^2}\,|\mathbf{x}|^2) + U\right)\,\psi(t,\mathbf{x})\,.$$

Geometrical digression

In the nonrelativistic limit $c \to \infty$, the metric and its inverse become degenerate

$$-\frac{1}{c^2}\,\eta_{\mu\nu}dx^\mu dx^\nu \to (dt)^2\,, \qquad \eta^{\mu\nu}\partial_\mu\partial_\nu \to \delta^{ij}\partial_i\partial_j\,,$$

which defines the two "metrics" (for time and space) of the Galilei space-time, the flat Newtonian space-time.

The Galilei and Newton-Hooke transformations actually preserve the time interval dt and also the spatial metric $\delta_{ij}dx^i dx^j$ on each simultaneity leaf (i.e. for $dt = 0$).

The Newton-Hooke space-times are Newtonian space-times endowed with the same metrics but different (not flat) torsionless affine connection

$$\Gamma^i_{00} = -\frac{k}{\tau^2}\,x^i$$

equal to minus the gravitational force experienced by a free particle (i.e. in free fall).

Definition (Negro, del Olmo & Rodriguez-Marco, 1997)

A nonrelativistic conformal algebra is a Lie algebra such that

(i) a nonrelativistic kinematical algebra is a proper subalgebra,

(ii) space and time reversal Π and T are automorphisms,

(iii) it admits a faithful vector-field realization such that the conformal equivalence classes (i.e. modulo conformal factors) of the time interval dt and the space metric $\delta_{ij}dx^i dx^j$ on each simultaneity leaf are preserved.

Classification (Negro, del Olmo & Rodriguez-Marco, 1997)

Theorem

For space of dimension $d \geqslant 3$, there is a countable class of finite-dimensional nonrelativistic conformal algebras: for a given positive integer $2\ell > 0$, there is only one inequivalent ***nonrelativistic ℓ-conformal (Galilei ⇔ Newton-Hooke) algebra***

$$\mathfrak{cgal}_{2\ell}(d) \cong \mathfrak{cnh}^{\pm}_{2\ell}(d)$$

such that $(dt)^{2\ell}$ and the space metric $\delta_{ij}dx^i dx^j$ transform with the same conformal factor. The inverse $z = 1/\ell$ is called the ***dynamical exponent*** *of the nonrelativistic ℓ-conformal algebra.*
Any (half)integer-conformal Galilei algebras admit only (one non)trivial central extension.

The ℓ-conformal Galilei algebra $\mathfrak{cgal}_{2\ell}(d)$ can also be obtained as the Inönu-Wigner contraction of the ℓ-conformal Newton-Hooke algebra $\mathfrak{cnh}_{\pm,\ell}(d)$.

Standard realization of ℓ-conformal Galilei algebra

ℓ-conformal Galilei algebra: $\mathfrak{cgal}_{2\ell}(d) = \mathbb{R}^{(2\ell+1)d} \ni \big(\mathfrak{o}(d) \oplus \mathfrak{sp}(2,\mathbb{R})\big)$

$\mathbb{R}^{(2\ell+1)d}$: Space-translation, Galilean boost and acceleration

$$\hat{C}_i^{(n)} = -i\, t^n \partial_i , \qquad (n = 0, 1, \ldots, 2\ell)$$

$\mathfrak{o}(d)$: Rotation

$$\hat{J}_{ij} = -i(x_i \partial_j - x_j \partial_i),$$

$\mathfrak{sp}(2,\mathbb{R})$: Time-translation, scale transformation and expansion

$$\hat{P}_t = i\partial_t,$$

$$\hat{D} = i\left(t\,\partial_t + \ell\, x^i \partial_i \right),$$

$$\hat{C} = i\left(t^2 \partial_t + 2\ell\, t x^i \partial_i \right).$$

Structure of ℓ-conformal Galilei algebra

ℓ-conformal Galilei algebra: $\mathfrak{cgal}_{2\ell}(d) = \mathbb{R}^{(2\ell+1)d} \ni \big(\mathfrak{o}(d) \oplus \mathfrak{sp}(2,\mathbb{R})\big)$

The adjoint action of the subalgebra $\mathfrak{o}(d) = \text{span}_{\mathbb{R}}\{\mathsf{J}_{ij}\}$ decomposes into the sum of the irreducible

- trivial representation on the module spanned by H, D or C:

$$[\mathsf{J},\mathsf{H}] = 0\,, \qquad [\mathsf{J},\mathsf{D}] = 0\,, \qquad [\mathsf{J},\mathsf{C}] = 0\,,$$

- fundamental representation on the module spanned by the $\mathsf{C}_i^{(n)}$ for fixed integer n:

$$[\mathsf{J},\mathsf{C}^{(n)}] \sim \mathsf{C}^{(n)}\,,$$

- adjoint representation on the module spanned by the J_{ij}:

$$[\mathsf{J},\mathsf{J}] \sim \mathsf{J}\,,$$

Structure of ℓ-conformal Galilei algebra

ℓ-conformal Galilei algebra: $\mathfrak{cgal}_{2\ell}(d) = \mathbb{R}^{(2\ell+1)d} \ni \big(\mathfrak{o}(d) \oplus \mathfrak{sp}(2,\mathbb{R})\big)$

The adjoint action of the subalgebra $\mathfrak{sp}(2,\mathbb{R}) = \text{span}_{\mathbb{R}}\{\mathsf{H},\mathsf{D},\mathsf{C}\}$ decomposes into the sum of the irreducible

- trivial representation on the module spanned by the J_{ij}:

$$[\mathsf{J},\mathsf{H}] = 0\,, \qquad [\mathsf{J},\mathsf{D}] = 0\,, \qquad [\mathsf{J},\mathsf{C}] = 0\,,$$

- spin-ℓ representation on the module spanned by the $\mathsf{C}_i^{(n)}$ for fixed integer i:

$$[\mathsf{D},\mathsf{C}_i^{(n)}] = (n-\ell)\mathsf{C}_i^{(n)}\,, \qquad [\mathsf{H},\mathsf{C}_i^{(n)}] \sim \mathsf{C}_i^{(n-1)}\,, \qquad [\mathsf{C},\mathsf{C}_i^{(n)}] \sim \mathsf{C}_i^{(n+1)}\,.$$

- adjoint representation on the subalgebra $\mathfrak{sp}(2,\mathbb{R})$ itself:

$$[\mathsf{D},\mathsf{H}] \sim \mathsf{H}\,, \qquad [\mathsf{D},\mathsf{C}] \sim \mathsf{C}\,, \qquad [\mathsf{H},\mathsf{C}] \sim \mathsf{D}\,.$$

Standard realization of conformal Galilei algebra

1-conformal Galilei algebra: $\mathfrak{cgal}_2(d) = \mathbb{R}^{3d} \ni \big(\mathfrak{o}(d) \oplus \mathfrak{sp}(2,\mathbb{R})\big)$

$\mathbb{R}^{3d}$: Space-translation, Galilean boost and acceleration

$$\hat{C}_i^{(0)} = -i\partial_i = \hat{P}_i, \quad \hat{C}_i^{(1)} = -it\partial_i = -\hat{K}_i, \quad \hat{C}_i^{(2)} = -i\,t^2\partial_i = \hat{C}_i,$$

$\mathfrak{o}(d)$: Rotation

$$\hat{J}_{ij} = -i(x_i\partial_j - x_j\partial_i),$$

$\mathfrak{sp}(2,\mathbb{R})$: Time-translation, scale transformation and expansion

$$\hat{P}_t = i\partial_t,$$
$$\hat{D} = i\left(t\,\partial_t + x^i\partial_i\right),$$
$$\hat{C} = i\left(t^2\partial_t + 2\,tx^i\partial_i\right).$$

Conformal Galilei algebra

Usually, the 1-conformal Galilei algebra $\mathfrak{cgal}_2(d)$ is simply called the conformal Galilei algebra $\mathfrak{cgal}(d)$ because it corresponds to a dynamical exponent $z = 1$ as in relativistic physics and it is indeed the Inönu-Wigner contraction of the relativistic conformal algebra

$$\mathfrak{o}(d+1,2) \stackrel{c\to\infty}{\longrightarrow} \mathfrak{cgal}(d)$$

with the following identification for the generators of

- Space translations: P_i,
- Time translation: $\mathsf{P}_0 \sim \frac{1}{c}\,\mathsf{H}$,
- Rotations: J_{ij},
- (Lorentz $\to$ Galilei) boosts: $\mathsf{J}_{0i} \sim c\,\mathsf{K}_i$,
- Dilation: D,
- Spacelike conformal boosts $\to$ accelerations: $\mathsf{S}_i \sim c^2\,\mathsf{C}_i$,
- Timelike conformal boost $\to$ expansion: $\mathsf{S}_0 \sim c\,\mathsf{C}$.

The conformal Galilei algebra $\mathfrak{cgal}(d)$ only admit trivial central extensions. Heuristically, this comes from the fact that the relativistic conformal algebra is a symmetry of massless particles only.

Standard realization of $\frac{1}{2}$-conformal Galilei algebra

$\frac{1}{2}$-conformal Galilei algebra: $\mathfrak{cgal}_1(d) = \mathbb{R}^{2d} \ni \left(\mathfrak{o}(d) \oplus \mathfrak{sp}(2,\mathbb{R})\right)$

$\mathbb{R}^{2d}$: Space-translation and Galilean boost

$$\hat{C}_i^{(0)} = -i\partial_i = \hat{P}_i, \qquad \hat{C}_i^{(1)} = -it\partial_i = -\hat{K}_i,$$

$\mathfrak{o}(d)$: Rotation

$$\hat{J}_{ij} = -i(x_i\partial_j - x_j\partial_i),$$

$\mathfrak{sp}(2,\mathbb{R})$: Time-translation, scale transformation and expansion

$$\hat{P}_t = i\partial_t,$$

$$\hat{D} = i\left(2t\,\partial_t + x^i\partial_i\right),$$

$$\hat{C} = i\left(t^2\partial_t + tx^i\partial_i\right).$$

Schrödinger algebra

The central extension of the $\frac{1}{2}$-conformal Galilei algebra $\mathfrak{cgal}_1(d)$ is called the Schrödinger algebra $\mathfrak{sch}(d)$ because it corresponds to a dynamical exponent $z = 2$ characteristic of nonrelativistic particles and it is indeed the symmetry algebra of the Schrödinger equation for a free particle (with zero internal energy).

Standard realization of Schrödinger algebra

Schrödinger algebra: $\mathfrak{sch}(d) = \mathfrak{h}_d \ni \big(\mathfrak{o}(d) \oplus \mathfrak{sp}(2,\mathbb{R})\big)$

$\mathfrak{h}_d$: Time-translation, Galilei boost and mass

$$\hat{P}_i = -i\partial_i, \qquad \hat{K}_i = m x_i + i t \partial_i, \qquad \hat{M} = m,$$

$\mathfrak{o}(d)$: Rotation

$$\hat{M}_{ij} = -i(x_i\partial_j - x_j\partial_i),$$

$\mathfrak{sp}(2,\mathbb{R})$: Time-translation, scale transformation and expansion

$$\hat{P}_t = i\partial_t,$$

$$\hat{D} = i\left(2\,t\,\partial_t + x^i\partial_i + \frac{d}{2}\right),$$

$$\hat{C} = i\left(t^2\partial_t + t\left(x^i\partial_i + \frac{d}{2}\right)\right) + \frac{m}{2}\,x^2.$$

What are non-relativistic singletons?

Group-theoretical definitions

Free relativistic singleton
UIR of the Poincaré algebra $\mathfrak{io}(d,1)$ that can be lifted to a UIR of the conformal algebra $\mathfrak{o}(d+1,2)$.
⇔ Helicity representation labeled by zero mass and by spin (Angelopoulos, Flato, Fronsdal, Sternheimer, 1980).

Free non-relativistic singleton
UIR of the Bargmann algebra $\mathfrak{bar}(d)$ that can be lifted to a UIR of the Schrödinger algebra $\mathfrak{sch}(d)$.
⇔ Massive representations labeled by zero internal energy and by spin (Perroud, 1977).

In other words, the free non-relativistic singletons can be identified with the solutions of the free Schrödinger equation with zero internal energy

$$\left(i\partial_t + \frac{\Delta}{2m}\right)\psi(t,\mathbf{x}) = 0 \qquad \text{(Hagen-Niederer, 1972)}$$

Schrödinger algebra

Schrödinger algebra: $\mathfrak{sch}(d) = \mathfrak{h}_d \niplus \big(\mathfrak{o}(d) \oplus \mathfrak{sp}(2,\mathbb{R})\big)$

Standard representation as order-one differential operators acting on wave functions $\psi(t,\mathbf{x})$

Standard realization

Schrödinger algebra: $\mathfrak{sch}(d) = \mathfrak{h}_d \ni \big(\mathfrak{o}(d) \oplus \mathfrak{sp}(2,\mathbb{R})\big)$

$\mathfrak{h}_d$: Time-translation, Galilei boost and mass

$$\hat{P}_i = -i\partial_i, \qquad \hat{K}_i = m x_i + i t \partial_i, \qquad \hat{m} = m,$$

$\mathfrak{o}(d)$: Rotation

$$\hat{M}_{ij} = -i(x_i\partial_j - x_j\partial_i),$$

$\mathfrak{sp}(2,\mathbb{R})$: Time-translation, scale transformation and expansion

$$\hat{P}_t = i\partial_t,$$

$$\hat{D} = i\left(2\,t\,\partial_t + x^i\partial_i + \frac{d}{2}\right),$$

$$\hat{C} = i\left(t^2\partial_t + t\Big(x^i\partial_i + \frac{d}{2}\Big)\right) + \frac{m}{2}\,x^2.$$

Standard realization

Schrödinger algebra: $\mathfrak{sch}(d) = \mathfrak{h}_d \ni \big(\mathfrak{o}(d) \oplus \mathfrak{sp}(2,\mathbb{R})\big)$

Observation: (M. Valenzuela, 2009) Alternative representation as degree-two polynomials in the momenta and Galilean boost generators acting on wave functions solutions of free Schrödinger equation $(i\partial_t + \frac{\Delta}{2m})\psi(t,\mathbf{x}) = 0$.

Standard realization

Schrödinger algebra: $\mathfrak{sch}(d) = \mathfrak{h}_d \ni \big(\mathfrak{o}(d) \oplus \mathfrak{sp}(2,\mathbb{R})\big)$

$\mathfrak{h}_d$:

$$\hat{P}_i = \hat{P}_i(-t), \qquad \hat{K}_i = m\hat{X}_i(-t), \qquad \hat{m} = m,$$

$\mathfrak{o}(d)$:

$$\hat{M}_{ij} = \hat{X}^i(-t)\hat{P}^j(-t) - \hat{X}^j(-t)\hat{P}^i(-t),$$

$\mathfrak{sp}(2,\mathbb{R})$:

$$\hat{H} = \frac{\hat{P}^2(-t)}{2m},$$

$$\hat{D} = -\hat{X}^i(-t)\hat{P}_i(-t) + \frac{d}{2},$$

$$\hat{C} = \frac{m}{2}\,\hat{X}^2(-t).$$

Standard realization

Schrödinger algebra: $\mathfrak{sch}(d) = \mathfrak{h}_d \ni \big(\mathfrak{o}(d) \oplus \mathfrak{sp}(2,\mathbb{R})\big)$

Observation: If one changes the Hamiltonian $\hat{H} = \frac{\hat{P}^2}{2m}$ to

$$\hat{H}' = \hat{H} - \frac{k}{\tau^2}\hat{C} = \frac{1}{2m}(\hat{P}^2 - \frac{k}{\tau^2}\hat{X}^2)$$

and change accordingly the time dependence of $\hat{X}^i(-t)$ and $\hat{P}_j(-t)$ then the former representation of $\mathfrak{sch}(d)$ as degree-two polynomials in the momenta and Galilean boost generators produce a Newton-Hooke realization of $\mathfrak{sch}(d)$ on wave functions solutions of Schrödinger equation for a harmonic oscillator with zero internal energy

$$\left(i\partial_t + \frac{1}{2m}(\Delta + \frac{k}{\tau^2}\,|\mathbf{x}|^2)\right)\psi(t,\mathbf{x}) = 0\,, \qquad \text{(Niederer, 1972).}$$

Light-like dimensional reduction

Main idea behind the light-like dimensional reduction:

The kinetic operator of a relativistic theory

$$\Box - M^2 = -2\partial_+\partial_- + \Delta - M^2$$

when acting on eigenmodes of a light-like component of the momentum,

$$\Psi(x) = e^{-imx^-}\psi(x^+, x^i),$$

is proportional to the kinetic Schrödinger operator of a non-relativistic theory

$$i\partial_t + \Delta/2m + \mu$$

via the identification $x^+ = t$ and $M^2 = -\mu/2m$.

Light-like dimensional reduction

Main idea behind the light-like dimensional reduction:

(Group theory) The quadratic Casimir operators of the Poincaré and the Bargmann algebras are related

$$\hat{P}^\mu \hat{P}_\mu/2 = -\hat{P}_+\hat{P}_- + \hat{P}^i\hat{P}_i/2 = -\hat{m}\hat{P}_t + \hat{P}^i\hat{P}_i/2$$

upon the standard light-cone identification of the non-relativistic mass and Hamiltonian operators

$$\hat{P}_+ = \hat{m}\,, \qquad \hat{P}_- = \hat{P}_t\,.$$

The Bargmann (Schrödinger) algebra is isomorphic to the subalgebra of the Poincaré (conformal) algebra that commutes with $\hat{P}_+ = \hat{m}$. [Gomis and Pons, 1978 (Burdet, Perrin and Sorba, 1973)]