A_∞-STRUCTURES AND THETA SERIES

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A_∞-algebras

Appeared in the work of *Stasheff* in 1963 as generalizations of *dg-algebras*. (A_∞-algebras are also called *strongly homotopy associative algebras*). Currently, are popular in math/physics related to string theory.

Definition. A *dg-algebra* over a field k is a $\mathbb{Z}$-graded associative k-algebra A with a differential d of degree $+1$ such that the Leibnitz identity holds:

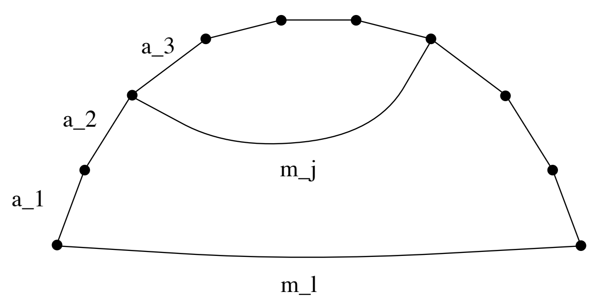
$$d(a \cdot b) = d(a) \cdot b + (-1)^{\deg a} a \cdot d(b).$$

Definition. An A_∞*-algebra* is a $\mathbb{Z}$-graded k-vector space A equipped with a collection of maps

$$m_n : A^{\otimes n} \to A, n \geq 1 \text{ of degree } 2 - n, \text{ such that}$$

$$\sum_{n=i+j+k} (-1)^{i+jk} m_{i+k+1}(\mathrm{id}^{\otimes i} \otimes m_j \otimes \mathrm{id}^{\otimes k}) = 0 \text{ for } n \geq 1.$$

In other words, we take the sum (with signs) of the expressions of the form



For example, for $n = 1$ and $n = 2$ this gives

$$m_1(m_1(a)) = 0, \;\; m_1(m_2(a, b)) = m_2(m_1(a), b) \pm m_2(a, m_1(b)),$$

i.e., m_1 is a differential of degree $+1$, and m_2 satisfies the Leibnitz identity.

For $n = 3$ the A_∞-constraint states that the product m_2 is associative up to an explicit homotopy:

$$m_2(m_2(a, b), c) - m_2(a, m_2(b, c)) = m_1(m_3(a, b, c))$$
$$\pm m_3(m_1(a), b, c) \pm m_3(a, m_1(b), c) \pm m_3(a, b, m_1(c)).$$

Therefore, the cohomology space $H^*(A, m_1)$, with the product induced by m_2, is an associative algebra.

The A_∞-constraint simplifies in the case $m_1 = 0$. For example, the $n = 4$ case will look like

$$m_3(a,b,c)\cdot d \pm a\cdot m_3(b,c,d) = \\ m_3(a\cdot b,c,d) - m_3(a,b\cdot c,d) + m_3(a,b,c\cdot d).$$

The $n = 5$ case becomes

$$m_3(m_3(a,b,c),d,e) \pm m_3(a,m_3(b,c,d),e) \pm m_3(a,b,m_3(c,d,e)) = \\ m_4(a,b,c,d)\cdot e \pm a\cdot m_4(b,c,d,e) + m_4(a\cdot b,c,d,e) - \dots$$

Kadeishvili's Theorem (1982)

Let (B,d) be a dg-algebra. Then there is a natural structure of an A_∞-algebra on $A = H^*(B,d)$ with $m_1 = 0$, such that A and B are equivalent as A_∞-algebras.

A *morphism* $A \to B$ *between two* A_∞*-algebras* is given by a collection of maps $f_n : A^{\otimes n} \to B$ of degree $1-n$, satisfying certain system of equations involving products on A and B. There is also a notion of composition of such morphisms, of homotopy between morphisms, etc.

Explicit construction of equivalence in Kadeishvili's Theorem requires a choice of representatives for all cohomology classes given by a k-linear embedding $A \hookrightarrow B$, and also a choice of a projector $B \to A$ (*Merkulov, Kontsevich-Soibelman*)

Kadeishvili's Theorem explains the appearance of the *Massey products* on the cohomology of dg-algebras: these are "shades" of the A_∞-structure. The simplest *triple Massey product* is defined for a triple (a,b,c) in $A = H^*(B,d)$ such that $ab = 0$ and $bc = 0$. In this case $MP(a,b,c)$ is a coset modulo

$$A_{\deg a+\deg b-1}\cdot c + a\cdot A_{\deg b+\deg c-1} \subset A_{\deg a+\deg b+\deg c-1},$$

defined as follows: choose representative $\widetilde{a}$, $\widetilde{b}$ and $\widetilde{c}$ in $\ker(d) \subset B$. Then $\widetilde{a}\widetilde{b} = d(x)$, $\widetilde{b}\widetilde{c} = d(y)$ for some $x,y \in B$. Now set $MP(a,b,c)$ to be coset containing the class of $x\widetilde{c} - (-1)^{\deg a}\widetilde{a}y \in \ker(d)$. The same coset is obtained from the element $m_3(a,b,c)$.

In fact, Massey products can be viewed as homotopy invariants of A_∞-structures. For example, if $m_1 = 0$ and m_2 is fixed then a homotopy could change m_3 to

$$m_3'(a,b,c) = m_3(a,b,c) + f_2(a,b)\cdot c \pm a\cdot f_2(b,c) \\ + f_2(a\cdot b,c) - f_2(a,b\cdot c)$$

A_∞-categories

Similarly to A_∞-algebra can define A_∞-*categories*. Higher products will look like

$$\mathrm{Hom}(F_0, F_1) \otimes \mathrm{Hom}(F_1, F_2) \otimes \ldots \otimes \mathrm{Hom}(F_{n-1}, F_n) \stackrel{m_n}{\to} \mathrm{Hom}(F_0, F_n).$$

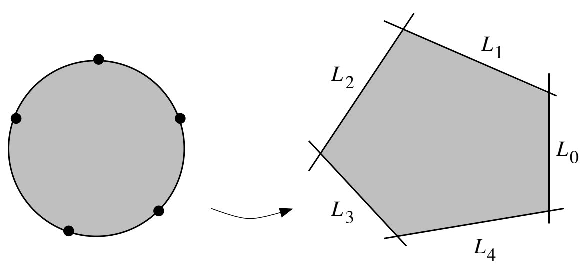
E.g., m_1 is the differential of degree 1 on morphism spaces. Taking cohomology with respect to m_1 get a usual category. Analogue of Kadeishvili's theorem holds in this setting: starting from a dg-category one gets an A_∞-structure on the cohomology category.

Example. *Derived category* $D(\mathcal{A})$ of an abelian category $\mathcal{A}$ is the localization of the category of complexes over $\mathcal{A}$ with respect to *quasi-isomorphisms*, i.e., chain maps inducing isomorphism on cohomology. In the case when $\mathcal{A}$ has enough injective objects, using injective resolutions, we can equip $D^+(\mathcal{A})$ with an A_∞-structure (canonically up to equivalence).

Fukaya category is an A_∞-category attached to a symplectic manifold (M, ω). Roughly speaking, objects are *Lagrangian submanifolds* $L \subset M$ (submanifolds of M of dimension $\dim M/2$ with $\omega|_L = 0$). Morphisms from L_1 to L_2 are elements of the space $\oplus_{p \in L_1 \cap L_2} \mathbb{C}[p]$, provided L_1 and L_2 are transversal.

The products m_n are determined by counting (pseudo-)holomorphic disks:

$$m_n([p_1], \ldots, [p_n]) = \sum_{p_0, \phi:(D, \partial D, (t_i)) \to (M, \cup L_i, (p_i))} \pm \exp(-\int_D \phi^* \omega)[p_0].$$



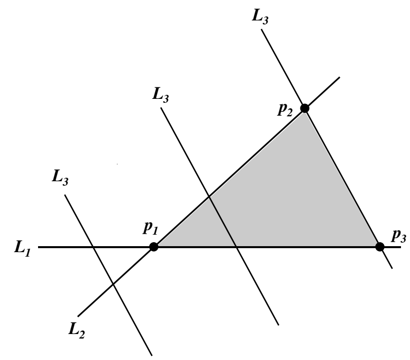
Homological Mirror Conjecture (HMC)

Kontsevich conjectured in 1994 an equivalence of the Fukaya category $\mathcal{F}(X)$ of a Calabi-Yau manifold X with the A_∞-enhancement of the derived category $D(Y)$ of coherent sheaves on the mirror dual manifold Y.

At present we have a lot of evidence for this conjecture.
One can weaken the conjecture by considering the usual categories associated with the above A_∞-categories. The hardest known case of the weak HMC is that of the smooth quartic surface in $\mathbb{P}^3$ (*Seidel 03*).

Now we will discuss the simplest case of the HMC, namely, that of a 2-*torus*.

Let $T = \mathbb{R}^2/\mathbb{Z}^2$ with the symplectic form $t dx \wedge dy$. Let L_1, L_2 and L_3 be geodesics circles on T (i.e., images of lines of rational slope). To calculate $m_2 : \mathrm{Hom}(L_1, L_2) \otimes \mathrm{Hom}(L_2, L_3) \to \mathrm{Hom}(L_1, L_3)$ in $\mathcal{F}(T)$ have to count holomorphic triangles bounded by L_1, L_2, L_3.



Computing the areas of the triangles we obtain that the coefficient of $[p_3]$ in $m_2([p_1], [p_2])$ has form

$$\sum_{n \in \mathbb{Z}} \exp(-(an+b)^2 t), \quad \text{where } a \in \mathbb{Q}, b \in \mathbb{R}.$$

Hence, the coefficients can be expressed in terms of the function

$$\theta(q, z) = \sum_{n \in \mathbb{Z}} q^{n^2/2} z^n$$

for $q = \exp(-at)$, $a \in \mathbb{Q}$.

This is famous *theta function*. For fixed q it satisfies the following quasi-periodicity condition in z:

$$\theta(qz) = q^{-1/2} z^{-1} \theta(z).$$

This means that it can be viewed as a global section of some algebraic line bundle over the elliptic curve $\mathbb{C}^*/q^{\mathbb{Z}} = \mathbb{C}/(\mathbb{Z} + \frac{it}{2\pi}\mathbb{Z})$.

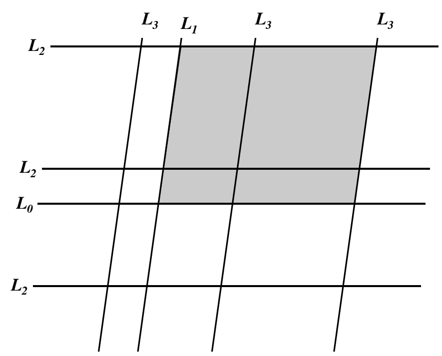
More generally, morphisms between vector bundles on an elliptic curve have canonical bases (as follows from classification of such bundles by Atiyah, and from the classical theory of theta functions). Matrix coefficients of the products m_2 with respect to these bases can be computed explicitly in terms of $\theta(q^a, z)$, for $a \in \mathbb{Q}$.

Elaborating on this we proved

Theorem*(P-Zaslow 98)* Weak HMC is true for 2-tori, i.e, the Fukaya category of a 2-torus is equivalent to the derived category of coherent sheaves on an elliptic curve.

Our proof uses the fact that in this case every object of the derived category is the direct sum of its cohomology sheaves, every sheaf is the direct sum of sheaves supported at points and vector bundles. Together with the Serre duality $\mathrm{Ext}^1(F,G) \simeq \mathrm{Hom}(G,F)^*$ this allows to reduce all computations to the composition of morphisms between vector bundles. So it all boils down to the addition theorem for theta functions.

What about higher Fukaya products? Here is an example:



The coefficients of this triple product are of the form

$$\sum_{m\geq 0,n\geq 0} q^{(m+a)(n+b)} - \sum_{m<0,n<0} q^{(m+a)(n+b)}.$$

Thus, we get values of the *Kronecker function*

$$F(q,z,w) = \sum_{m\geq 0,n\geq 0} q^{mn} z^m w^n - \sum_{m<0,n<0} q^{mn} z^m w^n,$$

where $|q| < |z|, |w| < 1$. In fact, the triple product we considered is an example of a *univalued* triple Massey product. Computing the corresponding Massey product for the derived category of an elliptic curve we get a certain ratio of theta-functions. So the HMS predicts the identity:

$$F(z,w) = c(q) \cdot \frac{\theta_{11}(zw)}{\theta_{11}(z)\theta_{11}(w)},$$

where $\theta_{11}(z) = \theta(-q^{-1/2}z)$. Such an identity indeed was discovered by *Kronecker* in 1881.

It turned out that to prove the strong HMC (with higher products) for an elliptic curve one doesn't really have to know that much about higher products (but one has to understand m_2 really well). A cohomology computation shows that the A_∞-structure on a large subcategory of the derived category of the elliptic curve is determined by m_2 almost uniquely - up to homotopy and certain rescaling. This leads to

Theorem *(P 2000)* Strong HMS is true for the subcategory of the Fukaya category of the 2-torus corresponding to nonvertical lines of rational slope: this A_∞-category is equivalent to the A_∞-subcategory of bundles.

Remark. To deal with the entire Fukaya category of the 2-torus one has to add torsion sheaves on the derived category side.

As an application of the HMS we can interpret certain interesting q-series as triple Massey products in the derived category of the elliptic curve. These series are called *indefinite theta series* because they have form

$$\sum_{m\geq 0,n\geq 0} f(m,n)q^{Q(m,n)} - \sum_{m<0,n<0} f(m,n)q^{Q(m,n)},$$

where $Q(m,n) = am^2 + 2bmn + cn^2$ is an indefinite quadratic form with $a,b,c>0$ (i.e., $b^2 > ac$).

Theorem *(Pasol-P 04)* The above series comes from some univalued Massey product on the elliptic curve if and only if the corresponding sums along all horizontal and vertical lines vanish:

$$\sum_m f(m,n_0)q^{Q(m,n_0)} = \sum_n f(m_0,n)q^{Q(m_0,n)} = 0.$$

Generalizations of Kronecker's identity coming from the HMS:

$$\sum^{\pm}(-1)^{m+n}q^{2mn+\frac{m^2+m+n^2+n}{2}} = \prod(1-q^n)^2 = q^{-\frac{1}{12}}\eta(q)^2$$

— this was also obtained by Kac-Peterson using representation theory;

$$\sum^{\pm}_{m+n \text{ odd}} (-1)^{\frac{m+n-1}{2}} q^{\frac{m^2+6mn+3n^2-1}{2}} = \prod(1+q^n)(1-q^{2n})(1-q^{3n})(1+q^{6n}).$$

Our series can be rewritten as linear combinations of *Hecke's indefinite theta series*:

$$\Theta_\Lambda(q) = \sum_{\lambda\in(\Lambda+c)\cap C/G} \mathrm{sign}(\lambda)q^{Q(\lambda)},$$

where (Λ, Q) is a rank-2 indefinite lattice, $c \in \Lambda\otimes\mathbb{Q}$, $G \subset \mathrm{Aut}_+(\Lambda, Q)$. This leads to a geometric interpretation of such series. In particular, we hope to resolve some questions raised by Hecke about linear relations between his series.

Triple products and Yang-Baxter equations (YBE)

Quantum YBE (QYBE) for $R \in A\otimes A$, where $A = \mathrm{Mat}(N,\mathbb{C})$:

$$R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12},$$

where $R^{12} = R\otimes 1 \in A\otimes A\otimes A$, etc.

Spectral parameter: replace R^{ij} with $R(v_i - v_j)$.

Unitarity condition: $RR^{21} = 1\otimes 1$.

Came from exactly solvable two-dimensional lattice models. Plays an important role in Drinfeld's theory of quantum groups.

If $R(h)$ is the deformation of the trivial solution:

$$R(h) = 1 \otimes 1 + hr + \dots$$

then r satisfies the *Classical YBE (CYBE)*:

$$[r^{12}, r^{13}] - [r^{23}, r^{12}] + [r^{13}, r^{23}] = 0,$$

and the unitarity $r^{21} = -r$ (as before, $r^{ij} = r^{ij}(v_i - v_j)$).

It turns out that the triple products on the elliptic curves corresponding to the quadruple of objects $(E_1, \mathcal{O}_{p_1}, E_2, \mathcal{O}_{p_2})$, where E_1 and E_2 are stable vector bundles from the same component of the moduli space, p_1 and p_2 are points, give solutions of the *Associative YBE (AYBE)*:

$$r^{12}(-u')r^{13}(u+u') - r^{23}(u+u')r^{12}(u) + r^{13}(u)r^{23}(u') = 0.$$

Namely, using natural bases in the spaces $\mathrm{Hom}(E_i, \mathcal{O}_{p_j})$ and Serre duality we can write the triple product as

$$m_3 : V \otimes V^* \otimes V \to V, \text{ or equivalently } r : V \otimes V \to V \otimes V.$$

The equation on r comes from the A_∞-constraint for $n = 5$. Also, get the unitarity: $r^{21}(-u) = -r(u)$.

Taking limit as $u \to 0$ of a solution of the AYBE and projecting to traceless matrices we get solutions of the CYBE (provided the limit exists).
Natural questions: 1) which sl_n-solutions of the CYBE lift to the AYBE? 2) What is the relation to the QYBE?

Consider nondegenerate unitary solutions of the AYBE of the form

$$r(u) = \frac{1 \otimes 1}{u} + r_0 + ur_1 + \dots$$

Let $\overline{r}_0$ be the projection of r_0 to sl_n.
Theorem *(P 06)* (i) In this situation $\overline{r}_0(v)$ is a nondegenerate solution of the CYBE, so it falls within *Belavin-Drinfeld classification*: it is either rational, or trigonometric, or elliptic.
(ii) If $\overline{r}_0(v)$ has a period (=not rational) then $r(u)$ satisfies the QYBE.
(iii) The only elliptic solutions are those coming from m_3 on elliptic curve (given by Belavin's elliptic R-matrix).
(iv) There is a classification of trigonometric solutions in terms of combinatorial data similar to Belavin-Drinfeld triples.

It turned out that not all nondegenerate trigonometric solutions of the CYBE can be lifted to the AYBE (*Schedler*). Also, not all trigonometric solutions of the AYBE can be obtained from m_3 on nodal degenerations of elliptic curves.

Problems

What geometric data is responsible for all trigonometric solutions of the AYBE?

Compute the superpotential for the elliptic curve (generating series for all higher products).

Investigate the A_∞-structure for higher genus curves. E.g., considering triple Massey products for the quadruple of objects $(\mathcal{O}_C, \mathcal{O}_{p_1}, L, \mathcal{O}_{p_2})$, where L is a line bundle of degree $g-1$, one obtains *Fay's trisecant identity* for theta functions on the Jacobian of C (*P 03*).

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