GLUING STABILITY CONDITIONS

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Stability conditions

Definition. A stability condition σ is given by a pair (Z, P), where $Z : K_0(\mathcal{D}) \to \mathbb{C}$ is a homomorphism from the Grothendieck group $K_0(\mathcal{D})$ of $\mathcal{D}$, and P is a slicing. Such a slicing is given by a collection of subcategories $P(\phi)$ of semistable objects of phase ϕ for each $\phi \in \mathbb{R}$, where $\mathrm{Hom}(P(\phi_1), P(\phi_2)) = 0$ for $\phi_1 > \phi_2$, and $P(\phi)[1] = P(\phi + 1)$. For an object $E \in P(\phi)$ we will use the notation $\phi(E) = \phi$. Similarly to the case of vector bundles, for each object E of $\mathcal{D}$ there should exist a *Harder-Narasimhan filtration* (HN-filtration), i.e., a collection of exact triangles building E from the semistable factors $E_1, \ldots, E_n$ (called the *HN-factors* of E), where $\phi(E_1) > \ldots > \phi(E_n)$ ($E_1 \to E$ is an analog of the subbundle of maximal phase, etc.).

For each interval $I \subset \mathbb{R}$ we denote by $PI \subset \mathcal{D}$ the extension-closed subcategory generated by all the subcategories $P(\phi)$ for $\phi \in I$. For example, $P(0, 1]$ denotes the subcategory corresponding to the interval $(0, 1]$.

If $\sigma = (Z, P)$ is a stabiity condition then $P(0, 1]$ is a heart of a bounded nondegenerate t-structure on $\mathcal{D}$ with $\mathcal{D}^{\leq 0} = P(0, +\infty)$ and $\mathcal{D}^{\geq 0} = P(-\infty, 1]$ (the *heart* of σ).

To give a stability condition is the same as to give a heart $H \subset \mathcal{D}$ together with a homomorphism $Z : K_0(H) \to \mathbb{C}$ such that for every nonzero object $E \in H$ one has either $\Im Z(E) > 0$ or $Z(E) \in \mathbb{R}_{<0}$. These data should satisfy the Harder-Narasimhan property.

A stability condition $\sigma = (Z, P)$ is called *locally finite* if there exists $\eta > 0$ such that for every $\phi \in \mathbb{R}$ the quasi-abelian category $P(\phi - \eta, \phi + \eta)$ is of finite length. Denote by $\mathrm{Stab}(\mathcal{D})$ the space of locally finite stability conditions. It is equipped with a natural topology.

Theorem (Bridgeland) For every connected component $\Sigma \subset \mathrm{Stab}(\mathcal{D})$ there exists a linear subspace $V \subset \mathrm{Hom}(K_0(\mathcal{D}), \mathbb{C})$ and a linear topology on V, such that the projection $\Sigma \to V$ is a local homeomorphism.

Also can consider *numerical* stabilities, by requiring Z to factor through numerical Grothendieck group.

There is a canonical action of $\widetilde{\mathrm{GL}_2^+(\mathbb{R})}$ on $\mathrm{Stab}(\mathcal{D})$, and the commuting action of $\mathrm{Auteq}(\mathcal{D})$. The first action is compatible with the natural action of $\mathrm{GL}_2^+(\mathbb{R})$ on the set of central charges.

Example: action of $\mathbb{R} \subset \widetilde{\mathrm{GL}_2^+(\mathbb{R})}$ covering rotations $S^1 \subset \mathrm{GL}_2^+(\mathbb{R})$ For $a \in \mathbb{R}$ and $\sigma = (Z, P)$ one has $R_a\sigma = (r_{-\pi a} \circ Z, P')$, where $P'(t) = P(t + a)$, $r_{-\pi a}$ is the rotation in $\mathbb{C} = \mathbb{R}^2$ through the angle $-\pi a$. For $a \in (0, 1)$ the new heart $P'(0, 1] = P(a, a + 1]$ is obtained from $P(0,1]$ by tilting with respect to the torsion pair $(P(a,1],P(0,a])$ (recall the def.). In the case of elliptic curve, categories we get in this way from standard stability can be interpreted as hol. bundles on nc torus.

Examples. 1. Derived category of a curve of genus ≥ 1. Then have Mumford's stability σ_{st} with $Z(E)=-d(E)+ir(E)$. In this case the action of $\widetilde{\mathrm{GL}_2^+}(\mathbb{R})$ is transitive (Macri). In the case of $\mathbb{P}^1$ the space Stab is also described: it has "ends" corresponding to open subsets where $\mathcal{O}(n)$ and $\mathcal{O}(n+1)$ are semistable. Common intersection is the orbit of the standard one.

2. For K3 surfaces Bridgeland constructed stability with $Z(E)=(\exp(\beta+i\omega),v(E))$ (where ω is in the ample cone) and described a connected component in Stab.

Lemma: suppose have two stabilities with the same charge such that $P_1(0,1]\subset P_2(-1,2]$. Then they are the same.

Proof: observe that the condition is symmetric. Given $E\in P_1(0,1]$, there is an exact triangle

$$F\to E\to G\to F[1]$$

with $F\in P_2(1,2]$ and $G\in P_2(-1,1]$. Observe that $F\in P_1(>0)$ and $G\in P_1(\leq 2)$. Since F is an extension of E by $G[-1]$, we derive that $F\in P_1(0,1]$. But the intersection $P_1(0,1]\cap P_2(1,2]$ is trivial (since $Z_1=Z_2$), so $F=0$. This proves that $E\in P_2(-1,1]$.

Similarly, considering an exact triangle

$$F\to E\to G\to F[1]$$

with $F\in P_2(0,1]$ and $G\in P_2(-1,0]$, we prove that $E\in P_2(0,1]$.

Gluing

Suppose $\mathcal{D}$ has a *semiorthogonal decomposition* $\mathcal{D}=\langle\mathcal{D}_1,\mathcal{D}_2\rangle$. By definition, this means that $\mathcal{D}_1$ and $\mathcal{D}_2$ are triangulated subcategories in $\mathcal{D}$ such that $\mathrm{Hom}(E_2,E_1)=0$ for every $E_1\in\mathcal{D}_1$ and $E_2\in\mathcal{D}_2$, and for every object $E\in\mathcal{D}$ there exists an exact triangle

$$E_2\to E\to E_1\to E_2[1] \tag{1}$$

with $E_1\in\mathcal{D}_1$, $E_2\in\mathcal{D}_2$. Assume we are given hearts of t-structures $H_1\subset\mathcal{D}_1$ and $H_2\subset\mathcal{D}_2$. In this situation there exists a glued t-structure on $\mathcal{D}$. Under the additional assumption that

$$\mathrm{Hom}^{\leq 0}(H_1,H_2)=0 \tag{2}$$

the corresponding glued heart H will be the smallest full subcategory of $\mathcal{D}$, closed under extensions and containing H_1 and H_2.

If we have stability conditions on $\mathcal{D}_1$ and $\mathcal{D}_2$ with the above hearts then we can define a central charge Z on $\mathcal{D}$ uniquely, so that it restricts to the given central charges on $\mathcal{D}_1$ and $\mathcal{D}_2$. In order for the pair (H,Z) to determine a stability condition on $\mathcal{D}$ one should check the Harder-Narasimhan property.

Theorem. Let $(\mathcal{D}_1, \mathcal{D}_2)$ be a semiorthogonal decomposition of a triangulated category $\mathcal{D}$. Suppose (σ_1, σ_2) is a pair of *reasonable* stability conditions on $\mathcal{D}_1$ and $\mathcal{D}_2$, with the slicings P_i and central charges Z_i $(i = 1, 2)$, and let a be a real number in $(0, 1)$. Assume:

(1) $\mathrm{Hom}_{\mathcal{D}}^{\leq 0}(P_1(0,1], P_2(0,1]) = 0$;
(2) $\mathrm{Hom}_{\mathcal{D}}^{\leq 0}(P_1(a,a+1], P_2(a,a+1]) = 0$;

Then there exists a stability σ glued from σ_1 and σ_2. Furthermore, σ is reasonable. For fixed a the gluing map is continuous.
Definition: A stability condition $\sigma = (Z, P)$ on $\mathcal{D}$ is called *reasonable* if

$$\inf_{E \text{ semistable}, E \neq 0} |Z(E)| > 0$$

Easy to see: If σ is reasonable then every category $P(t, t+\eta)$ for $0 < \eta < 1$ is of finite length.

Also true: this property propagates on the connected component.

Note: the categories $P(t, t+\eta)$ for $\eta < 1$ are *quasi-abelian* (pull-back of strict epi is strict epi, push-out of strict mono is strict mono). In particular, for such categories if fg is strict mono then g is strict mono.

Idea of proof: define torsion pair $P(a, 1]$, $P(0, a]$ on the glued heart, where $P(a, 1]$ is glued from $P_1(a, 1]$ and $P_2(a, 1]$, etc. Then check HN-property separately on $P(a, 1]$ and $P(0, a]$.

Double coverings

Let $\pi : X \to Y$ be a double covering of smooth projective varieties X and Y, ramified along a smooth divisor R in Y. Then we have an action of $\mathbb{Z}_2$ on X such that the nontrivial element acts by the corresponding involution $\tau : X \to X$. Let us denote by $\mathcal{D}_{\mathbb{Z}_2}(X)$ the corresponding bounded derived category of $\mathbb{Z}_2$-equivariant coherent sheaves on X.

Let us denote by $i : R \to X$ (resp., $j : R \to Y$) the closed embedding of the ramification divisor into X (resp., Y). For every sheaf F on R we equip i_*F with the trivial $\mathbb{Z}_2$-equivariant structure. This gives a functor $i_* : \mathcal{D}(R) \to \mathcal{D}_{\mathbb{Z}_2}(X)$. On the other hand, for a coherent sheaf F on Y we have a natural $\mathbb{Z}_2$-equivariant structure on π^*F, so we obtain a functor $\pi^* : \mathcal{D}(Y) \to \mathcal{D}_{\mathbb{Z}_2}(X)$.

Theorem The functors $i_* : \mathcal{D}(R) \to \mathcal{D}_{\mathbb{Z}_2}(X)$ and $\pi^* : \mathcal{D}(Y) \to \mathcal{D}_{\mathbb{Z}_2}(X)$ are fully faithful. We have two canonical semiorthogonal decompositions of $\mathcal{D}_{\mathbb{Z}_2}(X)$:

$$\mathcal{D}_{\mathbb{Z}_2}(X) = \langle \pi^*\mathcal{D}(Y), i_*\mathcal{D}(R) \rangle = \langle \zeta \otimes i_*\mathcal{D}(R), \pi^*\mathcal{D}(Y) \rangle$$

where ζ is the nontrivial character of $\mathbb{Z}_2$.

Now we consider the case when X and Y are curves. In this case the ramification divisor R consists of points $p_1, \ldots, p_n$, and the category $\mathcal{D}(R)$ is generated by the orthogonal exceptional objects $\mathcal{O}_{p_1}, \ldots, \mathcal{O}_{p_n}$. Recall that the category $\mathcal{D}(X)$ has a standard stability condition σ_{st} with $Z_{st} = -\deg + i \,\mathrm{rk}$ and $P_{st}(0, 1] = \mathrm{Coh}(X)$. There is an induced stability condition on $\mathcal{D}_{\mathbb{Z}_2}(X)$ with the heart $\mathrm{Coh}_{\mathbb{Z}_2}(X)$ that we still denote by σ_{st}. We have the $\widetilde{\mathrm{GL}_2^+(\mathbb{R})}$-orbit of σ_{st}

$$O \subset \mathrm{Stab} = \mathrm{Stab}_{\mathcal{N}}(\mathcal{D}_{\mathbb{Z}_2}(X)).$$

But there are more points in Stab around O (dimension of Stab over $\mathbb{C}$ is $n+2$).

For a subset $I \subset \{1, \dots, n\}$ let us denote by $\mathcal{D}(I) \subset \mathcal{D}_{\mathbb{Z}_2}(X)$ the full triangulated subcategory generated by $\pi^*\mathcal{D}(Y)$ and $\mathcal{O}_{p_i}$ with $i \in I$. Consider a semiorthogonal decomposition

$$\mathcal{D}_{\mathbb{Z}_2}(X) = \langle \mathcal{D}(I), \langle \mathcal{O}_{p_i}, i \notin I \rangle \rangle.$$

Can use the t-structure on $\mathcal{D}(I)$ with the heart $\mathrm{Coh}(I) = \mathrm{Coh}(X) \cap \mathcal{D}(I)$ to obtain glued stabilities on $\mathcal{D}_{\mathbb{Z}_2}(X)$. Namely, choosing positive numbers $\mathbf{n} = (n_i)$ for $i \notin I$, we can get by gluing stability conditions σ with the hearts

$$H(I; \mathbf{n}) = [\mathrm{Coh}(I), [\mathcal{O}_{p_i}[-n_i], i \notin I]],$$

and central charge Z satisfying

(1) $\Im Z(\mathcal{O}_X) > 0$, and $Z(\mathcal{O}_{\pi^{-1}(y)}) \in \mathbb{R}_{<0}$ for any point $y \in Y$;
(2) $Z(\mathcal{O}_{p_i}[-n_i]) \in \mathfrak{h}'$ for $i \notin I$;
(3) $Z(\mathcal{O}_{p_i}) \in \mathbb{R}_{<0}$ and $Z(\zeta \otimes \mathcal{O}_{p_i}) \in \mathbb{R}_{<0}$ for $i \in I$,

where $\mathfrak{h}' \subset \mathbb{C}$ denotes the union of the upper half-plane with $\mathbb{R}_{<0}$. All the objects $\mathcal{O}_{\pi^{-1}(y)}$ for $y \in Y$ are σ-semistable (of phase 1).

Theorem Let $U \subset \mathrm{Stab}$ denote the set of locally finite stability conditions $\sigma = (Z, P)$ such that

(1) $\mathcal{O}_{\pi^{-1}(y)}$ is stable of phase ϕ_σ for every $y \in Y \setminus R$;
(2) $\mathcal{O}_{p_i}, \zeta \otimes \mathcal{O}_{p_i}$ are semistable with the phases in $(\phi_\sigma - 1, \phi_\sigma + 1)$ for all $i = 1, \dots, n$.

Then every point in U is obtained from one of the above glued stability conditions with all $n_i = 1$, by the action of an element of $\mathbb{R} \times \mathrm{Pic}_{\mathbb{Z}_2}(X)$, where $\mathbb{R}$ acts on $\mathrm{Stab}_{\mathcal{N}}(\mathcal{D}_{\mathbb{Z}_2}(X))$ by rotations (shifts of phases). The subset U is open in $\mathrm{Stab}_{\mathcal{N}}(\mathcal{D}_{\mathbb{Z}_2}(X))$. The natural map $U \to \overline{U}$ is a universal covering of $\overline{U}$, and $\overline{U} = U/\mathbb{Z}$, where $1 \in \mathbb{Z}$ acts on the stability space by shifting phases by 2. Furthermore, U is contractible.

Stratification near standard orbit

Consider the walls

$$\overline{W}_i = \{Z \mid Z(\mathcal{O}_{2p_i}) = cZ(\mathcal{O}_{p_i}) \text{ for } c \in \mathbb{R}\} \subset \overline{U}$$

and the corresponding walls W_i in U (all $\widetilde{\mathrm{GL}_2^+(\mathbb{R})}$-invariant). Then $\cap_{i=1}^n W_i$ has real codimension n in U and is a retract of O in Stab. The complement $U \subset \cup_{i=1}^n W_i$ is the union of connected components C_I indexed by subsets $I \subseteq \{1, 2, \cdots, n\}$ where for all $Z \in C_I$,

$$\phi(O_{p_i}) < \phi(O_{2p_i}), i \in I, \quad \phi(\mathcal{O}_{p_i}) > \phi(\mathcal{O}_{2p_i}), i \notin I.$$

Proposition. The action of $\mathrm{Pic}_{\mathbb{Z}_2}(X)$ is transitive on the set of chambers (C_I) with stabilizer $\pi^* \mathrm{Pic}(Y)$ (note that $\mathrm{Pic}_{\mathbb{Z}_2}(X)/\pi^* \mathrm{Pic}(Y) \cong (\mathbb{Z}_2)^n$).

Proposition. Suppose $Y = \mathbb{P}^1$. Then for all points $\sigma \in U \setminus \cap W_i$ some rotation of σ is glued from an exceptional collection.

Non-rational case

Assume now that $Y \not\simeq \mathbb{P}^1$.

Problem: given an exact triangle $Y \to E \to X \to X[1]$ in $D^b(\mathcal{A})$, where $\mathcal{A}$ is an abelian category of homological dimension 1, $E \in \mathcal{A}$ and $\mathrm{Hom}^{\leq 0}(Y, X) = 0$, try to restrict possible X and Y.

Gorodentsev, Kuleshov, Rudakov: if there are no rigid objects in $\mathcal{A}$ then X and Y are automatically in $\mathcal{A}$.

In our case there are few rigid objects (e.g. $\mathcal{O}_{p_i}$).

For each i consider $\mathcal{D}_{p_i} \subset \mathcal{D}_{\mathbb{Z}_2}(X)$ of sheaves supported on p_i. Set $\mathcal{S}_i = \mathrm{Stab}(\mathcal{D}_{p_i})$.

Lemma. Every stability condition on $\mathcal{D}_{\mathbb{Z}_2}(X)$ restricts to a stability condition on $\mathcal{D}_{p_i}$ (recall that $g(Y) \geq 1$).

Proposition. There exists a noncompact simply connected Riemann surface Σ such that $\mathcal{S}_i \simeq \mathbb{C} \times \Sigma$. The projection $f_i : \mathcal{S}_i \to \mathbb{C}$ satisfies $\exp(\pi f) = Z(\mathcal{O}_{2p_i})$.

Theorem. Assume that $g(Y) \geq 1$. Then the map

$$\mathrm{Stab} \to \mathcal{S}_1 \times \ldots \times \mathcal{S}_n \times \mathbb{C} : \sigma \mapsto (\sigma|_{\mathcal{D}_{p_1}}, \ldots, \sigma|_{\mathcal{D}_{p_n}}, Z(\mathcal{O}))$$

identifies Stab with a contractible open subset of the closed subset Θ consisting of $(\sigma_1, \ldots, \sigma_n, z)$ such that $f_1(\sigma_1) = \ldots = f_n(\sigma_n)$.