Chacon maps revisited

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Outline

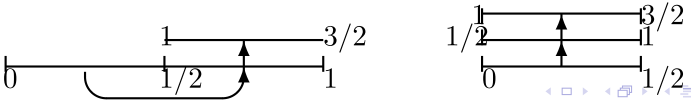
- Historical review
- Some Classical results from Chacon maps 3 to Chacon maps 2
- Our contribution and opens questions

Chacon map 2

Chacon introduce what it is nowadays called Chacon map 2 in his fameus 1967's paper : "Weakly mixing transformations which are not strongly mixing. Proc. Amer. Math. Soc. 22 1969, 559–562." The main goal of this construction is to give a concret and simple example which illustrates the Halmos-Rohklin Theorem .

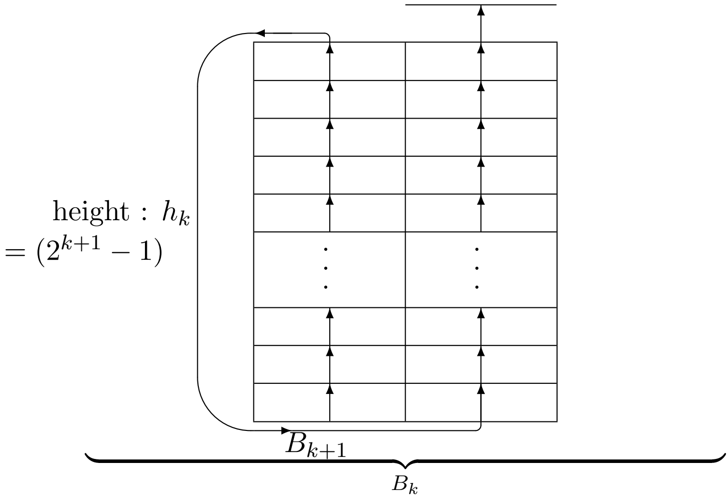
Halmos-Rohklin Theorem asserts that the weak mixing is generic but the mixing is meager.

Chacon map 2 is given by the cutting and stacking methods as follows



Chacon map 2

At the stage n



$\longrightarrow$ Basic well known properties of Chacon map 2

- Ergodicity: follows for the standard martingale argument, for any $f \in L^2(X)$, we have

$$\mathbb{E}(f|_{(T^iB_n)_{i=0}^{h_n-1}}) \xrightarrow[n\to\infty]{} f$$

- Weak mixing without mixing : is a consequence of the weak convergence of T^{h_n}. Precisely, for any $f \in L^2(X)$, we have

$$< U_T^{h_n} f, f > \xrightarrow[n\to\infty]{} \sum_{k=0}^{+\infty} \frac{1}{2^{k+1}} < U_T^k f, f >$$

Where U_T is a Koopman operator $f \mapsto f \circ T^{-1}$. Therefore if $\lambda \in \mathbb{T}$ is an eigenvalue then we have

$$\lambda^{h_n} \xrightarrow[n\to\infty]{} \sum_{k=0}^{+\infty} \frac{1}{2^{k+1}} \lambda^k = \frac{1}{2-\lambda}$$

- Lightly mixing: No other investigation on Chacon map 2 until 1991. Indeed, in 1991 N. Friedman and J. King established that Chacon map 2 can be considered as a concrete example to answer J. King question on the existence of rank one construction with lightly mixing and without mixing.

Definition

Let T be an invertible measure preserving map of probability space $(X, \mathcal{B}, \mu)$. T is lightly mixing if for all $A, B \in \mathcal{B}$ of positive measure we have

$$\liminf_{n \longrightarrow +\infty} \mu(T^n A \cap B) > 0.$$

The definition is inspired from the paper of England and Martin (1968) B.A.M.S., 505-507.

N. Friedman and J. King proved

Theorem (N. Friedman and J. King)

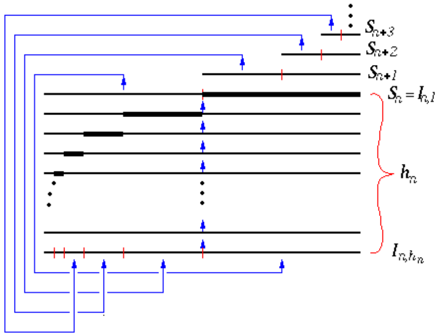
Chacon map 2 is lightly mixing i.e. for any $A, B \in \mathcal{B}$ of positive measure we have

$$\liminf_{n \longrightarrow +\infty} \mu(T^n A \cap B) > 0.$$

The proof of Friedman-King Theorem is based on the fact that

$$< U_T^{h_n} f, f > \xrightarrow[n \to \infty]{} \sum_{k=0}^{+\infty} \frac{1}{2^{k+1}} . < U_T^k f, f >$$

and the idea of the proof is here



with the symbolic language

Using the symbolic language the dynamic of the Chacon maps 2 can be describe by the following

$$\begin{aligned}
B_{n+1} &= B_n B_n s \\
B_{n+2} &= B_{n+1} B_{n+1} s = B_n B_n s B_n B_n ss \\
B_{n+3} &= B_n B_n s B_n B_n ss B_n B_n s B_n B_n sss \\
B_{n+4} &= B_n B_n s B_n B_n ss B_n B_n s B_n B_n sss B_n B_n s B_n B_n ss B_n B_n s B_n B_n ssss
\end{aligned}$$

The red color tell us that T and T^{-1} are isomorphic since the word are symmetric.

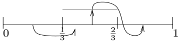
⟶ What about the other properties

Since then, the proprieties of Chacon maps 2 will be deduced from the general properties proved for rank one maps unless from the properties proved for the Chacon maps 3 until we discover recently in our work that these two maps are disjoint in the sens of Furstenberg which implies that they are non-isomophic. This insure that those two maps can be maked in some king of competition in order to be the preferable contre-example for some classical questions in ergodic theory. Indeed, the intereset in Chacon map 3 come exactly from the fact that this maps can be used to construct a contre-exemples much simpler in the classical ergodic theory.

Chacon map 3

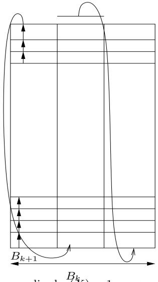
Chacon map 3 is inspired by two papers of Chacon published in 1966 and 1969 :
"Transformations having continuous spectrum", J. Math. Meca., 1966 and "Approximation of transformations with continuous spectrum", Pacific J. Math.,1969.

First Stage :



Let B_k be the base of the $k-$th tower. Thus, we have

At k-t stage :



The measure of space is renormalised $\mu(X) = 1$.

Properties of Chacon map 3

The main goal behind introducing Chacon map 3 is to anwser to Halmos question.

Question (4 of Halmos)

Does every transformation with continuous spectrum have a square root? (page 97 of Ergodic Halmos book.)

The construction done in those two papers of Chacon gives a negative anwser to Halmos question.
Since then, The construction has been simplified and widely popularized by Friedman,del Junco, Rudolph, Petersen and many authors giving what it's nowadays known as Chacon map.
In contrary to Chacon map 2, the properties of Chacon map 3 has been intensively investigated by many authors leading to many nice theorems and proofs.

- Ergodicity: follows as for chacon map 2.
- Weak mixing without mixing : is a consequence of the weak convergence of T^{h_n}. Precisely, for any $f \in L^2(X)$, we have

$$< U_T^{h_n} f, f > \xrightarrow[n \to \infty]{} \frac{1}{2}(< f, f > + < U_T f, f >) \qquad (1)$$

Therefore Chacon map 3 is $\frac{1}{2}$-rigid. This property insure that the spectrum of Chacon map 3 is singular.
This fact can be connected and add to the following well known Baxter theorem showing that there exists a rank one maps with singular spectrum. Indeed, Baxter prove
- Baxter Theorem : Every transformation preserving measure α-rigid with $\alpha > \frac{1}{2}$ has a singular spectrum.
Ryzhikov extended this result by proving
- Ryzhikov Theorem : Every transformation preserving measure α-rigid with $\alpha > \frac{1}{2}$ is spectraly disjoint from any mixing map.

We deduce also from (1) the following

Lemma

Chacon map 3 is not lightly mixing.

Proof. Take $B = A^c \cap TA^c$.

Therefore

Theorem

Chacon maps 2 and 3 are not isomorphic.

We have more by del Junco-Rudolph Theorem

Theorem

Chacon maps 2 and 3 are disjoint in the sens of Furstenberg.

In fact, by del Junco-Rahe-Swanson

Theorem (del Junco-Rahe-Swanson,1979)

Chacon map 3 has a minimal self joining of any order.

As Corollary we have

Corollary (del Junco)

The centralizer of Chacon map 3 is trivial and Chacon map 3 has no root.

We shall show that The same results holds for Chacon map 2.

The proof of del Junco-Rahe-Swanson Theorem is based on the fact that any non diagonal joining λ is $id \times T$ invariant which implies that λ is the product since the identity is disjoint from any ergodic map. the same proof yields

Theorem (Fieldsteel)

Chacon map 3 is not isomorphic to its inverse.

and by our works

Theorem (E.A, Lemańczyk and de la Rue)

Chacon map 2 has a minimal self joining of any order.

As Corollary we obtain the following extention of del Junco Theorem on the centralizer of Chacon map 3.

Corollary

The centralizer of Chacon map 2 is trivial and Chacon map 2 has no root.

Von Neumann uniform ergodicity of the powers :

In 1992 Adams and Friedman explored a news properties of Chacon map 3. There main goal is to construct a straicase mixing maps. For that the authors introduced the notion of Von Neumann uniform ergodicity of the powers.

Definition

The transformation preserving measure T is Von Neumann uniformly ergodic over its powers if for any measurable set A we have

$$\left\| \frac{1}{N} \sum_{j=1}^{N} \mathbb{1}_A(T^{jk}x) - \mu(A) \right\|_1 \xrightarrow[N\to\infty]{} 0,$$

uniformly for any $k \neq 0$.

This property implies that T is mildly mixing (T is mildly mixing if the only rigid functions is constant functions).

Adams and Friedman proved

Theorem (Adams and Friedman,1992)

Chacon map 3 is Von Neumann uniformly ergodic over its powers.

Chacon map 3 is used to produce also a sequence of low-discrepancy (a discrepancy of the sequence $x = (x_i)_{i\in\mathbb{N}}$ is given by $D_N(x) = \sup_{J \text{interval} \subset \mathbb{T}} \left| \frac{1}{N} \sum_{k=1}^{N} \mathbb{1}_J(x_i) - \lambda(J) \right|$). Van de Coprut sequence (which is a orbite of odometer) is known to have a low-discrepancy which means that ($D_N(x) \leq C.\frac{\log(N)}{N}$, C is a positive constant). Bob Burton and Aimée obtain the following

Theorem (Burton-Aimée,2003)

The orbit of generic points in Chacon map 3 has a low-discrepancy.

Spectral properties and Riesz Products Business

Ageev-Ryzhikov, Parreau-Lemańczyk Result:
In 1999 Ageev, Ryzhikov, Parreau and Lemańczyk proved independently that the spectral type σ of Chacon map 3 satisfy the convolution singularity which means that $\sigma * \sigma$ and σ are singular. This gives a answer by Ageev and Ryzhikov to the Rohklin question on the genericity of homogneous spectra of mutiplicity two. Ageev, Ryzhikov-Parreau and Lemańczyk result gives also a new examples answering a Kolomogrov question on the nature of spectral type.

Question (Kolomogrov)

Does the spectral type of a transformation preserving measure σ satisfy the group property which means does σ satisfy

$$\sigma * \sigma \approx \sigma?$$

Oseledets, Stepin and Katok-Stepin Theorems

- The first answer was given by Oseledets, Stepin, Katok, Katok and Stepin in 1967. Indeed, on one hand Katok and Stepin construct a two points extension of irrationnal rotation satisfying the convoltions singularity and on other hand they introduce the notion of κ-mixing.

Definition

A map preserving measure T is κ-mixing, $\kappa \in]0,1[$, if there exists a sequence of integers (n_k) such that for any measurable set A

$$\mu(T^{n_k}A \cap A) \xrightarrow[k\to+\infty]{} \kappa\mu(A)^2 + (1-\kappa)\mu(A).$$

- Using the notion of generalized character, it is easy to see that the spectral type of κ-mixing has a strong independent powers which means that $\sigma^{(m)} = \underbrace{\sigma * \sigma * \cdots * \sigma} \perp \sigma^{(n)}$ provided that $n \neq m$.

Prihodk'o-Ryzhikov Theorem

In 2000 Prihodk'o-Ryzhikov establish that the spectral type of Chacon map 3 has a property of the strong independent powers

Theorem (Prihodk'o-Ryzhikov,2000)

Let σ be a spectral type of Chacon map 3. Then, for any $n \neq$ $\sigma^{(m)}$ and $\sigma^{(n)}$ are mutually singular.

The proof is based on the convergence of T^{kh_n} to sequences of polynomials.
In our work by adapting some classical results from the theory of classical Riesz Product we extend Prihodk'o-Ryzhikov theorem to Chacon map 2.

Riesz Products

- Let B_n be the base of the tower as the stage n. Then $B_n = B_{n+1} \cup T^{h_n} B_{n+1}$ and $h_{n+1} = 2h_n + 1$. Therefore $d\sigma_n = \frac{1}{2}|1 + z^{h_n}|^2 d\sigma_{n+1}$, where σ_n is a spectral measure of $\frac{\mathbb{1}_{B_n}}{\sqrt{\mu(B_n)}}$.
  $\rightarrow$ Iterating this property we obtain that the following relation

$$d\sigma_0 = \prod_{n=1}^{N} (1 + \cos(h_n\theta) d\sigma_n,$$

- It follows that the spectral type of chacon map 2 is given by

$$\sigma_0 = W^* \lim \prod_{n=1}^{N} (1 + \cos(h_n\theta) d\theta,$$

Where W^* lim is a limit in the weak* topology.

Techniques

1. We use the following Peyriére criterion adapted in our situation. We stress that the classical Peyriére criterion concerne the sequence 2^n.

2. **Theorem (Peyriére, 1975)**

Let

$$\sigma_a = W^* \lim \prod_{n=1}^{N} \left(1 + \Re\left(a_j e^{ih_n\theta}\right)\right) d\theta,$$

and

$$\sigma_b = W^* \lim \prod_{n=1}^{N} \left(1 + \Re\left(b_j e^{ih_n\theta}\right)\right) d\theta,$$

be two Riesz product with $|a_j| \leq 1$ and $|b_j| \leq 1$. Then

$$\sum_{n\geq 0} |\widehat{\sigma_a}(h_n) - \widehat{\sigma_b}(h_n)| = +\infty \Longrightarrow \sigma_a \perp \sigma_b.$$

1. D-ergodicity in Harmonic Analysis

The ergodicity is usually defined for invariant measures or at least quasi-invariant measures. But, in the context of Riesz product Brown and Moran introduce the following notion of ergodicity.
Let μ a probability measure and D a countable subgroup of $\mathbb{T}$.

Definition

μ is said to be D-ergodic if $\mu(A) \in \{0, 1\}$ for every D-invariant set (i.e. $A + d = A$, for any $d \in D$.)

By Host-Parreau argument we prove the following

- The spectral type σ of Chacon map 2 is not ergodic.
- and is singular with respect to its translation (This gives a positive answer to Choksi-Nadkarni question), i.e. $\sigma \perp \delta_x * \sigma$, for any $x \neq 0$.

2. Strong independence of the powers

We extended Padé Theorem by applying again Peyriére criterion. Indeed, the convolution powers of σ satisfy

$$\sum_{j\geq 0} |(\widehat{\sigma}(h_j))^n - (\widehat{\sigma}(h_j))^m|^2 = +\infty,$$

for any $n \neq m$.

Therefore we have

Theorem (E. A., Lemańczyk and de la Rue)

The spectral type of Chacon map 2 has a property of strong independence of powers.

Idea of the proof of non ergodicity :

We apply the following result of Host-Parreau in our context

ergodicity criterion

Let μ a probability measure on the circle. μ is ergodic if and only if, for any measure $0 < \nu \ll \mu$ and $0 < \nu' \ll \mu$ there exists x such that $\delta_x * \nu$ is not singular to ν'.

$\longrightarrow$ By showing that the group of quasi-invariance of σ is trivial i.e.

$$H(\sigma) = \{x \in \mathbb{T} \quad : \quad \delta_x * \sigma \approx \sigma\} = \{0\}.$$

Von Neuman Uniform ergodicity does'nt implies midly mixing

In connection with our work, it is legitimate to ask on the relation between the Von Neuman Uniform ergodicity and the midly mixing property.
Indeed, it is clear from the definition that Von Neumann uniform ergodicity implies midly mixing.
We have been able to prove that the inverse is not true. Indeed we show more by proving the following

Theorem (E. A., Lemańczyk and de la Rue)

There exist a partiely mixing maps without Von Neumann uniform ergodicity.

The proof is inspired from the work of del Junco-Lemanczyk on the κ-mixing combined with the Adams-Friedman methods.

Open problems

- del Junco-Rahe-Swanson asked if $T \times T$ is Loosely Bernouilli. This questions still open for two Chacon maps. The best known result is the Mentzen-Lemańczyk theorem which say that $T \times \tau$ is not localy rank one provided that τ is aperiodic and $T \times \tau$.
- From our works it is legitimate to ask if Chacon map 2 is Von Neumann uniformly ergodic.
- the discrepancy of the orbit of generic of Chacon map 2.
- What about the ergodicity of the spectral type of Chacon map 3.
- Does the Chacon maps belonging in the class of κ-mixing maps?

Thanks very much.
Spassiba